# A Theory of Multi-Scale, Curvature- and Torsion-Based Shape <br> Representation for Space Curves 

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#### Abstract

This paper introduces a novel and new multi-scale shape representation technique for space curves which satisfies several criteria considered necessary for any shape representation method. This property makes the representation suitable for tasks which call for recognition of a noisy curve at any scale or orientation.

The method rests on the concept of describing a curve at varying levels of detail using features that are invariant with respect to transformations which do not change the shape of the curve. Three different ways of computing the representation are described in this paper. These three methods result in the following representations: the curvature and torsion scale space images, the renormalized curvature and torsion scale space images, and the resampled curvature and torsion scale space images.

The process of describing a curve at increasing levels of abstraction is referred to as the evolution of that curve. Several evolution properties of space curves are described in this paper. Some of these properties show that evolution is a physically plausible operation and characterize possible behaviours of space curves during evolution. Some show that the representations proposed in this paper in fact satisfy the required criteria. Others impose constraints on the location of a space curve as it evolves. Together, these evolution properties provide a theoretical foundation for the representation methods introduced in this paper.


## A. Introduction

Why study the problem of representing the shape of space curves? Space curves are useful to study for the following reasons:
a. Trajectories of objects in outer space and paths taken by atomic particles are space curves. Often, such an object or particle can be recognized when studying the shape of its path when subjected to specific forces.
b. Axes of generalized cones and cylinders [Agin \& Binford 1973] are also space curves. A generalized cone or cylinder representation of a three-dimensional object can itself be efficiently represented by its axes.
c. Bounding contours of objects that consist of flat or nearly flat surfaces are rich in information and can be used to represent the object effectively and economically. These bounding contours are space curves and can be extracted by thinning the object into lines and planes. An attempt to describe such objects using three-dimensional surfaces may not add much useful information but can significantly increase storage and processing requirements.

This paper introduces a novel theory of multi-scale shape representation for
space curves. A useful shape representation method in computational vision should make accurate and reliable recognition of an object possible. Therefore such a representation should satisfy a number of necessary criteria. The following is a list of such criteria:
a. Efficiency: The representation should be efficient to compute and store. This is important since it may be necessary for an object recognition system to perform real-time recognition.
b. Invariance: Uniform scaling, rotation and translation are considered to be the transformations which do not change the shape of an object. Therefore the representation should remain essentially invariant while the represented object undergoes one of these transformations.
c. Sensitivity: The degree of change to the shape of an object should correspond to the degree of the resulting change in its representation. Otherwise, a small change to the shape of an object may cause a large change in its representation which will make it impossible to detect two objects that are close in shape.
d. Uniqueness: There should be a one-to-one correspondence between objects and their representations. This requirement is only up to the class induced by criterion $\mathbf{b}$ above. If this criterion is not satisfied, it will be impossible to distinguish objects of different shapes which have the same representation.
e. Detail: The representation should contain information about the object at varying levels of detail. This is important since features on an object usually exist at different scales. Large-scale features describe the basic structure of the object while small-scale features describe fine detail on the object.
f. Robustness: Any arbitrary initial choices should not change the structure of the representation.
g. Local support: The representation should be computed using local support so that incomplete data only affects the representation locally.
h. Ease of implementation: It is advantageous to use a shape representation technique which is as easy as possible to implement. This will help minimize programming and debugging efforts.
i. Matchability: The representation technique should compute a data structure which lends itself easily to a matching algorithm. Such an algorithm would take two representations as input and return a description of the similarity (or
dissimilarity) of the shapes they represent. If this criterion is not satisfied, recognition of an object will not be possible even if its representation has been computed.

A number of shape representation methods for planar curves have been proposed in the computational vision literature. A number of these methods can be extended to apply to space curves but each fails to satisfy one or more of the criteria outlined above. The following is a summary of those methods and the criteria each fails to satisfy:
a. Hough transform: Has been used to detect lines [Hough 1962], circles [Duda \& Hart 1972] and arbitrary shapes [Ballard 1981]. Edge elements in the image vote for the parameters of the objects of which they are parts. The votes are accumulated in a parameter space. The peaks of the parameter space then indicate the parameters of the objects searched for. For any object more complicated than a line, the parameter space becomes multi-dimensional and therefore the Hough transform fails to satisfy the efficiency criterion. The parameters which define an object change when it undergoes rotation, uniform scaling or translation therefore the invariance criterion is not satisfied either. The local support criterion is also not satisfied since in order to obtain a distinguishable peak in the parameter space, the entire object must be present in the image.
b. Chain encoding [Freeman 1974, McKee \& Aggarwal 1977] and polygonal approximations [Pavlidis 1972, 1977]: The curve is approximated using a polygon or line segments which lie on a grid. These methods do not satisfy the invariance criterion since the approximating polygon rotates, scales and moves as the original curve rotates, scales and moves. Furthermore, the robustness criterion is not satisfied since changing the starting point on the curve can change the shape of the approximating polygon, and the sensitivity criterion is not satisfied since a small change in the shape of the curve can drastically change the shape of the approximating polygon.
c. Shape factors and quantitative measurements [Danielsson 1978]: The shape of the object is described using one or more global quantitative measurements of the object such as area, perimeter and compactness. The uniqueness criterion is not satisfied since there is a dramatic reduction in data. The detail criterion is not satisfied since only one scale is represented. Furthermore, the local support criterion is not satisfied since the entire object must be present in the image and the sensitivity criterion is not satisfied since even large changes in the shape of an object may cause no change in its representation.
d. Strip trees [Davis 1977, Ballard \& Brown 1982]: A strip tree is a set of
approximating polygons or polylines ordered such that each polygon or polyline approximates the curve with less approximation error than the previous polygon or polyline. This class also suffers from the shortcomings of class $\mathbf{b}$.
e. Splines [Ballard \& Brown 1982]: The curve is represented using a set of analytic and smooth curves. The invariance criterion is not satisfied since shapepreserving transformations of the curve change the parameters of the approximating splines. The uniqueness criterion is not satisfied since reconstruction of the original curve is not possible. The robustness criterion is also not satisfied since a particular choice of knot points results in a particular set of approximating splines.
f. Smoothing splines [Shahraray \& Anderson 1989]: The curve is parametrized to obtain two coordinate functions. Cross-validated regularization [Wahba 1977] is then used to arrive at an "optimal" smoothing of each coordinate function. The smoothed functions together define a new smooth curve. This method does not satisfy the efficiency criterion since cross-validation is quite expensive. It does not satisfy the detail criterion since the object is represented at only one scale. It does not satisfy the uniqueness criterion since the reconstruction of the original curve is not possible, It also does not satisfy the local support criterion since all data points must be present for cross-validation.
g. Fourier descriptors [Persoon \& Fu 1974]: The curve is represented by the coefficients of the Fourier expansion of a parametric representation of the curve. The invariance criterion is not satisfied by this class since shapepreserving transformations of the curve will change its Fourier coefficients. The local support criterion is also not satisfied since the entire curve must be available in order to compute its Fourier expansion.
h. Curvature primal sketch [Asada \& Brady 1984]: The curve is approximated using a library of well-defined, analytic curves. Then the curvature function of the approximating curve is computed and convolved with a Gaussian of varying standard deviation. This method does not satisfy the sensitivity criterion. If the original curve is noisy, then computing its curvature function is an error-prone process and the computed representation may change significantly. More will be said on this method in section 6.
i. Extended circular image [Horn \& Weldon 1986]: This representation is the two-dimensional equivalent of the extended Gaussian image. In the extended circular image, one is given the radius of curvature as a function of normal direction. The invariance criterion is not satisfied by this method since the representation rotates as the original curve rotates. The uniqueness criterion is
also not satisfied since the representation is one-to-one only for the class of simple and convex curves.
j. Volumometric diffusion [Koenderink \& van Doorn 1986]: A geometrical object is defined by way of its "characteristic function" $\chi(\mathbf{r})$ which equals unity when the point $\mathbf{r}$ belongs to the object and zero otherwise. The object is then blurred by requiring that its characteristic function satisfy the diffusion equation. The boundary of each blurred object is defined by the equation $\chi(\mathbf{r})=0.5$. The efficiency criterion is not satisfied by this method since an image must be convolved with a large number of two-dimensional Gaussian filters. The invariance criterion is also not satisfied since shape-preserving transformations of the object also change the blurred objects computed by this method.

A multi-scale representation for one-dimensional functions was first proposed by Stansfield [1980] and later developed by Witkin [1983]. The function $f(x)$ is convolved with a Gaussian function as its variance $\sigma^{2}$ varies from a small to a large value. The zero-crossings of the second derivative of each convolved function are extracted and marked in the $x-\sigma$ plane. The result is the scale space image of the function.

The curvature scale space image was introduced in [Mokhtarian \& Mackworth 1986] as a new shape representation for planar curves. The representation is computed by convolving a path-based parametric representation of the curve with a Gaussian function, as the standard deviation of the Gaussian varies from a small to a large value, and extracting the curvature zero-crossing points of the resulting curves. The representation is essentially invariant under rotation, uniform scaling and translation of the curve. This and a number of other properties makes it suitable for recognizing a noisy curve at any scale or orientation. This representation was further generalized to space curves by Mokhtarian [1988a]. The curvature and torsion scale space images of a space curve are computed by extracting the curvature level-crossings and torsion zero-crossings of the curve respectively at varying levels of detail. The process of describing a curve at increasing levels of abstraction is referred to as the evolution of that curve. The evolution of a space curve and the curvature and torsion scale space images are described in detail in section $\mathbf{B}$.

Mackworth and Mokhtarian [1988] introduced a modification of the curvature scale space image referred to as the renormalized curvature scale space image. This representation is computed in a similar fashion but the curve is reparametrized by arc length after convolution. As was demonstrated in [Mackworth \& Mokhtarian 1988], the renormalized curvature scale space image is more suitable for recognizing a curve with non-uniform noise added to it. However,
unlike the regular curvature scale space representation, the renormalized curvature scale space applies only to closed curves. The renormalized curvature and torsion scale space images are described in detail in section $\mathbf{C}$.

The resampled scale space images are significant further refinements of the renormalized scale space images which are based on the concept of arc length evolution. It is shown that the resampled scale space images are even more suitable than the renormalized scale space images for recognition of curves with added non-uniform noise. The arc length evolution of a space curve and the resampled curvature and torsion scale space images are described in detail in section $\mathbf{D}$.

Section $\mathbf{E}$ contains the evolution and arc length evolution properties of space curves. Almost all these properties are shown to be true about both evolution and arc length evolution. Together, these properties provide a theoretical foundation for the representation method proposed in this paper. The proofs of the lemmas and theorems of section $\mathbf{E}$ are contained in the appendix.

Section F presents several examples of space curves during evolution and scale space representations of those curves and presents a number of experiments carried out to demonstrate the stability of the proposed representations under various conditions of noise. Section $\mathbf{F}$ also discusses the significance of the evolution and arc length evolution properties described in section E. It argues that these properties show that:
a. Evolution and arc length evolution are physically plausible operations.
b. There exist strong constraints on the location of a space curve during evolution or arc length evolution.
c. The representations based on evolution and arc length evolution satisfy the criteria required of any shape representation method.
d. Behaviour of a space curve during evolution or arc length evolution can be completely characterized. For example, the shape of a space curve can be locally determined before and after the creation of a cusp point.

Section $\mathbf{G}$ presents the conclusions of this paper.

## B. The curvature and torsion scale space images

This section introduces the parametric representation of space curves and describes the Frenet Trihedron for space curves. Curvature and torsion of a space curve are then defined and geometrical interpretations given to them. Next, it is shown how to compute curvature and torsion on a space curve at varying levels of detail. A multi-scale representation for a space curve which combines
information about the curvature and torsion of the curve at varying levels of detail is then proposed.

## I. The parametric representation of a space curve

The set of points of a space curve are the values of the continuous, vectorvalued, locally one-to-one function [Goetz 1970]:

$$
\begin{equation*}
\mathbf{r}=\mathbf{r}(u)=(x(u), y(u), z(u)) \tag{1}
\end{equation*}
$$

where $x(u), y(u)$ and $z(u)$ are the components of $\mathbf{r}(u)$ and $u$ is a function of arclength $s$ of the curve. $s$ is also called the natural parameter. The function $\mathbf{r}(u)$ or the triple of functions $(x(u), y(u), z(u))$ is called a parametric representation of the curve.

## II. Frenet Trihedron and formulae for space curves

With every point $P$ of a space curve of class $C_{2}$ is associated an orthonormal triple of unit vectors: the tangent vector $t$, the principal normal vector $\mathbf{n}$ and the binormal vector $\mathbf{b}$ (Figure 1). The osculating plane at $P$ is defined to be the plane with the highest order of contact with the curve at $P$. The principal normal vector is the unit vector normal to the curve at $P$ which lies in the osculating plane. The binormal vector is the unit vector perpendicular to the osculating plane such that the three vectors $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ in that order form a positively oriented triple. The plane containing $\mathbf{t}$ and $\mathbf{n}$ is the osculating plane. The one containing $\mathbf{n}$ and $\mathbf{b}$ is the normal plane and the one containing $\mathbf{b}$ and $\mathbf{t}$ is the rectifying plane. The derivatives of $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ with respect to the arc-length parameter give us:

$$
\frac{d \mathbf{t}}{d s}=\kappa \mathbf{n}, \quad \frac{d \mathbf{n}}{d s}=-\kappa \mathbf{t}+\tau \mathbf{b}, \quad \frac{d \mathbf{b}}{d s}=-\tau \mathbf{n} .
$$

These formulae are called the Frenet or the Serret-Frenet formulae. The coefficients $\kappa$ and $\tau$ are called the curvature and torsion of the curve respectively.

Curvature is the instantaneous rate of change of the tangent vector to the curve with respect to the arc length parameter. Curvature has no sign. Torsion is the instantaneous rate of change of the binormal vector with respect to the arc length parameter. A sign is assigned to the absolute measure of torsion as following:

Let point $P$ correspond to value $s$ of the arc length parameter and let point $Q$ correspond to value $s+h$. Let line $l$ be the intersection of the osculating planes at $P$ and $Q$. Give line $l$ the orientation of a vector $\mathbf{w}$ on $l$ such that $\mathbf{t} . \mathbf{w}>0$. Consider the rotation about $l$ through a non-obtuse angle which superposes the osculating plane at $P$ on the osculating plane at $Q$. This rotation also superposes $\mathbf{b}(s)$ on $\mathbf{b}(s+h)$. If $\mathbf{b}(s), \mathbf{b}(s+h)$ and $\mathbf{w}$ form a positively oriented triple, then torsion has
positive sign, otherwise it has negative sign.
Since the curve is represented in parametric form, in order to compute curvature and torsion at each point on the curve, we need to express those quantities in terms of the derivatives of $x(),. y($.$) and z($.$) . In what follows, \mathbf{r}(u)$ represents the parametrization of a space curve with respect to an arbitrary parameter and $\rho(s)$ represents the parametrization of that curve with respect to the arc-length parameter. We next show how to express curvature and torsion on a space curve in terms of derivatives of its coordinate functions.

## III. Curvature

In case of an arc-length parametrization, we have:

$$
\kappa=|\ddot{\rho}|=\sqrt{(\ddot{x})^{2}+(\ddot{y})^{2}+(\ddot{z})^{2}} .
$$

Given an arbitrary parametrization of the curve:

$$
\kappa=\left|\mathbf{t}_{s}\right|=\frac{\left|\mathbf{t}_{u}\right|}{|\dot{\mathbf{r}}|}=\frac{\left|\frac{d}{d u}(\dot{\mathbf{r}} /|\dot{\mathbf{r}}|)\right|}{|\dot{\mathbf{r}}|}=\frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^{3}} .
$$

In coordinate form

$$
\kappa=\frac{\sqrt{A^{2}+B^{2}+C^{2}}}{\left((\dot{x})^{2}+(\dot{y})^{2}+(\dot{z})^{2}\right)^{3 / 2}}
$$

where

$$
A=\left|\begin{array}{cc}
\dot{y} & \dot{z} \\
\ddot{y} & \ddot{z}
\end{array}\right| \quad B=\left|\begin{array}{cc}
\dot{z} & \dot{x} \\
\ddot{z} & \ddot{x}
\end{array}\right| \quad C=\left|\begin{array}{cc}
\dot{x} & \dot{y} \\
\ddot{x} & \ddot{y}
\end{array}\right| .
$$

## IV. Torsion

We will first derive an expression for the torsion of a space curve with arclength parametrization. Multiplying both sides of the third Frenet formula by $n$ results in

$$
\tau=-\mathbf{b}_{s} \mathbf{n}=-(\mathbf{t} \times \mathbf{n})_{s} \mathbf{n}=-\left(\mathbf{t}_{s} \times \mathbf{n}\right) \mathbf{n}-\left(\mathbf{t} \times \mathbf{n}_{s}\right) \mathbf{n}=\mathbf{t n n}_{s}
$$

Note that $\mathbf{t n n}_{s}$ is the mixed product of vectors $\mathbf{t}, \mathrm{n}$ and $\mathbf{n}_{s}$ and is equal to $(\mathbf{t} \times \mathbf{n}) \mathbf{n}_{s}$. We now make use of

$$
\mathbf{t}=\dot{\rho}, \quad \mathbf{n}=\frac{\ddot{\rho}}{|\kappa|}, \quad \mathbf{n}_{s}=\frac{\dddot{\rho}}{\kappa}-\frac{\kappa_{s}}{\kappa^{2}} \ddot{\rho}
$$

to obtain

$$
\tau=\frac{\dot{\rho} \dddot{\rho} \dddot{\rho}}{\kappa^{2}}=\frac{\dot{\rho} \dddot{\rho} \ddot{\rho}}{(\ddot{\rho})^{2}}=\frac{\left|\begin{array}{lll}
\dot{x} & \dot{y} & \dot{z} \\
\ddot{x} & \ddot{y} & \ddot{z} \\
\dddot{x} & \dddot{y} & \dddot{z}
\end{array}\right|}{(\ddot{x})^{2}+(\ddot{y})^{2}+(\ddot{z})^{2}} .
$$

In case of an arbitrary parametrization, we make use of:

$$
\dot{\rho}=\dot{\mathbf{r}} \frac{d t}{d s}, \quad \ddot{\rho}=\ddot{\mathbf{r}}\left(\frac{d t}{d s}\right)^{2}+\dot{\mathbf{r}} \frac{d^{2} t}{d s^{2}}
$$

and

$$
\dddot{\boldsymbol{\rho}}=\dddot{\mathbf{r}}\left(\frac{d t}{d s}\right)^{3}+3 \ddot{\mathbf{r}}\left(\frac{d t}{d s}\right) \frac{d^{2} t}{d s^{2}}+\dot{\mathbf{r}} \frac{d^{3} t}{d s^{3}}
$$

to obtain

$$
\tau=\frac{\ddot{\mathbf{r}} \ddot{\mathbf{r}} \ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^{6}} \cdot \frac{|\dot{\mathbf{r}}|^{6}}{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}})^{2}}=\frac{\ddot{\mathbf{r}} \ddot{\mathbf{r}} \ddot{\mathbf{r}}}{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}})^{2}}
$$

In coordinate form

$$
\tau=\frac{\left|\begin{array}{ccc}
\dot{x} & \dot{y} & \dot{z} \\
\ddot{x} & \ddot{y} & \ddot{z} \\
\dddot{x} & \dddot{y} & \dddot{z}
\end{array}\right|}{A^{2}+B^{2}+C^{2}}
$$

where $A, B$ and $C$ are as before.

## V. Computing curvature and torsion at varying levels of detail

In order to compute $\kappa$ and $\tau$ at varying levels of detail of the curve $\Gamma$, functions $x(u), y(u)$ and $z(u)$ are convolved with a Gaussian kernel $g(u, \sigma)$ of width $\sigma$ [Marr \& Hildreth 1980]:

$$
g(u, \sigma)=\frac{1}{\sigma \sqrt{2 \pi}} e^{-\frac{u^{2}}{2 \sigma^{2}}}
$$

The convolved functions together define the evolved curve $\Gamma_{\sigma}$. The convolution of a function $f(u)$ and the Gaussian kernel is defined as:

$$
F(u, \sigma)=f(u) \circledast g(u, \sigma)=\int_{-\infty}^{\infty} f(v) \frac{1}{\sigma \sqrt{2 \pi}} e^{\frac{-(u-v)^{2}}{2 \sigma^{2}}} d v
$$

Furthermore, it is known that

$$
\dot{F}(u, \sigma)=\frac{\partial F(u, \sigma)}{\partial u}=f(u) \circledast \frac{\partial g(u, \sigma)}{\partial u}
$$

$$
\ddot{F}(u, \sigma)=\frac{\partial^{2} F(u, \sigma)}{\partial u^{2}}=f(u) \circledast \frac{\partial^{2} g(u, \sigma)}{\partial u^{2}}
$$

and

$$
\dddot{F}(u, \sigma)=\frac{\partial^{3} F(u, \sigma)}{\partial u^{3}}=f(u) \circledast \frac{\partial^{3} g(u, \sigma)}{\partial u^{3}} .
$$

These properties of convolution can be used to compute curvature and torsion on evolved versions of a space curve. Let

$$
\begin{aligned}
& X(u, \sigma)=x(u) \circledast g(u, \sigma) \\
& Y(u, \sigma)=y(u) \circledast g(u, \sigma)
\end{aligned}
$$

and

$$
Z(u, \sigma)=z(u) \circledast g(u, \sigma)
$$

It follows that curvature on an evolved curve $\Gamma_{\sigma}$ is given by

$$
\kappa=\frac{\sqrt{D^{2}+E^{2}+F^{2}}}{\left(\dot{X}(u, \sigma)^{2}+\dot{Y}(u, \sigma)^{2}+\dot{Z}(u, \sigma)^{2}\right)^{3 / 2}}
$$

where

$$
\begin{aligned}
& D=\left|\begin{array}{ll}
\dot{Y}(u, \sigma) & \dot{Z}(u, \sigma) \\
\ddot{Y}(u, \sigma) & \ddot{Z}(u, \sigma)
\end{array}\right| \\
& E=\left|\begin{array}{ll}
\dot{\ddot{Z}}(u, \sigma) & \dot{X}(u, \sigma) \\
\ddot{Z}(u, \sigma) & \dot{X}(u, \sigma)
\end{array}\right|
\end{aligned}
$$

and

$$
F=\left|\begin{array}{ll}
\dot{X}(u, \sigma) & \dot{Y}(u, \sigma) \\
\ddot{X}(u, \sigma) & \dot{Y}(u, \sigma)
\end{array}\right|
$$

and torsion on evolved curve $\Gamma_{\sigma}$ is given by

$$
\tau=\frac{\left|\begin{array}{lll}
\dot{X}(u, \sigma) & \dot{Y}(u, \sigma) & \dot{Z}(u, \sigma) \\
\dot{X}(u, \sigma) & \ddot{Y}(u, \sigma) & \ddot{Z}(u, \sigma) \\
\ddot{X}(u, \sigma) & \ddot{Y}(u, \sigma) & \ddot{Z}(u, \sigma)
\end{array}\right|}{D^{2}+E^{2}+F^{2}}
$$

where $D, E$ and $F$ are as before.

## VI. A multi-scale representation for space curves

The curvature and torsion functions of a space curve specify that curve uniquely up to rotation and translation [Do Carmo 1976]. We therefore propose a representation for a space curve that consists of the curvature scale space and
torsion scale space images of the curve. This representation is a generalization of the curvature scale space representation proposed for planar curves in [Mokhtarian \& Mackworth 1986]. The scale space image was first proposed as a representation for one-dimensional signals in [Stansfield 1980] and developed in [Witkin 1983].

To compute the torsion scale space image of a space curve $\Gamma=(x(u), y(u), z(u))$, evolved curves $\Gamma_{\sigma}$ are computed as $\sigma$ varies from a small to a large value. The torsion zero-crossing points of each $\Gamma_{\sigma}$ are extracted and recorded in the $u-\sigma$ space. The smallest value of $\sigma$ used is slightly larger than zero and the largest value of $\sigma$ used in the smallest value of $\sigma$ that results in a $\Gamma_{\sigma}$ with very few torsion zero-crossing points.

The curvature scale space image of a space curve is constructed in a similar fashion. The only difference is that level-crossings rather than zero-crossings are searched for. This is because the curvature of a space curve has only magnitude and no sign. Some care should be given to choosing a suitable value for level $L$. If $L$ is too large, the number of level-crossing points found on curves $\Gamma_{\sigma}$ drops to zero quickly as $\sigma$ increases and the resulting curvature scale space image will not be very rich and therefore not suitable for matching. If $L$ is too small, the resulting curvature scale space image will contain excessive detail. The actual value used for $L$ is the average of curvature values of points of $\Gamma_{\sigma_{0}}$ where $\sigma_{0} \in\left[0, \sigma_{t}\right]$ and $\sigma_{t}$ is the largest value of $\sigma$ used to compute the torsion scale space image of $\Gamma$. Using such a value ensures that the resulting curvature scale space image will be sufficiently rich for matching and will represent roughly the same range of values of $\sigma$ represented in the torsion scale space image of $\Gamma$.

In order to match a space curve against another, the torsion scale space images of both are constructed and matched against each other using the algorithm described in [Mokhtarian \& Mackworth 1986]. If the resulting cost of match is low, then one curve is transformed according to the transformation parameters predicted by the match so that both curves exist at the same scale. The curvature scale space images of both curves are then constructed and matched using the same algorithm. The final cost of match is a combination of the two costs.

## C. The renormalized curvature and torsion scale space images

Mackworth and Mokhtarian [1988] observed that although $w$ is the normalized arc length parameter on the original curve $\Gamma$, the parameter $u$ is not, in general, the normalized arc length parameter on the evolved curve $\Gamma_{\sigma}$. This can lead to poor matches when local shape differences exist between two curves. To overcome this problem, we propose the renormalized curvature and torsion scale space images.

Let

$$
\mathbf{R}(u, \sigma)=(X(u, \sigma), Y(u, \sigma), Z(u, \sigma))
$$

and

$$
w=\Phi_{\sigma}(u)
$$

where

$$
\Phi_{\sigma}(u)=\frac{\int_{0}^{u}\left|\mathbf{R}_{v}(v, \sigma)\right| d v}{\int_{0}^{1}\left|\mathbf{R}_{v}(v, \sigma)\right| d v} .
$$

Now define

$$
\begin{align*}
& \hat{X}(w, \sigma)=X\left(\Phi_{\sigma}^{-1}(w), \sigma\right) \\
& \hat{Y}(w, \sigma)=Y\left(\Phi_{\sigma}^{-1}(w), \sigma\right)  \tag{2}\\
& \hat{Z}(w, \sigma)=Z\left(\Phi_{\sigma}^{-1}(w), \sigma\right)
\end{align*}
$$

Functions $\hat{X}(w, \sigma), \hat{Y}(w, \sigma)$ and $\hat{Z}(w, \sigma)$ defined by equations (2) define the renormalized version of each $\Gamma_{\sigma}$. That is, each evolved curve $\Gamma_{\sigma}$ is reparametrized by its normalized arc length parameter $w$.

Notice that

$$
\begin{aligned}
& \Phi_{\sigma}(0)=0 \\
& \Phi_{\sigma}(1)=1
\end{aligned}
$$

and

$$
\frac{d \Phi_{\sigma}(u)}{d u}=\frac{\left|\mathbf{R}_{u}(u, \sigma)\right|}{\int_{0}^{1}\left|\mathbf{R}_{v}(v, \sigma)\right| d v}>0 \quad \text { at non-singular points. }
$$

Also

$$
\Phi_{0}(u)=u
$$

$\Phi_{\sigma}(u)$ deviates from the identity function $\Phi_{\sigma}(u)=u$ only to the extent to which the scale-related statistics deviate from stationarity along the original curve.

The function defined implicitly by

$$
\kappa(w, \sigma)=c
$$

is the renormalized curvature scale space image of $\Gamma$ and the function defined implicitly by

$$
\tau(w, \sigma)=0
$$

is the renormalized torsion scale space image of $\Gamma$.

## D. The resampled curvature and torsion scale space images

Note that as a space curve evolves according to the process defined in section B, the parametrization of its coordinate functions $x(u), y(u)$ and $z(u)$ does not change. In other words, the function mapping values of the parameter $u$ of the original coordinate functions $x(u), y(u)$ and $z(u)$ to the values of the parameter $u$ of the smoothed coordinate functions $X(u, \sigma), Y(u, \sigma)$ and $Z(u, \sigma)$ is the identity function.

For both theoretical and practical reasons, it is interesting to generalize the definition of evolution so that the mapping function can be different from the identity function. Again let $\Gamma$ be defined by:

$$
\Gamma=\{(x(w), y(w), z(w)) \mid w \in[0,1]\}
$$

The generalized evolution which maps $\Gamma$ to $\Gamma_{\sigma}$ is now defined by:

$$
\Gamma \longrightarrow \Gamma_{\sigma}=\{(X(W, \sigma), Y(W, \sigma), Z(W, \sigma)) \mid W \in[0,1]\}
$$

where

$$
\begin{aligned}
& X(W, \sigma)=x(W) \circledast g(W, \sigma) \\
& Y(W, \sigma)=y(W) \circledast g(W, \sigma)
\end{aligned}
$$

and

$$
Z(W, \sigma)=z(W) \circledast g(W, \sigma)
$$

Note that

$$
W=W(w, \sigma)
$$

and

$$
W\left(w, \sigma_{0}\right)
$$

where $\sigma_{0}$ is any value of $\sigma$, is a continuous and monotonic function of $w$. This condition is necessary to ensure physical plausibility since $W$ is the parameter of the evolved curve $\Gamma_{\sigma}$.

A specially interesting case is when $W$ always remains the arc length parameter as the curve evolves. When this criterion is satisfied, the evolution of $\Gamma$ is referred to as arc length evolution. An explicit formula for $W$ can be derived [Gage \& Hamilton 1986].

Recall equation (1)

$$
\mathbf{r}(u)=(x(u), y(u), z(u))
$$

The Frenet equations for a planar curve are given by

$$
\begin{gathered}
\frac{\partial \mathbf{t}}{\partial u}=\left|\frac{\partial \mathbf{r}}{\partial u}\right| \kappa \mathbf{n} \\
\frac{\partial \mathbf{n}}{\partial u}=-\left|\frac{\partial \mathbf{r}}{\partial u}\right| \kappa \mathbf{t}+\left|\frac{\partial \mathbf{r}}{\partial u}\right| \tau \mathbf{b}
\end{gathered}
$$

Let $t=\sigma^{2} / 2$. Observe that

$$
\frac{\partial}{\partial t}\left(\left|\frac{\partial \mathbf{r}}{\partial u}\right|^{2}\right)=\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u}\right)=2\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial^{2} \mathbf{r}}{\partial u \partial t}\right)
$$

Note that

$$
\frac{\partial \mathbf{r}}{\partial u}=\left|\frac{\partial \mathbf{r}}{\partial u}\right| \mathbf{t}
$$

and

$$
\frac{\partial \mathbf{r}}{\partial t}=\kappa \mathbf{n}
$$

since the Gaussian function satisfies the heat equation. It follows that

$$
\frac{\partial}{\partial t}\left(\left|\frac{\partial \mathbf{r}}{\partial u}\right|^{2}\right)=2\left(\left|\frac{\partial \mathbf{r}}{\partial u}\right| \mathbf{t} \cdot \frac{\partial}{\partial u}(\kappa \mathbf{n})\right)=2\left(\left|\frac{\partial \mathbf{r}}{\partial u}\right| \mathbf{t} \cdot\left(\frac{\partial \kappa}{\partial u} \mathbf{n}-\left|\frac{\partial \mathbf{r}}{\partial u}\right| \kappa^{2} \mathbf{t}+\left|\frac{\partial \mathbf{r}}{\partial u}\right| \kappa \tau \mathbf{b}\right)\right)=-2\left|\frac{\partial \mathbf{r}}{\partial u}\right|^{2} \kappa^{2} .
$$

Therefore

$$
2\left|\frac{\partial \mathbf{r}}{\partial u}\right| \frac{\partial}{\partial t}\left|\frac{\partial \mathbf{r}}{\partial u}\right|=-2\left|\frac{\partial \mathbf{r}}{\partial u}\right|^{2} \kappa^{2}
$$

or

$$
\frac{\partial}{\partial t}\left|\frac{\partial \mathbf{r}}{\partial u}\right|=-\left|\frac{\partial \mathbf{r}}{\partial u}\right| \kappa^{2}
$$

Let $L$ denote the length of the curve. Now observe that

$$
\frac{\partial L}{\partial t}=\int_{0}^{L} \frac{\partial}{\partial t}\left|\frac{\partial \mathbf{r}}{\partial u}\right| d u=-\int_{0}^{L}\left|\frac{\partial \mathbf{r}}{\partial u}\right| \kappa^{2} d u=-\int_{0}^{1} \kappa^{2} d w
$$

Since the value $w_{0}$ of the normalized arc length parameter $w$ at a point $P$ measures the length of the curve from the starting point to point $P$, it follows that

$$
\frac{\partial W}{\partial t}=-\int_{0}^{W} \kappa^{2}(W, t) d W
$$

and therefore

$$
\begin{equation*}
W(w, t)=-\int_{0}^{t} \int_{0}^{W} \kappa^{2}(W, t) d W d t \tag{3}
\end{equation*}
$$

Note that

$$
W(w, 0)=w
$$

The function defined implicitly by

$$
\kappa(W, \sigma)=c
$$

is the resampled curvature scale space of $\Gamma$ and the function defined implicitly by

$$
\tau(W, \sigma)=0
$$

is the resampled torsion scale space of $\Gamma$.
Since the function $\kappa(W, t)$ in (3) is unknown, $W(w, t)$ can not be computed directly from (3). However, the resampled curvature and torsion scale space images can be computed in a simple way: A Gaussian filter based on a small value of the standard deviation is computed. The curve $\Gamma$ is parametrized by the normalized arc length parameter and convolved with the filter. The resulting curve is reparametrized by the normalized arc length parameter and convolved again with the same filter. This process is repeated until the curve has very few torsion zero-crossing points. The curvature level-crossings of each curve are marked in the resampled curvature scale space image and the torsion zerocrossings of each curve are marked in the resampled torsion scale space image.

## E. Evolution and arc length evolution properties of space curves

This section contains a number of results on evolution and arc length evolution of space curves as defined in section $\mathbf{D}$. The proofs can be found in the appendix.

The first five lemmas express a number of global properties of space curves during evolution and arc length evolution.

Lemma 1. Evolution and arc length evolution of a space curve are invariant under rotation, uniform scaling and translation of the curve.

Lemma 2. A closed space curve remains closed during evolution and arc length evolution.

Lemma 3. A connected space curve remains connected during evolution and arc length evolution.

Lemma 4. The center of mass of a space curve is invariant during evolution and
arc length evolution.

Lemma 5. Let $\Gamma$ be a closed space curve and let $G$ be its convex hull. $\Gamma$ remains inside $G$ during evolution and arc length evolution.

The following theorem also appeared in Mokhtarian [1989]. It concerns the uniqueness properties of the torsion scale space representations.

Theorem 1. Let $\Gamma$ be a space curve in $C_{3}$. Let $\tau(u)$ and $\kappa(u)$ represent the torsion and curvature functions of $\Gamma$ respectively. A single point on one torsion zero-crossing contour in the regular, resampled or renormalized torsion scale space image of $\Gamma$ determines function $\beta(u)=\tau(u) \kappa^{2}(u)$ uniquely modulo a scale factor (except on a set of measure zero).

The following theorem makes explicit the conditions under which new torsion zero-crossings will not be observed at the higher scales of torsion scale space images.

Theorem 2. Let $\Gamma$ be a space curve in $C_{3}$. If all evolved and arc length evolved curves $\Gamma_{\sigma}$ are in $C_{3}$ and torsion is bounded at every point of $\Gamma$ during evolution, then all extrema of contours in the regular, renormalized and resampled torsion scale space images of $\Gamma$ are maxima.

Theorems 3 and 4 first appeared in [Mokhtarian 1988b]. They concern the local behaviour of space curves just before and just after the formation of cusp points during evolution.

Theorem 3. Let $\Gamma=(x(w), y(w), z(w))$ be a space curve in $C_{1}$ and let $x(w), y(w)$ and $z(w)$ be polynomial functions of $w$. Let $\Gamma_{\sigma}=(X(W, \sigma), Y(W, \sigma), Z(W, \sigma))$ be an evolved or arc length evolved version of $\Gamma$ with a cusp point at $W_{0}$. There is a $\delta>0$ such that either $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of point $W_{0}$ or two projections of $\Gamma_{\sigma-\delta}$ intersect themselves in a neighborhood of point $W_{0}$.

Theorem 4: Let $\Gamma=(x(w), y(w), z(w))$ be a space curve in $C_{1}$ and let $x(w), y(w)$ and $z(w)$ be polynomial functions of $w$. Let $\Gamma_{\sigma}=(X(W, \sigma), Y(W, \sigma), Z(W, \sigma))$ be an evolved or arc length evolved version of $\Gamma$ with a cusp point at $W_{0}$, then either $\Gamma_{\sigma+\delta}$ has two new torsion zero-crossings in a neighborhood of $W_{0}$ or a torsion zero-crossing point exists at $W_{0}$ on $\Gamma_{\sigma-\delta}$ and $\Gamma_{\sigma+\delta}$.

The last theorem defines other conditions under which new torsion zero-crossings can appear on a space curve.

Theorem 5: New torsion zero-crossings can appear on a smooth space curve during evolution or arc length evolution in a neighborhood of a point of zero curvature.

## F. Examples, experiments and discussion

This section presents a number of space curves during evolution and the torsion scale space representations of those curves. It also presents a number of experiments to demonstrate the stability of the proposed representations under various noise conditions and contains a discussion of the practical significance of the lemmas and theorems of section $\mathbf{E}$.

Figure 2 shows a space curve depicting a fork. Figure 3 shows several evolved versions of the fork and figure 4 shows the torsion scale space image of the fork. Note that as $\sigma$, the width of the Gaussian function, increases, small-scale features on the curve disappear but the more basic features are preserved. Figure 5 shows a space curve depicting a bottle-opener. Several evolved versions of the bottleopener are shown in figure 6 and its torsion scale space image is shown in figure 7.

The third example is the curve shown in figure 8. This curve depicts an armchair. Figure 9 shows several evolved versions of the armchair and figure 10 shows the torsion scale space image of the armchair. Figure 11 shows the curvature scale space image of the armchair without any noise. The renormalized torsion scale space image of the armchair is shown in figure 12. Note that since the fork, the bottle-opener and the armchair are all depicted by symmetric curves, their torsion scale space images are all symmetric as well.

Experiments were also carried out to examine the behaviour of the proposed torsion scale space representations when input curves are corrupted by noise. Figure 13 shows the armchair with a significant amount of noise added to it. The direction as well as the magnitude of the noise is random. Figure 14 shows the torsion scale space image of armchair with noise. It can be observed that despite the addition of a considerable amount of noise, the torsion scale space image retains its basic structure very well. Figure 15 shows the armchair corrupted with severe random noise. The torsion scale space image of armchair with severe noise, shown in figure 16, no longer matches well with the torsion scale space image of the original armchair shown in figure 10. However, the resampled torsion scale space image of the armchair, shown in figure 17, and the resampled torsion scale space image of the armchair with severe noise, shown in figure 18, show a very good degree of match which is quite remarkable.

In general, it can be said that the regular torsion scale space image is suitable for use as a representation when the input curve has undergone transformations consisting of uniform scaling, rotation and translation and/or has been
corrupted by uniform noise. However, when non-uniform noise and/or local shape changes have altered the shape of the input curve, the renormalized or resampled torsion scale space images are better choices. The resampled torsion scale space image is the most robust one with respect to non-uniform noise and local shape changes and is rather insensitive towards small-scale differences in shape which may be an undesirable feature. The renormalized scale space image is more sensitive towards those small-scale shape differences but can only be computed for closed curves and requires more computation time.

The following is a discussion of the practical significance of the lemmas and theorems of section $\mathbf{E}$.

Lemma 1 showed that evolution and arc length evolution of a space curve are invariant under rotation, uniform scaling and translation of the curve. This shows that the regular, renormalized and resampled torsion scale space images of a space curve have the invariance property [Mokhtarian \& Mackworth 1986]. The invariance property is essential since it makes it possible to match a space curve to another of similar shape which has undergone a transformation consisting of arbitrary amounts of rotation, uniform scaling and translation.

Lemmas 2 and 3 showed that connectedness and closedness of a space curve are preserved during evolution and arc length evolution. These lemmas show that evolution and arc length evolution of a space curve are physically plausible operations. Consider a closed, connected space curve that represents the boundary of a three-dimensional object. If such a curve is not closed or connected after evolution or arc length evolution, then it can no longer admit a physically plausible interpretation.

Lemma 4 showed that the center of mass of a space curve does not move as the curve evolves and lemma 5 showed that a space curve remains inside its convex hull during evolution and arc length evolution. Together, lemmas 4 and 5 impose constraints on the physical location of a space curve as it evolves. These constraints become useful whenever the physical location of curves in a scene or their locations with respect to each other is important. A possible application area is stereo matching in which it is advantageous to carry out matching at coarser levels of detail first and then match at fine detail levels to increase accuracy.

Theorem 1 shows that a space curve can be reconstructed modulo the class represented by function $\beta(u)=\tau(u) \kappa^{2}(u)$, where $\tau(u)$ and $\kappa(u)$ are the torsion and curvature functions of the curve, from its torsion scale space image. Two space curves of different shapes are unlikely to belong to the same class represented by function $\beta(u)$ and therefore their torsion scale space images will usually be different.

Theorem 2 shows that if a space curve remains smooth during evolution and torsion remains bounded at each of its points, then no new torsion zero-crossings can exist at the higher scales of its torsion scale space image. Theorem 3 shows that space curves are very likely to remain smooth during evolution.

Theorems 3 and 4 locally characterize the behaviour of a space curve during evolution just before and just after the creation of a cusp point. Theorem 3 shows that a space curve either intersects itself or two of its projections intersect themselves just before the formation of a cusp point during evolution, in a neighborhood of the cusp point. These conditions are usually not satisfied during evolution of a space curve and therefore cusp points are unlikely to occur on space curves during evolution.

Theorem 4 shows that new torsion zero-crossings can indeed occur after the formation of a cusp point during evolution. This theorem complements the fact stated by theorem 2.

Theorem 5 showed that new torsion zero-crossings can indeed occur on a space curve that remains smooth during evolution at points of zero curvature. Together, theorems 4 and 5 describe all situations that can lead to creation of new torsion zero-crossings on a space curve during evolution. This enables one to make inferences about a space curve when new torsion zero-crossings are observed in its torsion scale space image.

We now present an evaluation of the curvature and torsion scale space representations according to the criteria proposed in section $\mathbf{A}$.

## Criterion a: Efficiency

The computation of the representations proposed here calls for evaluation of a large number of convolutions. This process can be rendered efficient using one or more of the following techniques:

## i. Fast Fourier transforms

ii. Parallelism
iii. Expression of convolutions involving Gaussians of large widths in terms of convolutions involving Gaussians of small widths only.

Furthermore, curvature and torsion scale space representations can be stored very efficiently as binary images.

## Criterion b: Invariance

Translation of the curve causes no change in the curvature and torsion scale space representations proposed here. Rotation causes only a horizontal shift in the curvature and torsion scale space representations and uniform scaling causes
the torsion scale space representation to undergo uniform scaling as well. If the represented curves are closed, then their torsion scale space representations can be normalized and invariance with respect to uniform scaling will also be achieved. If the represented curves are open, changes due to uniform scaling can be handled by the matching algorithm.

## Criterion c: Sensitivity

Small changes to the shape of the curve usually result in small changes in its representation since smaller values of the scale parameter will be sufficient to smooth out the change.

## Criterion d: Uniqueness

As argued earlier, theorem 1 shows that function $\beta(u)=\tau(u) \kappa^{2}(u)$ can be reconstructed from any of the torsion scale space representations of a space curve and therefore the torsion scale space representations nearly satisfy the uniqueness criterion.

## Criterion e: Detail

Since the curvature and torsion scale space representations combine information about the curve at varying levels of detail, criterion e is also satisfied.

## Criterion f: Robustness

The only arbitrary choice to be made when computing the curvature and torsion scale space representations is the starting point for parametrization on a closed curve. This only causes a horizontal shift in the curvature and torsion scale space representations but no structural change.

## Criterion g: Local support

All convolutions are carried out using Gaussian filters therefore criterion $\mathbf{g}$ is also satisfied.

## Criterion h: Ease of implementation

The procedures needed to compute the curvature and torsion scale space images are quite straightforward to implement. All that is needed is to prepare masks approximating derivatives of Gaussians and to convolve those with the coordinate functions of the input curve. Curvature level-crossings and torsion zero-crossings are then readily located and their locations stored in twodimensional arrays. Hence criterion $\mathbf{h}$ is also satisfied.

## Criterion i: Matchability

Curvature and torsion scale space images consist of contours which can be matched in a straightforward way to contours in other such scale space images. For a scale space matching algorithm, see [Mokhtarian \& Mackworth 1986].

It follows that the curvature and torsion scale space representations satisfy the necessary criteria for shape representation methods better than shape representation techniques previously proposed.

## G. Conclusions

This paper introduced a novel shape representation technique for space curves and proposed a number of necessary criteria that any useful shape representation scheme should satisfy. Those criteria are: efficiency, invariance, sensitivity, uniqueness, detail, robustness, local support, ease of implementation and matchability.

Three different ways of computing the representation were described. Each method relies on extracting features of the curve that are invariant under shape preserving transformations at varying scales. These methods result in: the curvature and torsion scale space images, the renormalized curvature and torsion scale space images and the resampled curvature and torsion scale space images. It was shown that each of those representations is suitable for a specific application.

A number of theoretical properties of those representations were also investigated. These properties together provide a sound foundation for the representations proposed in this paper. It was argued that each of the properties described in this paper has significant practical applications. Finally, it was shown that the proposed representations satisfy the criteria introduced earlier better than previously proposed shape representation techniques for space curves or those shape representation techniques which have been proposed for planar curves and can be extended to apply to space curves.

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## Appendix

Proof of lemma 1: It will be shown that arc length evolution is This appendix contains the proofs of the lemmas and theorems of section $\mathbf{E}$. The proofs of the lemmas have been given for arc length evolution only since the proofs for regular evolution are similar and simpler. Theorems 1 and 2 have been shown to hold for regular, renormalized and resampled torsion scale space images. Theorems 3,4 and 5 have been shown to hold for both regular and arc length evolution.
invariant under a general affine transform. Let $\Gamma_{\sigma}=(X(W, \sigma), Y(W, \sigma), Z(W, \sigma))$ be an arc length evolved version of $\Gamma=(x(w), y(w), z(w))$. If $\Gamma_{\sigma}$ is transformed according to an affine transform, then its new coordinates, $X_{1}, Y_{1}$ and $Z_{1}$ are given by

$$
\begin{aligned}
& X_{1}(W, \sigma)=a X(W, \sigma)+b Y(W, \sigma)+c Z(W, \sigma)+d \\
& Y_{1}(W, \sigma)=e X(W, \sigma)+f Y(W, \sigma)+g Z(W, \sigma)+h \\
& Z_{1}(W, \sigma)=i X(W, \sigma)+j Y(W, \sigma)+k Z(W, \sigma)+l
\end{aligned}
$$

Now suppose $\Gamma$ is transformed according to an affine transform and then evolved. The coordinates $X_{2}, Y_{2}$ and $Z_{2}$ of the new curve are

$$
\begin{aligned}
& X_{2}(W, \sigma)=(a x(W)+b y(W)+c z(W)+d) \circledast g(W, \sigma) \\
& Y_{2}(W, \sigma)=(e x(W)+f y(W)+g z(W)+h) \circledast g(W, \sigma) \\
& Z_{2}(W, \sigma)=(i x(W)+j y(W)+k z(W)+l) \circledast g(W, \sigma) .
\end{aligned}
$$

Since the convolution operator is distributive [Kecs 1982], it follows that

$$
\begin{aligned}
& X_{2}(W, \sigma)=X_{1}(W, \sigma) \\
& Y_{2}(W, \sigma)=Y_{1}(W, \sigma) \\
& Z_{2}(W, \sigma)=Z_{1}(W, \sigma)
\end{aligned}
$$

and the lemma follows.
Proof of lemma 2: Let $\Gamma=(x(w), y(w), z(w))$ be a closed curve and let $\Gamma_{\sigma}=(X(W, \sigma), Y(W, \sigma), Z(W, \sigma))$ be an arc length evolved version of $\Gamma$. On $\Gamma$ :

$$
(x(0), y(0), z(0))=(x(1), y(1), z(1)) .
$$

On $\Gamma_{\sigma}$ :

$$
(X(0, \sigma), Y(0, \sigma), Z(0, \sigma))=(X(1, \sigma), Y(1, \sigma), Z(1, \sigma)) .
$$

It follows that $\Gamma_{\sigma}$ is closed.
Proof of lemma 3: Let $\Gamma=(x(w), y(w), z(w))$ be a connected planar curve and $\Gamma_{\sigma}=(X(W, \sigma), Y(W, \sigma), Z(W, \sigma))$ be an arc length evolved version of that curve. Since $\Gamma$ is connected, $x(w), y(w)$ and $z(w)$ are continuous functions and therefore $X(W, \sigma), Y(W, \sigma)$ and $Z(W, \sigma)$ are also continuous. Let $W_{0}$ be any value of parameter $W$ and let $x_{0}, y_{0}$ and $z_{0}$ be the values of $X(W, \sigma), Y(W, \sigma)$ and $Z(W, \sigma)$ at $W_{0}$ respectively. If $W$ goes through an infinitesimal change, then $X(W, \sigma), Y(W, \sigma)$ and $Z(W, \sigma)$ will also go through infinitesimal changes:

$$
\begin{aligned}
& X\left(W_{0}, \sigma\right) \rightarrow x_{0}+\delta \\
& Y\left(W_{0}, \sigma\right) \rightarrow y_{0}+\xi \\
& Z\left(W_{0}, \sigma\right) \rightarrow z_{0}+\epsilon
\end{aligned}
$$

As a result, point $P\left(x_{0}, y_{0}\right)$ on $\Gamma_{\sigma}$ goes to point $P^{\prime}\left(x_{0}+\delta, y_{0}+\xi, z_{0}+\epsilon\right)$. Let the distance between $P$ and $P^{\prime}$ be $D$. Then

$$
D=\sqrt{\delta^{2}+\xi^{2}+\epsilon^{2}} \leq \delta \sqrt{2}
$$

assuming $|\delta|$ is the largest of $|\delta|,|\xi|$ and $|\epsilon|$. It follows that an infinitesimal change in parameter $W$ also results in an infinitesimal change in the value of the vector-valued function $\Gamma_{\sigma}$. Therefore $\Gamma_{\sigma}$ is a connected curve.

Proof of lemma 4: Let $M$ be the center of mass of $\Gamma=(x(w), y(w), z(w))$ with coordinates $\left(x_{M}, y_{M}, z_{M}\right)$. Then

$$
x_{M}=\frac{\int_{0}^{1} x(w) d w}{\int_{0}^{1} d w}=\int_{0}^{1} x(w) d w
$$

Let $\Gamma_{\sigma}=(X(W, \sigma), Y(W, \sigma), Z(W, \sigma))$ be an arc length evolved version of $\Gamma$ with $N=\left(X_{N}, Y_{N}, Z_{N}\right)$ as its center of mass. Observe that

$$
X_{N}=\int_{0}^{1} X(W, \sigma) d W=\int_{0}^{1} \int_{-\infty}^{\infty} g(v, \sigma) x(W-v) d v d W=\int_{-\infty}^{\infty} g(v, \sigma)\left(\int_{0}^{1} x(W-v) d W\right) d v
$$

$W$ covers $\Gamma_{\sigma}$ exactly once. Therefore

$$
\int_{0}^{1} x(W-v) d W=x_{M}
$$

So

$$
X_{N}=x_{M} .
$$

Similarly

$$
Y_{N}=y_{M}
$$

and

$$
Z_{N}=z_{M}
$$

It follows that $M$ and $N$ are the same point.
Proof of lemma 5: Since $G$ is simple and convex, every plane $P$ tangent to $G$ contains that curve in the left (or right) half-space it creates. Since $\Gamma$ is inside $G$, $\Gamma$ is also contained in the same half-space. Now rotate $P$ and $\Gamma$ so that $P$ becomes parallel to the $Y Z$-plane. $P$ is now described by the equation $x=c$. Since $P$ does not intersect $\Gamma$, it follows that $x\left(w_{0}\right) \geq c$ for every point $w_{0}$ on $\Gamma$. Let $\Gamma_{\sigma}$ be an arc length evolved version of $\Gamma$. Every point of $\Gamma_{\sigma}$ is a weighted average of all the points of $\Gamma$. Therefore $X\left(W_{0}, \sigma\right) \geq c$ for every point $W_{0}$ on $\Gamma_{\sigma}$ and $\Gamma_{\sigma}$ is also contained in the same half-space. This result holds for every plane tangent to $G$ therefore $\Gamma_{\sigma}$ is contained inside the intersection of all the left (or right) halfspaces created by the tangent planes of $G$. It follows that $\Gamma_{\sigma}$ is also inside $G$.

Proof of theorem 1: The proof will first be given for the regular torsion scale space of $\Gamma$, then the modifications needed to apply the same proof to the resampled and renormalized torsion scale space of $\Gamma$ will be explained.

Section $\mathbf{i}$ shows that the derivatives at a point on a torsion zero-crossing contour provide homogeneous equations in the moments of the coordinate functions of the curve. Section ii shows that the moments are related to the coefficients of expansion of the coordinate functions of the curve in functions related to the Hermite polynomials. Section iii shows that the moments at one torsion zero-crossing point can be related to the moments at other torsion zero-crossing points. Section iv shows that the cubic equations obtained in section $\mathbf{i}$ can be converted into a homogeneous linear system of equations which can be solved uniquely for function $\tau(u) \kappa^{2}(u)$.

## i. Constraints from the torsion zero-crossing contours

Let $\Gamma=(x(u), y(u), z(u))$ be the arc-length parametrization of the curve with Fourier transform $\tilde{\Gamma}=(\tilde{x}(\omega), \tilde{y}(\omega), \tilde{z}(\omega))$. The Fourier transform of the Gaussian filter $G(u, t)=\frac{1}{\sqrt{2 t}} e^{-u^{2} / 4 t}$ is $\tilde{G}(\omega)=e^{-\omega^{2} t}$.

Let $\Gamma_{t_{0}}=\left(x\left(u, t_{0}\right), y\left(u, t_{0}\right), z\left(u, t_{0}\right)\right)$ be a curve obtained by convolving $x(u)$, $y(u)$ and $z(u)$ with $G\left(u, t_{0}\right)$. Assume that $\Gamma_{t_{0}}$ is in $C_{\infty}$. Such a $t_{0}$ exists since $\Gamma$ is in $C_{1}$. Assume that $\kappa(u, t) \neq 0$ on the torsion zero-crossing contours in a
neighborhood of $t_{0}$. It follows that the torsion zero-crossings are given by solutions of $\beta(u, t)=0$ where [Goetz 1970]

$$
\begin{equation*}
\beta(u, t)=\dot{x}(\ddot{y} \dddot{z}-\dddot{y} \ddot{z})-\dot{y}(\ddot{x} \dddot{z}-\dddot{x} \ddot{z})+\dot{z}(\ddot{x} \dddot{y}-\dddot{x} \ddot{y}) \tag{4}
\end{equation*}
$$

where. represents derivative with respect to $u$. Note that on $\Gamma(t=0), \beta(u, t)$ is given by

$$
\begin{equation*}
\beta(u, t)=\tau(u, t) \kappa^{2}(u, t) . \tag{5}
\end{equation*}
$$

Using the convolution theorem, $\dot{x}(u, t), \dot{y}(u, t)$ and $\dot{z}(u, t)$ can be expressed as following:

$$
\begin{aligned}
& \dot{x}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}(\dot{i \omega}) \tilde{x}(\omega) d \omega \\
& \dot{y}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}(i \omega) \tilde{y}(\omega) d \omega \\
& \dot{z}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}(\dot{\omega}) \tilde{z}(\omega) d \omega
\end{aligned}
$$

and therefore

$$
\begin{aligned}
& \ddot{x}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}\left(-\omega^{2}\right) \tilde{x}(\omega) d \omega \\
& \ddot{y}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}\left(-\omega^{2}\right) \tilde{y}(\omega) d \omega \\
& \ddot{z}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}\left(-\omega^{2}\right) \tilde{z}(\omega) d \omega \\
& \dddot{x}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}\left(-\omega^{3}\right) \tilde{x}(\omega) d \omega \\
& \dddot{y}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}\left(-\omega^{3}\right) \tilde{y}(\omega) d \omega \\
& \dddot{z}(u, t)=\int e^{-\omega^{2} t} e^{i \omega u}\left(-\omega^{3}\right) \tilde{z}(\omega) d \omega .
\end{aligned}
$$

Note that the moment of order $k$ of the function $f(\omega)=e^{-\omega^{2} t} e^{i \omega u}(i \omega) \tilde{x}(\omega)$ is defined by:

$$
M_{k}=\int_{-\infty}^{\infty}(i \omega)^{k} e^{-\omega^{2} t} e^{i \omega u}(i \omega) \tilde{x}(\omega) d \omega
$$

the moment of order $k$ of the function $f^{\prime}(\omega)=e^{-\omega^{2} t} e^{i \omega u}(i \omega) \tilde{y}(\omega)$ is defined by:

$$
M_{k}^{\prime}=\int_{-\infty}^{\infty}(\dot{\nu})^{k} e^{-\omega^{2} t} e^{i \omega u}(\dot{v}) \tilde{y}(\omega) d \omega
$$

and the moment of order $k$ of the function $f^{\prime \prime}(\omega)=e^{-\omega^{2} t} e^{i \omega u}(\dot{\psi}) \tilde{z}(\omega)$ is defined by:

$$
M_{k}^{\prime \prime}=\int_{-\infty}^{\infty}(\dot{w})^{k} e^{-\omega^{2} t} e^{\dot{i} u}(\dot{\psi}) \tilde{z}(\omega) d \omega
$$

Therefore equation (5) can be written as:

$$
\beta(u, t)=M_{0} M_{1}^{\prime} M_{2}^{\prime \prime}-M_{0} M_{1}^{\prime \prime} M_{2}^{\prime}-M_{0}^{\prime} M_{1} M_{2}^{\prime \prime}+M_{0}^{\prime} M_{2} M_{1}^{\prime \prime}+M_{0}^{\prime \prime} M_{1} M_{2}^{\prime}-M_{0}^{\prime \prime} M_{2} M_{1}^{\prime} .
$$

The Implicit Function Theorem guarantees that the contours $u(t)$ are $C_{\infty}$ in a neighborhood of $t_{0}$. Let $\xi$ be a parameter of the torsion zero-crossing contour. Then

$$
\frac{d}{d \xi}=\frac{d u}{d \xi} \frac{\partial}{\partial u}+\frac{d t}{d \xi} \frac{\partial}{\partial t} .
$$

On the torsion zero-crossing contour, $\beta=0$ and $\frac{d^{k}}{d \xi^{k}} \beta=0$ for all integers $k$. Furthermore, since the torsion zero-crossing contour is known, all the derivatives of $u$ and $t$ with respect to $\xi$ are known as well. We now compute the derivatives of $\beta$ with respect to $\xi$ at $\left(u_{0}, t_{0}\right)$. The first derivative is given by:

$$
\begin{equation*}
\frac{d}{d \xi} \beta\left(u_{0}, t_{0}\right)=\frac{d u}{d \xi} \frac{\partial \beta\left(u_{0}, t_{0}\right)}{\partial u}+\frac{d t}{d \xi} \frac{\partial \beta\left(u_{0}, t_{0}\right)}{\partial t} \tag{6}
\end{equation*}
$$

where

$$
\frac{\partial \beta\left(u_{0}, t_{0}\right)}{\partial u}=M_{3}^{\prime \prime} M_{0} M_{1}^{\prime}-M_{3}^{\prime} M_{0} M_{1}^{\prime \prime}-M_{3}^{\prime \prime} M_{0}^{\prime} M_{1}+M_{3} M_{0}^{\prime} M_{1}^{\prime \prime}+M_{3}^{\prime} M_{0}^{\prime \prime} M_{1}-M_{3} M_{0}^{\prime \prime} M_{1}^{\prime}
$$

and

$$
\begin{aligned}
\frac{\partial \beta\left(u_{0}, t_{0}\right)}{\partial t} & =M_{3}^{\prime} M_{0} M_{2}^{\prime \prime}+M_{4}^{\prime \prime} M_{0} M_{1}^{\prime}-M_{3}^{\prime \prime} M_{0} M_{2}^{\prime}-M_{4}^{\prime} M_{0} M_{1}^{\prime \prime}-M_{3} M_{0}^{\prime} M_{2}^{\prime \prime}-M_{4}^{\prime \prime} M_{0}^{\prime} M_{1} \\
& +M_{4} M_{0}^{\prime} M_{1}^{\prime \prime}+M_{3}^{\prime \prime} M_{0}^{\prime} M_{2}+M_{3} M_{0}^{\prime \prime} M_{2}^{\prime}+M_{4}^{\prime} M_{0}^{\prime \prime} M_{1}-M_{4} M_{0}^{\prime \prime} M_{1}^{\prime}-M_{3}^{\prime} M_{0}^{\prime \prime} M_{2}
\end{aligned}
$$

and the second derivative is given by:
$\frac{\partial^{2} \beta}{\partial \xi^{2}}=\frac{d^{2} u}{d \xi^{2}} \frac{\partial \beta}{\partial u}+\frac{d^{2} t}{d \xi^{2}} \frac{\partial \beta}{\partial t}+\left(\frac{d u}{d \xi}\right)^{2} \frac{\partial^{2} \beta}{\partial u^{2}}+2 \frac{d u}{d \xi} \frac{d t}{d \xi} \frac{\partial^{2} \beta}{\partial u \partial t}+\left(\frac{d t}{d \xi}\right)^{2} \frac{\partial^{2} \beta}{\partial t^{2}}$
where

$$
\begin{aligned}
\frac{\partial^{2} \beta}{\partial u^{2}} & =M_{4}^{\prime \prime} M_{0} M_{1}^{\prime}+M_{2}^{\prime} M_{3}^{\prime \prime} M_{0}-M_{4}^{\prime} M_{0} M_{1}^{\prime \prime}-M_{2}^{\prime \prime} M_{3}^{\prime} M_{0}-M_{4}^{\prime \prime} M_{0}^{\prime} M_{1}-M_{2} M_{3}^{\prime \prime} M_{0}^{\prime} \\
& +M_{4} M_{0}^{\prime} M_{1}^{\prime \prime}+M_{2}^{\prime \prime} M_{3} M_{0}^{\prime}+M_{4}^{\prime} M_{0}^{\prime \prime} M_{1}+M_{2} M_{3}^{\prime} M_{0}^{\prime \prime}-M_{4} M_{0}^{\prime \prime} M_{1}^{\prime}-M_{2}^{\prime} M_{3} M_{0}^{\prime \prime} \\
\frac{\partial^{2} \beta}{\partial u \partial t} & =M_{5}^{\prime \prime} M_{0} M_{1}^{\prime}+M_{2} M_{3}^{\prime \prime} M_{1}^{\prime}-M_{5}^{\prime} M_{0} M_{1}^{\prime \prime}-M_{2} M_{3}^{\prime} M_{1}^{\prime \prime}-M_{5}^{\prime \prime} M_{0}^{\prime} M_{1}-M_{2}^{\prime} M_{3}^{\prime \prime} M_{1} \\
& +M_{5} M_{0}^{\prime} M_{1}^{\prime \prime}+M_{2}^{\prime} M_{3} M_{1}^{\prime \prime}+M_{5}^{\prime} M_{0}^{\prime \prime} M_{1}+M_{2}^{\prime \prime} M_{3}^{\prime} M_{1}-M_{5} M_{0}^{\prime \prime} M_{1}^{\prime}-M_{2}^{\prime \prime} M_{3} M_{1}^{\prime}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^{2} \beta}{\partial t^{2}} & =M_{5}^{\prime} M_{0} M_{2}^{\prime \prime}+2 M_{4}^{\prime \prime} M_{3}^{\prime} M_{0}+M_{6}^{\prime \prime} M_{0} M_{1}^{\prime}+M_{2} M_{4}^{\prime \prime} M_{1}^{\prime}-M_{5}^{\prime \prime} M_{0} M_{2}^{\prime}-2 M_{4}^{\prime} M_{3}^{\prime \prime} M_{0} \\
& -M_{6}^{\prime} M_{0} M_{1}^{\prime \prime}-M_{2} M_{4}^{\prime} M_{1}^{\prime \prime}-M_{5} M_{0}^{\prime} M_{2}^{\prime \prime}-2 M_{4}^{\prime \prime} M_{3} M_{0}^{\prime}-M_{6}^{\prime \prime} M_{0}^{\prime} M_{1}-M_{2}^{\prime} M_{4}^{\prime \prime} M_{1} \\
& +M_{6} M_{0}^{\prime} M_{1}^{\prime \prime}+M_{2}^{\prime} M_{4} M_{1}^{\prime \prime}+2 M_{3}^{\prime \prime} M_{4} M_{0}^{\prime}+M_{5}^{\prime \prime} M_{0}^{\prime} M_{2}+M_{5} M_{0}^{\prime \prime} M_{2}^{\prime}+2 M_{4}^{\prime} M_{3} M_{0}^{\prime \prime} \\
& +M_{6}^{\prime} M_{0}^{\prime \prime} M_{1}+M_{2}^{\prime \prime} M_{4}^{\prime} M_{1}-M_{6} M_{0}^{\prime \prime} M_{1}^{\prime}-M_{2}^{\prime \prime} M_{4} M_{1}^{\prime}-2 M_{3}^{\prime} M_{4} M_{0}^{\prime \prime}-M_{5}^{\prime} M_{0}^{\prime \prime} M_{2} .
\end{aligned}
$$

Since the parametric derivatives along the torsion zero-crossing contours are zero, equation (6) is equal to zero. Note that equation (6) is in the first five moments of functions $f(\omega), f^{\prime}(\omega)$ and $f^{\prime \prime}(\omega)$ and equation (7) is in the first seven moments of those functions. In general, the $k+1$ st equation, $\frac{d^{k}}{d \xi^{k}} \beta(u, t)=0$ is a cubic equation in the first $2 k+3$ moments of each of the functions $f(\omega), f^{\prime}(\omega)$ and $f^{\prime \prime}(\omega)$.

It follows that the first $n+1$ equations at $\left(u_{0}, t_{0}\right)$ are in a total of $3(2 n+3)=6 n+9$ moments. Our axes are again chosen such that $u_{0}=0$. The next section shows that the moments of $f(\omega), f^{\prime}(\omega)$ and $f^{\prime \prime}(\omega)$ are the coefficients $a_{k}, b_{k}$ and $c_{k}$ in the expression of functions $\dot{x}(u), \dot{y}(u)$ and $\dot{z}(u)$ in functions $\phi_{k}(u, \sigma)$ related to Hermite polynomials. Therefore we have $n+1$ equations in the first $6 n+9$ coefficients $a_{k}, b_{k}$ and $c_{k}$. To determine the $a_{k}, b_{k}$ and $c_{k}$, we need $5 n+8$ additional and independent equations which can be provided by considering six neighboring torsion zero-crossing contours at $\left(u_{1}, t_{0}\right),\left(u_{2}, t_{0}\right),\left(u_{3}, t_{0}\right),\left(u_{4}, t_{0}\right)$, $\left(u_{5}, t_{0}\right)$ and ( $u_{6}, t_{0}$ ).

## ii. The moments and the coefficients of expansion of $\dot{x}(u), \dot{y}(u)$ and $\dot{z}(u)$

This section shows that the moments and the moment-triples in equations $\frac{d^{k}}{d \xi^{k}} \beta(u, t)$ are related respectively to the coefficients of the expression of the
functions $\dot{x}(u), \dot{y}(u)$ and $\dot{z}(u)$ and function $\beta(u)$ in functions related to the Hermite polynomials.

Expand function

$$
\dot{x}(u)=\frac{d}{d u} x(u)
$$

in terms of the functions $\phi_{k}(u, \sigma)$ related to the Hermite polynomials $H_{k}(u)$ by

$$
\begin{gathered}
\phi_{k}(u, \sigma)=(-1)^{k} \frac{\sigma^{k-1}}{(\sqrt{2})^{k+1} \sqrt{\pi}} H_{k}\left(\frac{u}{\sigma \sqrt{2}}\right) \\
H_{k}(u)=(-1)^{k} e^{u^{2}} \frac{d^{k}}{d u^{k}} e^{-u^{2}} \\
\dot{x}(u)=\sum a_{k}(\sigma) \phi_{k}(u, \sigma)
\end{gathered}
$$

The coefficients $a_{k}(\sigma)$ of the expansion are given by

$$
\left.a_{k}(\sigma)=<w_{k}(u, \sigma), \dot{x}(u)\right\rangle
$$

where $<,>$ denotes inner product in $L^{2}$ and $\left\{w_{k}(u, \sigma)\right\}$ is the set of functions biorthogonal to $\left\{\phi_{k}(u, \sigma)\right\}$. The $\left\{\phi_{k}(u, \sigma)\right\}$ are given explicitly by

$$
\phi_{k}(u, \sigma)=\frac{\sigma^{2 k-1}}{k!\sqrt{2 \pi}} e^{\frac{u^{2}}{2 \sigma^{2}}} \frac{d^{k}}{d u^{k}} e^{-\frac{u^{2}}{2 \sigma^{2}}}
$$

and the $w_{k}(u, \sigma)$ by

$$
w_{k}(u, \sigma)=(-1)^{k} \frac{d^{k}}{d u^{k}} e^{-\frac{\mathrm{u}^{2}}{2 \sigma^{2}}}
$$

Since

$$
\dot{x}(u)=\frac{1}{\sqrt{2 \pi}} \int e^{i \omega u}(\dot{w}) \tilde{x}(\omega) d \omega
$$

the $a_{k}$ are given by

$$
a_{k}(\sigma)=\frac{1}{\sqrt{2 \pi}}(-1)^{k} \int<\frac{d^{k}}{d u^{k}} e^{-\frac{u^{2}}{2 \sigma^{2}}}, e^{i \omega u}>(\dot{i} \omega) \tilde{x}(\omega) d \omega
$$

The inner product is just the inverse Fourier transform of $w_{k}(u, \sigma)$. Therefore

$$
a_{k}(\sigma)=\int(i \omega)^{k} e^{\frac{-\omega^{2} \sigma^{2}}{2}}(\dot{\omega}) \tilde{x}(\omega) d \omega
$$

which is equal to $M_{k}$ modulo a factor $e^{i \omega u}$, since $t=\sigma^{2} / 2$.
Similarly, the functions

$$
\begin{aligned}
& \dot{y}(u)=\frac{d}{d u} y(u) \\
& \dot{z}(u)=\frac{d}{d u} z(u)
\end{aligned}
$$

can be expanded in terms of the functions $\phi_{k}(u, \sigma)$ by

$$
\begin{aligned}
& \dot{y}(u)=\sum b_{k}(\sigma) \phi_{k}(u, \sigma) \\
& \dot{z}(u)=\sum c_{k}(\sigma) \phi_{k}(u, \sigma)
\end{aligned}
$$

and it again follows that

$$
\begin{aligned}
& b_{k}(\sigma)=\int(i \omega)^{k} e^{\frac{-\omega^{2} \sigma^{2}}{2}}(i \omega) \tilde{y}(\omega) d \omega \\
& c_{k}(\sigma)=\int(i \omega)^{k} e^{\frac{-\omega^{2} \sigma^{2}}{2}}(i \omega) \tilde{z}(\omega) d \omega
\end{aligned}
$$

which are equal to $M_{k}^{\prime}$ and $M_{k}^{\prime \prime}$ respectively modulo a factor $e^{i \omega u}$.
Furthermore, $a_{k}^{\prime}(\sigma), b_{k}^{\prime}(\sigma)$ and $c_{k}^{\prime}(\sigma)$, the coefficients of expansion of functions $\ddot{x}(u), \ddot{y}(u)$ and $\ddot{z}(u)$ in $\phi_{k}(u, \sigma)$ respectively, can be seen to be related to $a_{k}(\sigma)$, $b_{k}(\sigma)$ and $c_{k}(\sigma)$ according to the following relationships:

$$
\begin{align*}
a_{k-1}^{\prime}(\sigma) & =a_{k}(\sigma) \\
b_{k-1}^{\prime}(\sigma) & =b_{k}(\sigma)  \tag{8}\\
c_{k-1}^{\prime}(\sigma) & =c_{k}(\sigma)
\end{align*}
$$

and $a_{k}^{\prime \prime}(\sigma), b_{k}^{\prime \prime}(\sigma)$ and $c_{k}^{\prime \prime}(\sigma)$, the coefficients of expansion of functions $\dddot{x}(u), \dddot{y}(u)$ and $\dddot{z}(u)$ in $\phi_{k}(u, \sigma)$ respectively, can be seen to be related to $a_{k}(\sigma), b_{k}(\sigma)$ and $c_{k}(\sigma)$ by the following relationships:

$$
\begin{align*}
& a_{k-2}^{\prime \prime}(\sigma)=a_{k}(\sigma) \\
& b_{k-2}^{\prime \prime}(\sigma)=b_{k}(\sigma)  \tag{9}\\
& c_{k-2}^{\prime \prime}(\sigma)=c_{k}(\sigma) .
\end{align*}
$$

Now observe that the function $\tau(u) \kappa^{2}(u)$ can be expressed as:

$$
\begin{aligned}
\tau(u) \kappa^{2}(u) & =\dot{x}(u)(\ddot{y}(u) \ddot{z}(u)-\dddot{y}(u) \ddot{z}(u)) \\
& -\dot{y}(u)(\ddot{x}(u) \dddot{z}(u)-\dddot{x}(u) \ddot{z}(u)) \\
& +\dot{z}(u)(\ddot{x}(u) \ddot{y}(u)-\dddot{x}(u) \ddot{y}(u))
\end{aligned}
$$

$$
\begin{aligned}
& =\dot{x}(u) \ddot{y}(u) \dddot{z}(u)-\dot{x}(u) \dddot{y}(u) \ddot{z}(u)-\dot{y}(u) \ddot{x}(u) \not \ddot{z}^{\prime}(u) \\
& +\dot{y}(u) \dddot{x}(u) \ddot{z}(u)+\dot{z}(u) \ddot{x}(u) \dddot{y}(u)-\dot{z}(u) \dddot{x}(u) \ddot{y}(u) \\
& =\sum a_{i}(\sigma) \phi_{i}(u, \sigma) \sum b_{i}^{\prime}(\sigma) \phi_{i}(u, \sigma) \sum c_{i}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \\
& -\sum a_{i}(\sigma) \phi_{i}(u, \sigma) \sum b_{i}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \sum c_{i}^{\prime}(\sigma) \phi_{i}(u, \sigma) \\
& -\sum b_{i}(\sigma) \phi_{i}(u, \sigma) \sum a_{i}^{\prime}(\sigma) \phi_{i}(u, \sigma) \sum c_{i}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \\
& +\sum b_{i}(\sigma) \phi_{i}(u, \sigma) \sum a_{i}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \sum c_{i}^{\prime}(\sigma) \phi_{i}(u, \sigma) \\
& +\sum c_{i}(\sigma) \phi_{i}(u, \sigma) \sum a_{i}^{\prime}(\sigma) \phi_{i}(u, \sigma) \sum b_{i}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \\
& -\sum c_{i}(\sigma) \phi_{i}(u, \sigma) \sum a_{i}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \sum b_{i}^{\prime}(\sigma) \phi_{i}(u, \sigma) \\
& =\sum \sum \sum a_{i}(\sigma) b_{j}^{\prime}(\sigma) c_{k}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \phi_{j}(u, \sigma) \phi_{k}(u, \sigma) \\
& -\sum \sum \sum a_{i}(\sigma) b_{j}^{\prime \prime}(\sigma) c_{k}^{\prime}(\sigma) \phi_{i}(u, \sigma) \phi_{j}(u, \sigma) \phi_{k}(u, \sigma) \\
& -\sum \sum \sum b_{i}(\sigma) a_{j}^{\prime}(\sigma) c_{k}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \phi_{j}(u, \sigma) \phi_{k}(u, \sigma) \\
& +\sum \sum \sum b_{i}(\sigma) a_{j}^{\prime \prime}(\sigma) c_{k}^{\prime}(\sigma) \phi_{i}(u, \sigma) \phi_{j}(u, \sigma) \phi_{k}(u, \sigma) \\
& +\sum \sum \sum c_{i}(\sigma) a_{j}^{\prime}(\sigma) b_{k}^{\prime \prime}(\sigma) \phi_{i}(u, \sigma) \phi_{j}(u, \sigma) \phi_{k}(u, \sigma) \\
& \\
& +\sum \phi_{i}(u, \sigma) \phi_{j}(u, \sigma) \phi_{k}(u, \sigma) \\
& -\sum \sum \sum c_{i}(\sigma) a_{j}^{\prime \prime}(\sigma) b_{k}^{\prime}(\sigma) \phi_{i}(u, \sigma) \phi_{j}(u, \sigma) \phi_{k}(u, \sigma) \\
& =\sum \sum \sum \sum a_{i}(\sigma) b_{j}^{\prime}(\sigma) c_{k}^{\prime \prime}(\sigma)+b_{i}(\sigma) a_{j}^{\prime \prime}(\sigma) c_{k}^{\prime}(\sigma)+c_{i}(\sigma) a_{j}^{\prime}(\sigma) b_{k}^{\prime \prime}(\sigma) \\
&
\end{aligned}
$$

Using (8) and (9) we obtain

$$
\begin{aligned}
\tau(u) \kappa^{2}(u)=\sum \sum \sum & \left(a_{i}(\sigma) b_{j+1}(\sigma) c_{k+2}(\sigma)+b_{i}(\sigma) a_{j+2}(\sigma) c_{k+1}(\sigma)+c_{i}(\sigma) a_{j+1}(\sigma) b_{k+2}(\sigma)\right. \\
& \left.-a_{i}(\sigma) b_{j+2}(\sigma) c_{k+1}(\sigma)-b_{i}(\sigma) a_{j+1}(\sigma) c_{k+2}(\sigma)-c_{i}(\sigma) a_{j+2}(\sigma) b_{k+1}(\sigma)\right) \\
& \phi_{i}(u, \sigma) \phi_{j}(u, \sigma) \phi_{k}(u, \sigma)
\end{aligned}
$$

It follows that if the triples $a_{i}(\sigma) b_{j}(\sigma) c_{k}(\sigma)$ in the equation above are all known, the function $\beta(u)=\tau(u) \kappa^{2}(u)$ can be reconstructed.

## iii. Combining information from more than one contours

To solve the system of equations obtained in section $i$, we need to obtain additional equations from other points of the torsion scale space image and relate them to the equations obtained from the first point. Suppose additional equations are obtained in the moments of functions $e^{-\omega^{2} t} e^{\dot{k} \omega u^{\prime}}(i \omega) \tilde{x}(\omega), e^{-\omega^{2} t} e^{i \omega u^{\prime}}(i \omega) \tilde{y}(\omega)$ and $e^{-\omega^{2} t} e^{i \omega u^{\prime}}(i \omega) \tilde{z}(\omega)$ at point $\left(u^{\prime}, t_{0}\right)$. We have

$$
\begin{aligned}
& \dot{x}\left(u+u^{\prime}\right)=\int e^{i \omega u} e^{i \omega u^{\prime}}(\dot{w}) \tilde{x}(\omega) d \omega=\sum d_{k} \phi_{k}(u) \\
& \dot{y}\left(u+u^{\prime}\right)=\int e^{i \omega u} e^{i \omega u^{\prime}}(\dot{w}) \tilde{y}(\omega) d \omega=\sum e_{k} \phi_{k}(u)
\end{aligned}
$$

and

$$
\dot{z}\left(u+u^{\prime}\right)=\int e^{i \omega u} e^{i \omega u^{\prime}}(\dot{u}) \tilde{z}(\omega) d \omega=\sum f_{k} \phi_{k}(u)
$$

Now observe that

$$
\begin{aligned}
& \sum d_{k} \phi_{k}(u)=\sum a_{k} \phi_{k}\left(u+u^{\prime}\right) \\
& \sum e_{k} \phi_{k}(u)=\sum b_{k} \phi_{k}\left(u+u^{\prime}\right)
\end{aligned}
$$

and

$$
\sum f_{k} \phi_{k}(u)=\sum c_{k} \phi_{k}
$$

That is, $\phi_{k}\left(u+u^{\prime}\right)$ can be expressed as a linear combination of $\phi_{j}(u)$ with $j \leq k$ as has been shown in [Yuille and Poggio 1983].

## iv. Reconstructing the function $\tau(u) \kappa^{2}(u)$

It was shown in section $\mathbf{i}$ that seven points from seven torsion scale space contours give us $6 n+9$ cubic equations in the first $2 n+3$ moments of each of the functions $f(\omega), f^{\prime}(\omega)$ and $f^{\prime \prime}(\omega)$. Section iii showed that the moments of order $k$ of any function at $u+u^{\prime}$ can be expressed as a linear combination of the moments of order less than or equal to $k$ of that function at $u$. Therefore we obtain a system of homogeneous cubic equations in the first $6 n+9$ coefficients of functions $\dot{x}(u)$, $\dot{y}(u)$ and $\dot{z}(u)$ using seven points from the torsion scale space image of $\Gamma$ (Note that only three equations from the seventh point need be used). That system has at least one solution since the moments of order higher than $2 n+2$ of $f(\omega), f^{\prime}(\omega)$ and $f^{\prime \prime}(\omega)$ are assumed to be zero. However, the solution obtained from a cubic system of equations is in general not unique.

Equations (6) and (7) can be converted into homogeneous linear equations by assuming that each moment-triple appearing in those equations is a new variable. Tables 1-7 show the moment-triples in equations (6) and (7). The + signs designate the moment-triples in equation (6) and the + and the x signs together
designate the moment-triples in equation (7). Each table shows those momenttriples which share the same $M_{k}^{\prime \prime}, 0 \leq k \leq 6$.

|  | $M_{0}^{\prime}$ | $M_{1}^{\prime}$ | $M_{2}^{\prime}$ | $M_{3}^{\prime}$ | $M_{4}^{\prime}$ | $M_{5}^{\prime}$ | $M_{6}^{\prime}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{0}$ |  |  |  |  |  |  |  |  |
| $M_{1}$ |  |  |  |  | + | + | x | x |
| $M_{2}$ |  |  |  | + |  | x |  |  |
| $M_{3}$ |  | + | + |  | x |  |  |  |
| $M_{4}$ |  | + |  | x |  |  |  |  |
| $M_{5}$ |  | x | x |  |  |  |  |  |
| $M_{6}$ | x |  |  |  |  |  |  |  |

Table 1. Moment-triples sharing $M_{0}^{\prime \prime}$

|  | $M_{0}^{\prime}$ | $M_{1}^{\prime}$ | $M_{2}^{\prime}$ | $M_{3}^{\prime}$ | $M_{4}^{\prime}$ | $M_{5}^{\prime}$ | $M_{6}^{\prime}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{0}$ |  |  |  |  | + |  | x |
| $M_{1}$ |  |  |  | x | x |  |  |
| $M_{2}$ |  |  |  |  |  |  |  |
| $M_{3}$ | + | x |  |  |  |  |  |
| $M_{4}$ |  | x |  |  |  |  |  |
| $M_{5}$ | x |  |  |  |  |  |  |
| $M_{6}$ |  |  |  |  |  |  |  |

Table 3. Moment-triples sharing $M_{2}^{\prime \prime}$

|  | $M_{0}^{\prime}$ | $M_{1}^{\prime}$ | $M_{2}^{\prime}$ | $M_{3}^{\prime}$ | $M_{4}^{\prime}$ | $M_{5}^{\prime}$ | $M_{6}^{\prime}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{0}$ |  |  | + |  | x |  |  |  |
| $M_{1}$ | + |  | x |  |  |  |  |  |
| $M_{2}$ |  | x |  |  |  |  |  |  |
| $M_{3}$ | x |  |  |  |  |  |  |  |
| $M_{4}$ |  |  |  |  |  |  |  |  |
| $M_{5}$ |  |  |  |  |  |  |  |  |
| $M_{6}$ |  |  |  |  |  |  |  |  |

Table 5. Moment-triples sharing $M_{4}^{\prime \prime}$

|  | $M_{0}^{\prime}$ | $M_{1}^{\prime}$ | $M_{2}^{\prime}$ | $M_{3}^{\prime}$ | $M_{4}^{\prime}$ | $M_{5}^{\prime}$ | $M_{6}^{\prime}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{0}$ |  |  |  | + | + | x | x |
| $M_{1}$ |  |  |  |  |  |  |  |
| $M_{2}$ |  |  |  | x | x |  |  |
| $M_{3}$ | + |  | x |  |  |  |  |
| $M_{4}$ | + |  | x |  |  |  |  |
| $M_{5}$ | x |  |  |  |  |  |  |
| $M_{6}$ | x |  |  |  |  |  |  |

Table 2. Moment-triples sharing $M_{1}^{\prime \prime}$

|  | $M_{0}^{\prime}$ | $M_{1}^{\prime}$ | $M_{2}^{\prime}$ | $M_{3}^{\prime}$ | $M_{4}^{\prime}$ | $M_{5}^{\prime}$ | $M_{6}^{\prime}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{0}$ |  | + | + |  | x |  |  |
| $M_{1}$ | + |  | x |  |  |  |  |
| $M_{2}$ | + | x |  |  |  |  |  |
| $M_{3}$ |  |  |  |  |  |  |  |
| $M_{4}$ | x |  |  |  |  |  |  |
| $M_{5}$ |  |  |  |  |  |  |  |
| $M_{6}$ |  |  |  |  |  |  |  |

Table 4. Moment-triples sharing $M_{3}^{\prime \prime}$

|  | $M_{0}^{\prime}$ | $M_{1}^{\prime}$ | $M_{2}^{\prime}$ | $M_{3}^{\prime}$ | $M_{4}^{\prime}$ | $M_{5}^{\prime}$ | $M_{6}^{\prime}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{0}$ |  | x | x |  |  |  |  |
| $M_{1}$ | x |  |  |  |  |  |  |
| $M_{2}$ | x |  |  |  |  |  |  |
| $M_{3}$ |  |  |  |  |  |  |  |
| $M_{4}$ |  |  |  |  |  |  |  |
| $M_{5}$ |  |  |  |  |  |  |  |
| $M_{6}$ |  |  |  |  |  |  |  |

Table 6. Moment-triples sharing $M_{5}^{\prime \prime}$

|  | $M_{0}^{\prime}$ | $M_{1}^{\prime}$ | $M_{2}^{\prime}$ | $M_{3}^{\prime}$ | $M_{4}^{\prime}$ | $M_{5}^{\prime}$ | $M_{6}^{\prime}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{0}$ |  | x |  |  |  |  |  |
| $M_{1}$ | x |  |  |  |  |  |  |
| $M_{2}$ |  |  |  |  |  |  |  |
| $M_{3}$ |  |  |  |  |  |  |  |
| $M_{4}$ |  |  |  |  |  |  |  |
| $M_{5}$ |  |  |  |  |  |  |  |
| $M_{6}$ |  |  |  |  |  |  |  |

Table 7. Moment-triples sharing $M_{6}^{\prime \prime}$

Note that all other moment-triples in tables 1-7 can be computed from the existing ones using the following relationships:

$$
\begin{aligned}
M_{i} M_{j}^{\prime} M_{k}^{\prime \prime} & =\frac{M_{i} M_{j-1}^{\prime} M_{k}^{\prime \prime} \cdot M_{i+1} M_{j}^{\prime} M_{k}^{\prime \prime}}{M_{i+1} M_{j-1}^{\prime} M_{k}^{\prime \prime}}=\frac{M_{i} M_{j-1}^{\prime} M_{k}^{\prime \prime} \cdot M_{i-1} M_{j}^{\prime} M_{k}^{\prime \prime}}{M_{i-1} M_{j-1}^{\prime} M_{k}^{\prime \prime}} \\
& =\frac{M_{i-1} M_{j}^{\prime} M_{k}^{\prime \prime} \cdot M_{i} M_{j+1}^{\prime} M_{k}^{\prime \prime}}{M_{i-1} M_{j+1}^{\prime} M_{k}^{\prime \prime}}=\frac{M_{i+1} M_{j}^{\prime} M_{k}^{\prime \prime} \cdot M_{i} M_{j+1}^{\prime} M_{k}^{\prime \prime}}{M_{i+1} M_{j+1}^{\prime} M_{k}^{\prime \prime}} \\
& =\frac{M_{i} M_{j-1}^{\prime} M_{k}^{\prime \prime} \cdot M_{i} M_{j}^{\prime} M_{k-1}^{\prime \prime}}{M_{i} M_{j-1}^{\prime} M_{k-1}^{\prime \prime}}=\frac{M_{i} M_{j+1}^{\prime} M_{k}^{\prime \prime} \cdot M_{i} M_{j}^{\prime} M_{k-1}^{\prime \prime}}{M_{i} M_{j+1}^{\prime} M_{k-1}^{\prime \prime}} \\
& =\frac{M_{i} M_{j-1}^{\prime} M_{k}^{\prime \prime} \cdot M_{i} M_{j}^{\prime} M_{k+1}^{\prime \prime}}{M_{i} M_{j-1}^{\prime} M_{k+1}^{\prime \prime}}=\frac{M_{i} M_{j}^{\prime} M_{k+1}^{\prime \prime} \cdot M_{i} M_{j+1}^{\prime} M_{k}^{\prime \prime}}{M_{i} M_{j+1}^{\prime} M_{k+1}^{\prime \prime}} \\
& =\frac{M_{i} M_{j}^{\prime} M_{k-1}^{\prime \prime} \cdot M_{i-1} M_{j}^{\prime} M_{k}^{\prime \prime}}{M_{i-1} M_{j}^{\prime} M_{k-1}^{\prime \prime}}=\frac{M_{i-1} M_{j}^{\prime} M_{k}^{\prime \prime} \cdot M_{i} M_{j}^{\prime} M_{k+1}^{\prime \prime}}{M_{i-1} M_{j}^{\prime} M_{k+1}^{\prime \prime}} \\
& =\frac{M_{i} M_{j}^{\prime} M_{k+1}^{\prime \prime} \cdot M_{i+1} M_{j}^{\prime} M_{k}^{\prime \prime}}{M_{i+1} M_{j}^{\prime} M_{k+1}^{\prime \prime}}=\frac{M_{i+1} M_{j}^{\prime} M_{k}^{\prime \prime} \cdot M_{i} M_{j}^{\prime} M_{k-1}^{\prime \prime}}{M_{i+1} M_{j}^{\prime} M_{k-1}^{\prime \prime}} .
\end{aligned}
$$

Again we proceed to compute the first $n$ derivatives at point ( $u_{0}, t_{0}$ ) on one of the torsion zero-crossing contours. We now obtain $n+1$ homogeneous linear equations in some of the moment-triples $M_{i} M_{j}^{\prime} M_{k}^{\prime \prime}$ by assuming that each moment-triple is a new variable.

Since this system is in terms of the first $2 n+3$ moments of functions $f(\omega)$, $f^{\prime}(\omega)$ and $f^{\prime \prime}(\omega)$, it will contain $O\left(n^{3}\right)$ moment-triples. Therefore additional equations are required to constrain the system. To obtain those equations, we proceed as follows:

Assume that moments of order higher than $2 n+2$ are zero. Compute derivatives
of order higher than $n$ at $\left(u_{0}, t_{0}\right)$ but set moments of order higher than $2 n+2$ to zero in the resulting equations. If a sufficient number of derivatives are computed at ( $u_{0}, t_{0}$ ), the number of equations obtained will be equal to the number of moment-triples and our linear system will be constrained.

It follows from an assumption of generality that the system will have a unique zero eigenvector and therefore a unique solution modulo scaling. Once the moment-triples in the system are known, all other moment-triples can be computed from the known ones using the relationships given above. Since all the moment-triples $M_{i} M_{j}^{\prime} M_{k}^{\prime \prime}$ together determine function of $\beta(u)$, it follows that function $\beta(u)$ can be determined modulo constant scaling.

Yuille and Poggio [1983] have shown that a 1-D signal can be reconstructed using two points from its scale space image. Note that our result implies that only one point is sufficient for the reconstruction of that signal.

The theorem has now been proven for the regular torsion scale space image. To prove the same result about the resampled torsion scale space image, recall that derivatives at one point (at any scale) on any torsion zero-crossing contour in the torsion scale space of $\Gamma$ were computed and it was shown that the resulting equations can be solved for the coefficients of expansion of the function $\beta(u)$ in functions related to the Hermite polynomials.

As before, we choose a point on a zero-crossing contour at any scale of the resampled curvature scale space image of $\Gamma$ and compute the necessary derivatives. The value of $\sigma$ in the resulting equations is then set to zero. Consequently, the arc length evolved curve $\Gamma_{\sigma}$, where $\sigma$ corresponds to the scale at which the derivatives were computed, is reconstructed modulus uniform scaling, rotation and translation.

The next step is to recover the original curve $\Gamma$ modulo function $\beta(u)$. This is done by applying reverse arc length evolution to $\Gamma_{\sigma}$. Let the arc length evolved curve $\Gamma_{\sigma}$ be defined by:

$$
\Gamma_{\sigma}=\{(X(W, \sigma), Y(W, \sigma), Z(W, \sigma) \mid W \in[0,1]\}
$$

A reverse arc length evolved curve $\Gamma$ is defined by:

$$
\Gamma=\{(x(w), y(w), z(w)) \mid w \in[0,1]\}
$$

where

$$
\begin{aligned}
& x(w)=X(w, \sigma) \circledast D_{N}(w, \sigma) \\
& y(w)=Y(w, \sigma) \circledast D_{N}(w, \sigma)
\end{aligned}
$$

and

$$
z(w)=Z(w, \sigma) \circledast D_{N}(w, \sigma)
$$

where $D_{N}$ is a deblurring operator defined in [Hummel et al. 1987] and

$$
w(W, t)=\int_{0}^{t} \int_{0}^{w} \kappa^{2}(w, t) d w d t .
$$

where $t=\sigma^{2} / 2$. As a result, $\Gamma$ is recovered modulo function $\beta(u)$.
To prove the same result about the renormalized torsion scale space image, evolved curve $\Gamma_{\sigma}$ is again reconstructed, then each of its coordinate functions is deblurred by convolving it with the deblurring operator $D_{N}$. Once again $\Gamma$ is recovered modulo function $\beta(u)$.

Proof of theorem 2: Since by assumption all evolved and arc length evolved curves $\Gamma_{\sigma}$ are in $C_{3}$, the conditions of the implicit function theorem are satisfied on contours $\tau(u, \sigma), \tau(w, \sigma)$ and $\tau(W, \sigma)=0$ in the regular, renormalized and resampled torsion scale space images of $\Gamma$. Since the proofs are identical, the theorem will be proven here for the regular torsion scale space image.

The torsion of each evolved curve $\Gamma_{\sigma}=(x(u, \sigma), y(u, \sigma), z(u, \sigma))$ is given by:

$$
\tau(u, \sigma)=\frac{\dddot{z} \ddot{x} \ddot{y}-\dddot{z} \ddot{y} \ddot{x}+\dddot{y} \ddot{z} \ddot{x}-\dddot{y} \ddot{x} \ddot{z}+\dddot{x} \ddot{y} \ddot{z}-\dddot{x} \ddot{z} \ddot{y}}{(\ddot{y} \ddot{z}-\dot{z} \ddot{y})^{2}+(\dot{z} \ddot{x}-\dot{x} \ddot{z})^{2}+(\dot{x} \ddot{y}-\ddot{y} \ddot{x})^{2}}
$$

where . represents derivative with respect to $u$. On any contour in the torsion scale space image of $\Gamma$ :

$$
\tau(u, \sigma)=0
$$

It follows from the assumption that all $\Gamma_{\sigma}$ are in $C_{3}$ that:

$$
\beta(u, t)=\dddot{z} \dot{x} \ddot{y}-\dddot{z} \dddot{y} \ddot{x}+\dddot{y} \dot{z} \ddot{x}-\dddot{y} \ddot{x} \ddot{z}+\dddot{x} \ddot{y} \ddot{z}-\dddot{x} \dot{z} \ddot{y}
$$

where again . represents derivative with respect to $u$ and $t=\sigma^{2} / 2$. The functions $x(u, t), y(u, t)$ and $z(u, t)$ satisfy the heat equation:

$$
\begin{aligned}
& x_{u u}(u, t)=x_{t}(u, t) \\
& y_{u u}(u, t)=y_{t}(u, t) \\
& z_{u u}(u, t)=z_{t}(u, t) .
\end{aligned}
$$

Since evolved curves $\Gamma_{\sigma}$ are all in $C_{3}$, the conditions of the implicit function theorem are satisfied on contours $\beta(u, t)=0$ :

$$
t=t(u)
$$

$$
\dot{t}(u)=\frac{d t}{d u}=\frac{-\beta_{u}}{\beta_{t}} .
$$

The theorem will be proven if it is shown that if $\dot{t}(u)=0$ at any point on a torsion zero-crossing contour, then $\ddot{t}(u)<0$ at that point. Observe that $\dot{t}(u)=0$ if and only if $\beta_{u}(u, t)=0$. It follows that at a point where $\dot{t}(u)=0$ :

$$
\ddot{t}(u)=\frac{d}{d u}\left(\frac{-\beta_{u}}{\beta_{t}}\right)=\frac{\partial}{\partial u}\left(\frac{-\beta_{u}}{\beta_{t}}\right)+\frac{\partial}{\partial t}\left(\frac{-\beta_{u}}{\beta_{t}}\right) \frac{d t}{d u}=\frac{-\beta_{u u}}{\beta_{t}} .
$$

Therefore it must be shown that if $\beta(u, t)=\beta_{u}(u, t)=0$ then $\beta_{u u} / \beta_{t}>0$.
We will now derive explicit expressions for $\beta_{u u}$ and $\beta_{t}$. Differentiating the expression for $\beta(u, t)$ with respect to $u$ and simplifying yields:

$$
\beta_{u}(u, t)=z_{t t} x_{u} y_{t}-z_{t t} y_{u} x_{t}+y_{t t} z_{u} x_{t}-y_{t t} x_{u} z_{t}+x_{t t} y_{u} z_{t}-x_{t t} z_{u} y_{t}
$$

Differentiating the expression for $\beta_{u}$ with respect to $u$ and simplifying yields:

$$
\beta_{u u}=\Psi_{1}+\Psi_{2}
$$

where

$$
\Psi_{1}=z_{t t u} x_{u} y_{t}-z_{t t u} y_{u} x_{t}+y_{t t u} z_{u} x_{t}-y_{t t u} x_{u} z_{t}+x_{t t u} y_{u} z_{t}-x_{t t u} z_{u} y_{t}
$$

and

$$
\Psi_{2}=y_{t u} z_{t t} x_{u}-x_{t u} z_{t t} y_{u}+x_{t u} y_{t t} z_{u}-z_{t u} y_{t t} x_{u}+z_{t u} x_{t t} y_{u}-y_{t u} x_{t t} z_{u}
$$

Differentiating the expression for $\beta(u, t)$ with respect to $t$ and simplifying yields:

$$
\beta_{t}=\Psi_{1}-\Psi_{2}
$$

Let $P$ be a point on an evolved curve $\Gamma_{\sigma}$ where $\beta(u, t)=\beta_{u}(u, t)=0$. The coordinate functions of $\Gamma_{\sigma}$ can be locally approximated at $P$ using polynomial functions. Furthermore, assume that $u=0$ at point $P$. It follows that ( $u^{m}, u^{n}, u^{p}$ ) is a local approximation to $\Gamma_{\sigma}$ around $P$ where $m, n$ and $p$ are the lowest non-zero powers of the polynomials approximating functions $x(u, t), y(u, t)$ and $z(u, t)$ respectively. Also assume without loss of generality that $p>n>m$. Observe that

$$
\begin{gathered}
x_{u}=m u^{m-1} \\
x_{t}=m(m-1) u^{m-2} \\
x_{t u}=m(m-1)(m-2) u^{m-3} \\
x_{t t}=m(m-1)(m-2)(m-3) u^{m-4} \\
x_{t t u}=m(m-1)(m-2)(m-3)(m-4) u^{m-5}
\end{gathered}
$$

and that

$$
\begin{gathered}
y_{u}=n u^{n-1} \\
y_{t}=n(n-1) u^{n-2} \\
y_{t u}=n(n-1)(n-2) u^{n-3} \\
y_{t t}=n(n-1)(n-2)(n-3) u^{n-4} \\
y_{t t u}=n(n-1)(n-2)(n-3)(n-4) u^{n-5}
\end{gathered}
$$

and that

$$
\begin{gathered}
z_{u}=p u^{p-1} \\
z_{t}=p(p-1) u^{p-2} \\
z_{t u}=p(p-1)(p-2) u^{p-3} \\
z_{t t}=p(p-1)(p-2)(p-3) u^{p-4} \\
z_{t t u}=p(p-1)(p-2)(p-3)(p-4) u^{p-5}
\end{gathered}
$$

It now follows that at point $P$ :

$$
\frac{\beta_{u u}}{\beta_{t}}=\frac{\Xi_{1} u^{m+n+p-8}+\Xi_{2} u^{m+n+p-8}}{\Xi_{1} u^{m+n+p-8}-\Xi_{2} u^{m+n+p-8}}=\frac{\Xi_{1}+\Xi_{2}}{\Xi_{1}-\Xi_{2}}
$$

where

$$
\begin{aligned}
\Xi_{1} & =(p-1)(p-2)(p-3)(p-4)(n-1)-(p-1)(p-2)(p-3)(p-4)(m-1) \\
& +(n-1)(n-2)(n-3)(n-4)(m-1)-(n-1)(n-2)(n-3)(n-4)(p-1) \\
& +(m-1)(m-2)(m-3)(m-4)(p-1)-(m-1)(m-2)(m-3)(m-4)(n-1)
\end{aligned}
$$

and

$$
\begin{aligned}
\Xi_{2} & =(p-1)(p-2)(p-3)(n-1)(n-2)-(p-1)(p-2)(p-3)(m-1)(m-2) \\
& +(n-1)(n-2)(n-3)(m-1)(m-2)-(n-1)(n-2)(n-3)(p-1)(p-2) \\
& +(m-1)(m-2)(m-3)(p-1)(p-2)-(m-1)(m-2)(m-3)(n-1)(n-2)
\end{aligned}
$$

It can be shown that:

$$
\Xi_{1}=(p-n)(p-m)(n-m)\left(p^{2}+(n+m-10) p+n^{2}+m^{2}+m n-10 n-10 m+35\right)
$$

and that:

$$
\Xi_{2}=(p-n)(p-m)(n-m)(p(n+m)-3(n+m)-3 p+m n+7)
$$

It can now be concluded that to prove the theorem, it must be shown that:

$$
\left|\Xi_{1}\right| \geq\left|\Xi_{2}\right|
$$

or

$$
\left|\Delta_{1}\right| \geq\left|\Delta_{2}\right|
$$

where

$$
\Delta_{1}=p^{2}+n^{2}+m^{2}+n p+m p+m n-10 p-10 n-10 m+35
$$

and

$$
\Delta_{2}=n p+m p+m n-3 p-3 n-3 m+7 .
$$

We shall now use a case analysis to prove that the inequality above holds for all valid triples of values of $m, n$ and $p$. The analysis below shows that only triples of values which satisfy the inequality:

$$
p>m+n
$$

are valid:
Recall that $\left(u^{m}, u^{n}, u^{p}\right)$ was used to approximate the curve around point $P$. It follows that in a neighborhood of $P$, torsion is given by:

$$
\tau(u)=\frac{\lambda u^{p+n+m-6}}{\lambda_{1} u^{2(p+n-3)}+\lambda_{2} u^{2(p+m-3)}+\lambda_{3} u^{2(m+n-3)}}
$$

where $\lambda, \lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ are constants. The expression above is ambiguous at $u=0$. To resolve the ambiguity, l'Hopital's rule can be applied repeatedly. Since both the numerator and the denominator are polynomials, to have $\tau(u)=0$ at $u=0$, repeated application of l'Hopital's rule should result in:

$$
\lim _{u \rightarrow 0} \tau(u)=\frac{\psi u^{i}}{\xi+f(u)}
$$

where $\psi$ and $\xi$ are constants, $i>0$ and $f(u)=0$ at $u=0$. This can only happen when one of the following three conditions are met:
a. $p+n+m-6>2(p+n-3)$
b. $p+n+m-6>2(p+m-3)$
c. $p+n+m-6>2(m+n-3)$.

Conditions $\mathbf{a}$ and $\mathbf{b}$ are not possible since they violate the assumption that $p>n>m$. However, condition $\mathbf{c}$ is possible. It follows from this condition that:

$$
p>m+n .
$$

We can now proceed with the case analysis. All triples of values for $m, n$ and $p$ in which $m$ is even correspond to cusp points which are excluded by the
assumption that all evolved curves $\Gamma_{\sigma}$ are in $C_{3}$. Therefore we will consider only odd values of $m$.

Case 1. Suppose $m \geq 7$. Recall that $p>n>m$. It is easily seen that both $\Delta_{1}$ and $\Delta_{2}$ are positive. So the absolute value signs can be dropped and the inequality:

$$
\Delta_{1} \geq \Delta_{2}
$$

can be simplified. As a result, we must now show that the following inequality holds:

$$
p^{2}+n^{2}+m^{2} \geq 7 p+7 n+7 m-28
$$

Note that $m^{2} \geq 7 m, n^{2}>7 n$ and $p^{2}>7 p$. It follows that the inequality does hold.

Case 2. Suppose $m=5$. Again, it can be seen that both $\Delta_{1}$ and $\Delta_{2}$ are positive. We must again show that:

$$
p^{2}+n^{2}+m^{2} \geq 7 p+7 n+7 m-28
$$

Substitute $m=5$ in the above inequality. We now have:

$$
p^{2}+n^{2} \geq 7 p+7 n-18
$$

Since $n \geq 6, n^{2} \geq 7 n-18$ and since $p>11, p^{2}>7 p$. Hence the inequality again holds.

Case 3. Suppose $m=3$. Substitute this value for $m$ in $\Delta_{1}$. As a result, $\Delta_{1}$ simplifies to:

$$
p^{2}+n^{2}+n p-7 p-7 n+14
$$

Note that $n \geq 4$ and $p \geq 8$. So $p^{2}>7 p$. Hence to show that $\Delta_{1}$ is positive, it is sufficient to show that:

$$
n^{2}+n p-7 n+14>0
$$

Since $p \geq 8, n p \geq 8 n$. Therefore:

$$
n^{2}+n p-7 n+14 \geq n^{2}+8 n-7 n+14=n^{2}+n+14>0 .
$$

Now substitute $m=3$ in $\Delta_{2}$. As a result $\Delta_{2}$ simplifies to:

$$
n p+3 p+3 n-3 p-3 n-9+7=n p-2
$$

which is always positive. Therefore we must again show that:

$$
p^{2}+n^{2}+m^{2} \geq 7 p+7 n+7 m-28
$$

Substituting $m=3$ in the above inequality yields:

$$
p^{2}+n^{2} \geq 7 p+7 n-16
$$

Since $p \geq 8, p^{2}>7 p$ and it is sufficient to show that:

$$
n^{2} \geq 7 n-16
$$

It is easily seen that this inequality is satisfied for $n \geq 4$.
Case 4. Suppose $m=1$. Substituting this value in $\Delta_{1}$ simplifies it to:

$$
p^{2}+n^{2}+n p-9 p-9 n+26 .
$$

Since $p \geq 4, p^{2}-9 p \geq-20$. Hence to show that $\Delta_{1}$ is non-negative, it is sufficient to show that:

$$
n^{2}+n p-9 n+6 \geq 0
$$

Again since $p \geq 4$ :

$$
n^{2}+n p-9 n+6 \geq n^{2}+4 n-9 n+6=n^{2}-5 n+6
$$

which is non-negative for $n \geq 2$. Now substitute for $m=1$ in $\Delta_{2}$ to obtain:

$$
p n-2 p-2 n+4=p(n-2)-2 n+4
$$

Since $p \geq 4$

$$
p(n-2)-2 n+4 \geq 4(n-2)-2 n+4=4 n-8-2 n+4=2 n-4
$$

which is non-negative since $n \geq 2$. So $\Delta_{2}$ is also non-negative. Therefore we must again show that:

$$
p^{2}+n^{2}+m^{2} \geq 7 p+7 n+7 m-28
$$

Substitute for $m=1$ in the above expression to obtain:

$$
p^{2}+n^{2} \geq 7 p+7 n-22
$$

which is equivalent to:

$$
\left(p^{2}-7 p\right)+\left(n^{2}-7 n\right)+22 \geq 0
$$

If $n=2$, then $n^{2}-7 n=-10$ and $p \geq 4$. It follows from $p \geq 4$ that $p^{2}-7 p \geq-12$. As a result, the inequality above is satisfied. If $n>2$, then $n^{2}-7 n \geq-12$ and $p \geq 5$. It follows from $p \geq 5$ that $p^{2}-7 p \geq-10$. Therefore, the inequality above is again satisfied.

This completes the case analysis. We have shown that the inequality:

$$
\left|\Delta_{1}\right| \geq\left|\Delta_{2}\right|
$$

and therefore the inequality:

$$
\left|\Xi_{1}\right| \geq\left|\Xi_{2}\right|
$$

is satisfied for all valid triples of values of $m, n$ and $p$. Therefore $\beta_{u u} / \beta_{t}$ is always positive. Hence all extrema of contours in all torsion scale space images of $\Gamma$ are maxima.

Proof of theorem 3: It will be shown that this theorem holds for an arbitrary parametrization of $\Gamma_{\boldsymbol{\sigma}}$. Therefore it must also be true of arc length parametrization or close approximations.

Let $(X(u, \sigma), Y(u, \sigma), Z(u, \sigma))$ be an arbitrary parametrization of $\Gamma$ with a cusp point at $u_{0}$. It has been shown by [Hummel et al. 1987] that the class of polynomial functions is closed under convolution with a Gaussian. Therefore $X(u, \sigma), Y(u, \sigma)$ and $Z(u, \sigma)$ are also polynomial functions:

$$
\begin{aligned}
& X(u, \sigma)=a_{0}+a_{1} u+a_{2} u^{2}+a_{3} u^{3}+\cdots \\
& Y(u, \sigma)=b_{0}+b_{1} u+b_{2} u^{2}+b_{3} u^{3}+\cdots \\
& Z(u, \sigma)=c_{0}+c_{1} u+c_{2} u^{2}+c_{3} u^{3}+\cdots
\end{aligned}
$$

Let $\Gamma_{\sigma}$ go through the origin of the coordinate system at $u_{0}=0$. It follows that $a_{0}=b_{0}=c_{0}=0$. Every cusp point is also a singular point of the curve. Therefore $\Gamma_{\sigma}$ has a singularity at $u_{0}$. Now observe that

$$
\begin{aligned}
& X_{u}(u, \sigma)=a_{1}+2 a_{2} u+3 a_{3} u^{2}+4 a_{4} u^{3}+\cdots \\
& Y_{u}(u, \sigma)=b_{1}+2 b_{2} u+3 b_{3} u^{2}+4 b_{4} u^{3}+\cdots \\
& Z_{u}(u, \sigma)=c_{1}+2 c_{2} u+3 c_{3} u^{2}+4 c_{4} u^{3}+\cdots
\end{aligned}
$$

$X_{u}(u, \sigma), \quad Y_{u}(u, \sigma)$ and $Z_{u}(u, \sigma)$ are zero at a singular point. It follows that $a_{1}=b_{1}=c_{1}=0$. As a result, all powers of $u$ in $X(u, \sigma), Y(u, \sigma)$ and $Z(u, \sigma)$ are larger than one.

The following case analysis identifies the cases in which the singular point at $u_{0}$ is also a cusp point. Since $\Gamma_{\sigma}$ is examined in a small neighborhood of point $u_{0}=0$, it can be approximated using the lowest degree terms in $X(u, \sigma), Y(u, \sigma)$ and $Z(u, \sigma)$ :

$$
\Gamma_{\sigma}=\left(u^{m}, u^{n}, u^{p}\right)
$$

Assume without loss of generality that $p>n>m$. Observe that

$$
\mathbf{r}_{u}(u, \sigma)=\left(X_{u}(u, \sigma), Y_{u}(u, \sigma), Z_{u}(u, \sigma)\right)=\left(m u^{m-1}, n u^{n-1}, p u^{p-1}\right)
$$

Therefore

$$
\mathbf{r}_{u}(\epsilon, \sigma)=\left(m \epsilon^{m-1}, n \epsilon^{n-1}, p \epsilon^{p-1}\right)=\epsilon^{m-1}\left(m, n \epsilon^{n-m}, p \epsilon^{p-m}\right)
$$

and

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(m(-\epsilon)^{m-1}, n(-\epsilon)^{n-1}, p(-\epsilon)^{p-1}\right)
$$

Since $m, n$ and $p$ can be odd or even, the singular point at $u_{0}$ must be analyzed in each of eight possible cases:

1. $m, n$ and $p$ are even.
$m-1, n-1$ and $p-1$ are odd. So

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(-m \epsilon^{m-1},-n \epsilon^{n-1},-p \epsilon^{p-1}\right)=-\epsilon^{m-1}\left(m, n \epsilon^{n-m}, p \epsilon^{p-m}\right)
$$

Comparing $\mathbf{r}_{u}(\epsilon, \sigma)$ to $\mathbf{r}_{u}(-\epsilon, \sigma)$ shows that an infinitesimal change in parameter $u$ in a neighborhood of the singular point results in a large change in the direction of the tangent vector. Therefore this singularity is a cusp point.
2. $m$ and $n$ are even, $p$ is odd.
$m-1$ and $n-1$ are odd and $p-1$ is even. Therefore

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(-m \epsilon^{m-1},-n \epsilon^{n-1}, p \epsilon^{p-1}\right)=\epsilon^{m-1}\left(-m,-n \epsilon^{n-m}, p \epsilon^{p-m}\right)
$$

A comparison of $\mathbf{r}_{u}(\epsilon, \sigma)$ and $\mathbf{r}_{u}(-\epsilon, \sigma)$ again shows that an infinitesimal change in $u$ causes a large change in the tangent direction. Hence this singular point is also a cusp point.
3. $m$ is even, $n$ is odd and $p$ is even.
$m-1$ is odd, $n-1$ is even and $p-1$ is odd. Hence

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(-m \epsilon^{m-1}, n \epsilon^{n-1},-p \epsilon^{p-1}\right)=\epsilon^{m-1}\left(-m, n \epsilon^{n-m},-p \epsilon^{p-m}\right)
$$

An infinitesimal change in $u$ again results in a large change in the tangent direction. This singularity is a cusp point as well.
4. $m$ is even, $n$ and $p$ are odd.
$m-1$ is odd, $n-1$ and $p-1$ are even. So

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(-m \epsilon^{m-1}, n \epsilon^{n-1}, p \epsilon^{p-1}\right)=\epsilon^{m-1}\left(-m, n \epsilon^{n-m}, p \epsilon^{p-m}\right)
$$

A large change in the tangent direction is caused by an infinitesimal change in $u$. Therefore this singularity also corresponds to a cusp point.
5. $m$ is odd, $n$ and $p$ are even.
$m-1$ is even, $n-1$ and $p-1$ are odd. Therefore

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(m \epsilon^{m-1},-n \epsilon^{n-1},-p \epsilon^{p-1}\right)=\epsilon^{m-1}\left(m,-n \epsilon^{n-m},-p \epsilon^{p-m}\right)
$$

A comparison of $\mathbf{r}_{u}(\epsilon, \sigma)$ and $\mathbf{r}_{u}(-\epsilon, \sigma)$ now shows that an infinitesimal change in $u$
in the neighborhood of the singular point also results in an infinitesimal change in the tangent direction. Hence, this singular point is not a cusp point.
6. $m$ is odd, $n$ is even, $p$ is odd.
$m-1$ is even, $n-1$ is odd and $p-1$ is even. So

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(m \epsilon^{m-1},-n \epsilon^{n-1}, p \epsilon^{p-1}\right)=\epsilon^{m-1}\left(m,-n \epsilon^{n-m}, p \epsilon^{p-m}\right)
$$

The tangent direction changes only infinitesimally in the neighborhood of the singular point. Therefore this singularity is not a cusp point either.
7. $m$ and $n$ are odd, $p$ is even.
$m-1$ and $n-1$ are even and $p-1$ is odd. Hence

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(m \epsilon^{m-1}, n \epsilon^{n-1},-p \epsilon^{p-1}\right)=\epsilon^{m-1}\left(m, n \epsilon^{n-m},-p \epsilon^{p-m}\right)
$$

This singularity is again not a cusp point since the tangent direction changes only infinitesimally in its neighborhood.
8. $m, n$ and $p$ are odd.
$m-1, n-1$ and $p-1$ are even. Therefore

$$
\mathbf{r}_{u}(-\epsilon, \sigma)=\left(m \epsilon^{m-1}, n \epsilon^{n-1}, p \epsilon^{p-1}\right)=\epsilon^{m-1}\left(m, n \epsilon^{n-m}, p \epsilon^{p-m}\right)
$$

The last singular point is not a cusp point either since the changes in the tangent direction are again infinitesimal.

It follows from the case analysis above that only the singular points in cases 1-4 are cusp points. We next derive analytical expressions for the curve $\Gamma_{\sigma-\delta}$ so that it can be analyzed in a small neighborhood of the cusp point.

To deblur function $f(u)=u^{k}$, a rescaled version of that function is convolved with the function $\frac{2}{\sqrt{\pi}} e^{-u^{2}}\left(1-u^{2}\right)$. This function is an approximation to the deblurring operator derived in [Hummel et al. 1987] and is good for small amounts of deblurring. The convolution can be expressed as

$$
\left(D_{t} f\right)(u)=\int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}} e^{-v^{2}}\left(1-v^{2}\right) f(u+2 v \sqrt{t}) d v
$$

or

$$
\left(D_{t} f\right)(u)=\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^{2}}\left(1-v^{2}\right)(u+2 v \sqrt{t})^{k} d v
$$

where $t$ is the scale factor and controls the amount of deblurring. Solving the
integral above yields

$$
\begin{equation*}
\left(D_{t} f\right)(u)=\sum_{\substack{p=0 \\(p \text { even })}}^{k} 1.3 .5 \cdots(p-1) \frac{(2 t)^{p / 2} k(k-1) \cdots(k-p+1)}{p!}(1-p) u^{k-p} \tag{10}
\end{equation*}
$$

$\Gamma_{\sigma-\delta}$ can now be analyzed in each of the cases 1-4:
Case 1: $\Gamma_{\sigma}$ is approximated by $\left(u^{m}, u^{n}, u^{p}\right)$ where $m, n$ and $p$ are even.
To obtain analytical expressions for $\Gamma_{\sigma-\delta}$, we deblur each of its coordinate functions:

$$
\begin{aligned}
& \left(D_{t} x\right)(u)=u^{m}-c_{1} t u^{m-2}-\cdots-c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^{2}-c_{\frac{m}{2}} t^{\frac{m}{2}} \\
& \left(D_{t} y\right)(u)=u^{n}-c_{1}^{\prime} t u^{n-2}-\cdots-c_{\frac{n-2}{2}}^{\prime} t^{\frac{n-2}{2}} u^{2}-c_{\frac{n}{2}}^{\prime} t^{\frac{n}{2}} \\
& \left(D_{t} z\right)(u)=u^{p}-c_{1}^{\prime \prime} t u^{p-2}-\cdots-c_{\frac{p-2}{\prime}}^{\prime \prime} t^{\frac{p-2}{2}} u^{2}-c_{\frac{p}{2}}^{\prime \prime} t^{\frac{p}{2}}
\end{aligned}
$$

Note that all powers of $u$ are even and all constants are positive. It follows that

$$
\begin{aligned}
& \left(D_{t} \dot{x}\right)(u)=m u^{m-1}-(m-2) c_{1} t u^{m-3}-\cdots-2 c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u \\
& \left(D_{t} \dot{y}\right)(u)=n u^{n-1}-(n-2) c_{1}^{\prime} t u^{n-3}-\cdots-2 c_{\frac{n-2}{2}}^{\prime} t^{\frac{n-2}{2}} u \\
& \left(D_{t} \dot{z}\right)(u)=p u^{p-1}-(p-2) c_{1}^{\prime \prime t u^{p-3}}-\cdots-2 c_{\frac{p-2}{\prime \prime}}^{t^{\frac{p-2}{2}} u}
\end{aligned}
$$

contain only odd powers of $u$ and $\left(D_{t} \dot{\mathbf{r}}\right)(\epsilon)=-\left(D_{t} \dot{\mathbf{r}}\right)(-\epsilon)$. Hence there is also a cusp point on $\Gamma_{\sigma-\delta}$ at $u_{0}$. Since that cusp point is of the same kind as the cusp point on $\Gamma_{\sigma}$, it follows that a cusp point must also exist on $\Gamma$. This is a contradiction of the assumption that $\Gamma$ is in $C_{1}$. Therefore $\Gamma_{\sigma}$ can not have a cusp point of this kind at $u_{0}$.

Case 2: $\Gamma_{\sigma}$ is approximated by $\left(u^{m}, u^{n}, u^{p}\right)$ where $m$ and $n$ are even and $p$ is odd.
$\Gamma_{\sigma-\delta}$ is obtained by deblurring each of its coordinate functions:

$$
\begin{gathered}
\left(D_{t} x\right)(u)=u^{m}-c_{1} t u^{m-2}-\cdots-c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^{2}-c_{\frac{m}{2}} t^{\frac{m}{2}} \\
\left(D_{t} y\right)(u)=u^{n}-c_{1}^{\prime} t u^{n-2}-\cdots-c_{\frac{n-2}{2}}^{\prime} t^{\frac{n-2}{2}} u^{2}-c_{\frac{n}{2}}^{\prime} t^{\frac{n}{2}} \\
\left(D_{t} z\right)(u)=u^{p}-c_{1}^{\prime \prime} t u^{p-2}-\cdots-c_{\frac{p-1}{2}}^{\prime \prime} t^{\frac{p-1}{2}} u
\end{gathered}
$$

Note that $\left(D_{t} x\right)$ and $\left(D_{t} y\right)$ contain even powers of $u$ only, $\left(D_{t} z\right)$ contains odd powers of $u$ only and all constants are positive.

The deblurred curve intersects itself if there are two values of $u, u_{1}$ and $u_{2}$, such that

$$
\begin{aligned}
& x\left(u_{1}\right)=x\left(u_{2}\right) \\
& y\left(u_{1}\right)=y\left(u_{2}\right) \\
& z\left(u_{1}\right)=z\left(u_{2}\right)
\end{aligned}
$$

It follows from the first two constraints above that $u_{1}=-u_{2}$. Substituting for $u_{2}$ in the third constraint and simplifying yields:

$$
u_{1}^{p}-c_{1}^{\prime \prime} t u_{1}^{p-2}-\cdots-c_{\frac{p-1}{2}_{\prime \prime}^{t^{p-1}}}^{u_{1}=0}
$$

Now let $t=\delta$ to obtain

$$
\begin{equation*}
u_{1}^{p}-c_{1}^{\prime \prime} \delta u_{1}^{p-2}-\cdots-c_{\frac{p-1}{2}}^{\delta^{\frac{p-1}{2}}} u_{1}=0 \tag{11}
\end{equation*}
$$

The LHS of (11) is negative for very small values of $u_{1}$ since the first term will be smaller than all other terms, which are negative. As $u_{1}$ grows, the first term becomes larger than the sum of all other terms and the LHS of (11) becomes positive. Therefore there is a positive value of $u_{1}$ at which (11) is satisfied. Hence $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of $u_{0}$.

Case 3: $\Gamma_{\sigma}$ is approximated by $\left(u^{m}, u^{n}, u^{n}\right)$ where $m$ is even, $n$ is odd and $p$ is even.

As in the previous case, we obtain analytical expressions for $\Gamma_{\sigma-\delta}$ :

$$
\begin{gathered}
\left(D_{t} x\right)(u)=u^{m}-c_{1} t u^{m-2}-\cdots-c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^{2}-c_{\frac{m}{2}} t^{\frac{m}{2}} \\
\left(D_{t} y\right)(u)=u^{n}-c_{1}^{\prime} t u^{n-2}-\cdots-c_{\frac{n-1}{2}}^{\prime} t^{\frac{n-1}{2}} u \\
\left(D_{t} z\right)(u)=u^{p}-c_{1}^{\prime \prime} t u^{p-2}-\cdots-c_{\frac{p-2}{\prime}}^{\prime \prime} t^{\frac{p-2}{2}} u^{2}-c_{\frac{p}{2}}^{\prime} t^{\frac{p}{2}}
\end{gathered}
$$

In this case, $\left(D_{t} x\right)$ and $\left(D_{t} z\right)$ contain only even powers of $u$ and $\left(D_{t} y\right)$ contains only odd powers of $u$. Again, $\Gamma_{\sigma-\delta}$ can be shown to intersect itself if there are two values of $u, u_{1}$ and $u_{2}$, such that

$$
\begin{aligned}
& x\left(u_{1}\right)=x\left(u_{2}\right) \\
& y\left(u_{1}\right)=y\left(u_{2}\right) \\
& z\left(u_{1}\right)=z\left(u_{2}\right)
\end{aligned}
$$

It now follows from the first and third constraints above that $u_{1}=-u_{2}$. Substituting for $u_{2}$ in the second constraint, letting $t=\delta$ and simplifying yields

$$
\begin{equation*}
u_{1}^{n}-c_{1}^{\prime} \delta u_{1}^{n-2}-\cdots-c_{\frac{n-1}{2}}^{\prime} \delta^{\frac{n-1}{2}} u_{1}=0 \tag{12}
\end{equation*}
$$

An argument similar to the one used in the previous case shows that there exists a positive value of $u_{1}$ at which (12) is satisfied. Therefore $\Gamma_{\sigma-\delta}$ again intersects itself in a neighborhood of $u_{0}$.

Case 4: $\Gamma_{\sigma}$ is approximated by $\left(u^{m}, u^{n}, u^{p}\right)$ where $m$ is even and $n$ and $p$ are odd.
An analytical expression for $\Gamma_{\sigma-\delta}$ in a neighborhood of $u_{0}$ is given by

$$
\begin{gathered}
\left(D_{t} x\right)(u)=u^{m}-c_{1} t u^{m-2}-\cdots-c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^{2}-c_{\frac{m}{2}} t^{\frac{m}{2}} \\
\left(D_{t} y\right)(u)=u^{n}-c_{1}^{\prime} t u^{n-2}-\cdots-c_{\frac{n-1}{2}}^{t^{-2}} u \\
\left(D_{t} z\right)(u)=u^{p}-c_{1}^{\prime \prime} t u^{p-2}-\cdots-c_{\frac{p-1}{\prime \prime}}^{t^{\frac{p-1}{2}} u}
\end{gathered}
$$

All powers of $u$ in $\left(D_{t} x\right)$ are even and all powers of $u$ in $\left(D_{t} y\right)$ and $\left(D_{t} z\right)$ are odd.
As before, $\Gamma_{\sigma-\delta}$ intersects itself if the three constraints

$$
\begin{aligned}
& x\left(u_{1}\right)=x\left(u_{2}\right) \\
& y\left(u_{1}\right)=y\left(u_{2}\right) \\
& z\left(u_{1}\right)=z\left(u_{2}\right)
\end{aligned}
$$

are satisfied simultaneously. It follows from the first constraint that $u_{1}=-u_{2}$. Now substitute for $u_{2}$ in the second and third constraints, let $t=\delta$ and simplify:

$$
\begin{align*}
& u_{1}^{n}-c_{1}^{\prime} \delta u_{1}^{n-2}-\cdots-c_{\frac{n-1}{2}}^{\prime} \delta^{\frac{n-1}{2}} u_{1}=0  \tag{13}\\
& u_{1}^{p}-c_{1}^{\prime \prime} \delta u_{1}^{p-2}-\cdots-c_{\frac{p-1}{2}}^{\prime \prime} \delta^{p-1}  \tag{14}\\
& u_{1}=0
\end{align*}
$$

Each of the equations (13) and (14) is satisfied at a positive value of $u_{1}$ but, in general, the same value of $u_{1}$ will not satisfy both. It follows that, in this case, $\Gamma_{\sigma-\delta}$ does not intersect itself. However, an argument similar to the ones in the previous two cases shows that the planar curves defined by $\left(D_{t} x\right)$ and $\left(D_{t} y\right)$ and by $\left(D_{t} x\right)$ and $\left(D_{t} z\right)$, that is, the projections of $\Gamma_{\sigma-\delta}$ on the $X Y$ and $X Z$ planes respectively, do intersect themselves in a neighborhood of $u_{0}$.

This completes the proof of theorem 3.
Proof of theorem 4: It will be shown that this theorem holds for an arbitrary parametrization of $\Gamma_{\sigma}$. Therefore it must also be true of arc length parametrization or close approximations.

Let $(x(u), y(u), z(u))$ be an arbitrary parametrization of $\Gamma_{\sigma}$ with a cusp point at $u_{0}$. Using a case analysis similar to the one in the proof of theorem 3 to characterize all the possible singularities of $\Gamma_{\sigma}$ at $u_{0}$, we again conclude that only the singular points in cases 1-4 are cusp points.

We now derive analytical expressions for $\Gamma_{\sigma+\delta}$ so that it can be analyzed in a neighborhood of $u_{0}$. To blur function $f(u)=u^{k}$, we convolve a rescaled version of that function with the function $\frac{1}{\sqrt{\pi}} e^{-u^{2}}$, the blurring operator, as follows:

$$
F(u)=\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-v^{2}} f(u+2 v \sqrt{t}) d v
$$

or

$$
F(u)=\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^{2}}(u+2 v \sqrt{t})^{k} d v
$$

where $t$ is the scale factor and controls the amount of blurring. Solving the integral above yields

$$
\begin{equation*}
F(u)=\sum_{\substack{p=0 \\(p \text { even })}}^{k} 1.3 .5 \cdots(p-1) \frac{(2 t)^{p / 2} k(k-1) \cdots(k-p+1)}{p!} u^{k-p} \tag{15}
\end{equation*}
$$

An expression for $\Gamma_{\sigma+\delta}$ in a neighborhood of the cusp point can be obtained by blurring each of its coordinate functions. Furthermore, expressions for $\Gamma_{\sigma-\delta}$ in a neighborhood of the cusp point can be obtained by deblurring each of its coordinate functions according to (10).

Each of the cases $\mathbf{1 - 4}$ can now be analyzed in turn:
Case 1: $\Gamma_{\sigma}$ is approximated by $\left(u^{m}, u^{n}, u^{p}\right)$ where $m, n$ and $p$ are even.
An argument similar to the one used in case 1 of theorem 3 shows that this kind of cusp point can not arise during evolution of $\Gamma$.

Case 2: $\Gamma_{\sigma}$ is approximated by $\left(u^{m}, u^{n}, u^{p}\right)$ where $m$ and $n$ are even and $p$ is odd.
Observe that

$$
\begin{array}{ccl}
\dot{x}(u)=m u^{m-1} & \ddot{x}(u)=m(m-1) u^{m-2} & \dddot{x}(u)=m(m-1)(m-2) u^{m-3} \\
\dot{y}(u)=n u^{n-1} & \ddot{y}(u)=n(n-1) u^{n-2} & \dddot{y}(u)=n(n-1)(n-2) u^{n-3} \\
\dot{z}(u)=p u^{p-1} & \ddot{z}(u)=p(p-1) u^{p-2} & \dddot{z}(u)=p(p-1)(p-2) u^{p-3}
\end{array}
$$

Torsion on $\Gamma_{\sigma}$ is given by:

$$
\tau(u)=\frac{\dddot{z} \ddot{x} \ddot{y}-\dddot{z} \ddot{y} \ddot{x}+\dddot{y} \ddot{z} \ddot{x}-\dddot{y} \ddot{x} \ddot{z}+\dddot{x} \dddot{y} \ddot{z}-\dddot{x} \ddot{z} \ddot{y}}{(\ddot{y} \ddot{z}-\dot{z} \ddot{y})^{2}+(\dot{z} \ddot{x}-\dot{x} \ddot{z})^{2}+(\dot{x} \ddot{y}-\dot{y} \ddot{x})^{2}}
$$

or
$\tau(u)=\frac{m n p((p-1)(p-2)(n-m)+(n-1)(n-2)(m-p)+(m-1)(m-2)(p-n)) u^{p+n+m-6}}{A+B+C}$
where

$$
\begin{gathered}
A=\left((n p(p-1)-p n(n-1)) u^{p+n-3}\right)^{2} \\
B=\left((p m(m-1)-m p(p-1)) u^{p+m-3}\right)^{2}
\end{gathered}
$$

$$
C=\left((m n(n-1)-n m(m-1)) u^{m+n-3}\right)^{2}
$$

At $u=0$ (cusp point), $\tau$ is undefined. When $u$ is positive or negative, the sign of $\tau(u)$ depends on the sign of the coefficient of the numerator. Let $K$ be that coefficient divided by $m n p$. Observe that

$$
\begin{aligned}
K= & (p-1)(p-2)(n-m)+(n-1)(n-2)(m-p)+(m-1)(m-2)(p-n) \\
= & \left(p^{2}-3 p+2\right)(n-m)+\left(n^{2}-3 n+2\right)(m-p)+\left(m^{2}-3 m+2\right)(p-n) \\
= & n p^{2}-m p^{2}-3 p n+3 p m+2 n-2 m+m n^{2}-3 m n+2 m-p n^{2} \\
& +3 p n-2 p+p m^{2}-3 p m+2 p-n m^{2}+3 m n-2 n \\
= & (n-m) p^{2}+\left(m^{2}-n^{2}\right) p+m n^{2}-n m^{2} \\
= & (n-m) p^{2}+(m+n)(m-n) p+m n(n-m) \\
= & (n-m)\left(p^{2}-(m+n) p+m n\right) \\
= & (n-m)(p-m)(p-n)
\end{aligned}
$$

which is positive because of the assumption that $p>n>m$. Since $p+n+m-6$, the power of $u$ in the numerator, is odd, it follows that $\tau(u)$ is positive for positive $u$ and negative for negative $u$.

We now investigate $\tau(u)$ on $\Gamma_{\sigma+\delta}$. It follows from (15) that $\Gamma_{\sigma+\delta}$ is given by:

$$
\begin{gathered}
X(u)=u^{m}+c_{1} t u^{m-2}+\cdots+c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^{2}+c_{\frac{m}{2}} t^{\frac{m}{2}} \\
Y(u)=u^{n}+c_{1}^{\prime} t u^{n-2}+\cdots+c_{\frac{n-2}{2}}^{\prime} t^{\frac{n-2}{2}} u^{2}+c_{\frac{n}{2}}^{\prime} t^{\frac{n}{2}} \\
Z(u)=u^{p}+c_{1}^{\prime \prime} t u^{p-2}+\cdots+c_{\frac{p-1}{2}}^{\prime \prime} t^{\frac{p-1}{2}} u
\end{gathered}
$$

where all constants are positive, all powers of $u$ in $X(u)$ and $Y(u)$ are even, all powers of $u$ in $Z(u)$ are odd and $t$ equals $\delta$, a small constant. Note also that the last terms in $X(u)$ and $Y(u)$ do not contain any positive powers of $u$ but all terms in $Z(u)$ contain positive powers of $u$. It follows that the last terms in $\ddot{X}(u), \ddot{Y}(u)$, $\dot{Z}(u)$ and $\dddot{Z}(u)$. do not contain positive powers of $u$ whereas all terms in $\dot{X}(u)$, $\ddot{X}(u), \dot{Y}(u), \dddot{Y}(u)$ and $\ddot{Z}(u)$ contain positive powers of $u$. Therefore, at $u=0$, $\dot{X}(u)=\ddot{X}(u)=\dot{Y}(u)=\ddot{Y}(u)=\ddot{Z}(u)=0$ and $\tau=0$. As $u$ grows, the terms in $\dot{X}(u), \ddot{X}(u), \dddot{X}(u), Y(u), \ddot{Y}(u), \dddot{Y}(u), \dot{Z}(u), \ddot{Z}(u)$ and $\dddot{Z}(u)$ with the largest power of $u$ (which are also the only terms without $\delta$ ) become dominant and torsion is
again given by (16). It follows that $\tau(u)$ is positive for positive $u$ and negative for negative $u$ on $\Gamma_{\sigma+\delta}$. Since $\tau$ is zero at $u=0, \Gamma_{\sigma+\delta}$ has a torsion zero-crossing point at $u=0$.

We next investigate $\tau(u)$ on $\Gamma_{\sigma-\delta}$. From (10) it follows that $\Gamma_{\sigma-\delta}$ is given by:

$$
\begin{gathered}
\left(D_{t} x\right)(u)=u^{m}-d_{1} t u^{m-2}-\cdots-d_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^{2}-d_{\frac{m}{2}} t^{\frac{m}{2}} \\
\left(D_{t} y\right)(u)=u^{n}-d_{1}^{\prime} t u^{n-2}-\cdots-d_{\frac{n-2}{2}}^{\prime} t^{\frac{n-2}{2}} u^{2}-d_{\frac{n}{2}}^{\prime} t^{\frac{n}{2}} \\
\left(D_{t} z\right)(u)=u^{p}-d_{1}^{\prime \prime} t u^{p-2}-\cdots-d_{\frac{p-1}{2}} t^{\frac{p-1}{2}} u
\end{gathered}
$$

where all constants are positive, all powers of $u$ in $D_{t} x$ and $D_{t} y$ are even, all powers of $u$ in $D_{t} z$ are odd and $t$ equals $\delta$, a small constant. It again follows that $\tau=0$ at $u=0, \tau$ is positive for positive $u$ and negative for negative $u$. Therefore there is also a torsion zero-crossing point at $u=0$ on $\Gamma_{\sigma-\delta}$. It follows that there is a torsion zero-crossing point at $u_{0}$ on $\Gamma_{\sigma-\delta}$ before the formation of the cusp point and on $\Gamma_{\sigma+\delta}$ after the formation of the cusp point.

Case 3: $\Gamma_{\sigma}$ is approximated by $\left(u^{m}, u^{n}, u^{p}\right)$ where $m$ is even, $n$ is odd and $p$ is even.

The proof is analogous to that of case 2, and the same result follows.
Case 4: $\Gamma_{\sigma}$ is approximated by $\left(u^{m}, u^{n}, u^{p}\right)$ where $m$ is even, and $n$ and $p$ are odd.
At $u=0$, the cusp point, $\tau$ is undefined. At all other points, $\tau(u)$ is given by (16). Since the coefficient of the numerator of (16) is positive (as shown in the proof of case 2) and $p+n+m-6$, the power of $u$ in the numerator, is even, $\tau(u)$ is positive for positive and negative values of $u$ in the neighborhood of $u_{0}$ on $\Gamma_{\sigma}$. Therefore there are no torsion zero-crossing points in the neighborhood of $u_{0}$ on $\Gamma_{\sigma}$.

We now investigate $\tau(u)$ on $\Gamma_{\sigma+\delta}$. It follows from (15) that $\Gamma_{\sigma+\delta}$ is given by:

$$
\begin{gathered}
X(u)=u^{m}+c_{1} t u^{m-2}+\cdots+c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^{2}+c_{\frac{m}{2}} t^{\frac{m}{2}} \\
Y(u)=u^{n}+c_{1}^{\prime} t u^{n-2}+\cdots+c_{\frac{n-1}{2}}^{\prime} t^{\frac{n-1}{2}} u
\end{gathered}
$$

$$
Z(u)=u^{p}+c_{1}^{\prime \prime t u^{p-2}}+\cdots+c_{\frac{p-1}{2}}^{\prime \prime} t^{\frac{p-1}{2}} u
$$

where all constants are positive, all powers of $u$ in $X(u)$ are even, all powers of $u$ in $Y(u)$ and $Z(u)$ are odd and $t$ equals $\delta$, a small constant. Furthermore, note that the last term in $X(u)$ does not contain a positive power of $u$ but all terms in $Y(u)$ and $Z(u)$ contain positive powers of $u$. Therefore the last terms in $\ddot{X}(u)$, $\dot{Y}(u), \dddot{Y}(u), \dot{Z}(u)$ and $\dddot{Z}(u)$ do not contain positive powers of $u$ whereas all terms in $X(u), \ldots(u), \underline{Y}(u)$ and $Z(u)$ contain positive powers of $u$. Hence at $u=0$, $\dot{X}(u)=\ddot{X}(u)=\ddot{Y}(u)=\ddot{Z}(u)=0$ and

$$
\tau(u)=\frac{\dddot{Y}(u) \dot{Z}(u) \ddot{X}(u)-\dddot{Z}(u) \dot{Y}(u) \ddot{X}(u)}{(\dot{Z}(u) \ddot{X}(u))^{2}+(\dot{Y}(u) \ddot{X}(u))^{2}}=\frac{\ddot{X}(u)(\dddot{Y}(u) \dot{Z}(u)-\ddot{Z}(u) \dot{Y}(u))}{(\dot{Z}(u) \ddot{X}(u))^{2}+(\dot{Y}(u) \ddot{X}(u))^{2}}
$$

Since the denominator is positive and $\ddot{X}(u)$ is positive, to determine the sign of $\tau(u)$, we must determine the sign of the expression: $\dddot{Y}(u) \dot{Z}(u)-\dddot{Z}(u) \dot{Y}(u)$. At $u=0$, using (15) we conclude that the non-zero term of $\dot{Y}(u)$ is:

$$
c_{\frac{n-1}{2}}^{\prime} t^{\frac{n-1}{2}}=1.3 .5 \cdots(n-2) \frac{(2 t)^{\frac{n-1}{2}} n!}{(n-1)!}=1.3 .5 \cdots n 2^{\frac{n-1}{2}} \frac{n-1}{t^{2}}
$$

Similarly, at $u=0$, the non-zero term of $\dot{Z}(u)$ is:

$$
c_{\frac{p-1}{2}}^{\prime \prime} t^{\frac{p-1}{2}}=1.3 .5 \cdots p 2^{\frac{p-1}{2}} t^{\frac{p-1}{2}}
$$

Using (15), it follows that at $u=0$, the non-zero term of $\dddot{Y}(u)$ is:

$$
6 c_{\frac{n-3}{2}}^{\prime} t^{\frac{n-3}{2}}=6(1.3 .5 \cdots(n-4)) \frac{(2 t)^{\frac{n-3}{2}} n!}{6(n-3)!}=(1.3 .5 \cdots n)(n-1) 2^{\frac{n-3}{2} \frac{n-3}{t^{2}}}
$$

Similarly, at $u=0$, the non-zero term of $\dddot{Z}(u)$ is:

$$
6 c_{\frac{p-3}{2}} t^{\frac{p-3}{2}}=(1.3 .5 \cdots p)(p-1) 2^{\frac{p-3}{2}} t^{\frac{p-3}{2}}
$$

Therefore

$$
\begin{aligned}
\dddot{Y}(u) \dot{Z}(u)-\dddot{Z}(u) \dot{Y}(u)= & (1.3 .5 \cdots n)(n-1) 2^{\frac{n-3}{2} \frac{n-3}{t^{2}}}(1.3 .5 \cdots p) 2^{\frac{p-1}{2}} t^{\frac{p-1}{2}} \\
& -(1.3 .5 \cdots n) 2^{\frac{n-1}{2}} \frac{n-1}{t^{2}}(1.3 .5 \cdots p)(p-1) 2^{\frac{p-3}{2}} \frac{p-3}{2}
\end{aligned}
$$

$$
=(2 t)^{\frac{p+n-4}{2}}(1.3 .5 \cdots n)(1.3 .5 \cdots p)(n-p)
$$

and it follows that $\dddot{Y}(u) \dot{Z}(u)-\dddot{Z}(u) \dot{Y}(u)<0$ since $n<p$. Therefore $\tau(u)$ is negative at $u=0$ on $\Gamma_{\sigma+6}$. As $u$ grows the terms in $\dot{X}(u), \ddot{X}(u), \dddot{X}(u), \dot{Y}(u), \ddot{Y}(u)$, $\dddot{Y}(u), \dot{Z}(u), \ddot{Z}(u)$ and $\dddot{Z}(u)$ with the largest power of $u$ (which are also the only terms without $\delta$ ) become dominant and $\tau(u)$ is again given by (16). Since $p+n+m-6$, the power of $u$ in the numerator, in now even, $\tau(u)$ becomes positive as $u$ grows in absolute value. Therefore there exist two new torsion zero-crossings in a neighborhood of $u_{0}$ on $\Gamma_{\sigma+\delta}$.

This completes the proof of theorem 4.

Proof of theorem 5: It will be shown that this theorem holds for an arbitrary parametrization of $\Gamma_{\boldsymbol{\sigma}}$. Therefore it must also be true of arc length parametrization or close approximations.

Let $\Gamma=(x(u), y(u), z(u))$ be a space curve and let $x(u), y(u)$ and $z(u)$ be polynomial functions of $u$. Let $\Gamma_{\sigma}=(X(u, \sigma), Y(u, \sigma), Z(u, \sigma))$ be an evolved version of $\Gamma$ with a point of zero curvature at $u_{0}$. Assume without loss of generality that $u_{0}=0$ and that at $u_{0} \Gamma_{\sigma}$ goes through the origin of the coordinate system. It follows that $\Gamma_{\sigma}$ can be approximated in a neighborhood of $u_{0}$ by:

$$
\begin{equation*}
\Gamma_{\sigma}=\left(u^{m}, u^{n}, u^{p}\right) \tag{17}
\end{equation*}
$$

where $u^{m}, u^{n}$ and $u^{p}$ are the lowest degree terms in $X(u, \sigma), Y(u, \sigma)$ and $Z(u, \sigma)$ respectively. Assume without loss of generality that $p>n>m$.

Since $m, n$ and $p$ can be odd or even, point $u_{0}$ must be analyzed in each of eight possible cases. The analysis in the proof of theorem 3 showed that when $m$ is even, a cusp point exists on $\Gamma_{\sigma}$ at $u_{0}$. We will therefore look at the remaining four cases in which $m$ is odd:

Case 1. $m$ is odd and $n$ and $p$ are even.
Torsion on $\Gamma_{\sigma}$ is given by equation (16). Since $p+n+m-6$ is odd, torsion is positive for positive $u$ and negative for negative $u$ in a neighborhood of $u_{0}$. We now investigate torsion on $\Gamma_{\sigma+\delta}$ where $\delta$ is a small, positive number. Expressions for $X(u, i 5 m a), Y(u, \sigma)$ and $Z(u, \sigma)$ can be obtained using equation (15). Note that all powers of $u$ in $X(u, \sigma)$ are odd and all powers of $u$ in $Y(u, \sigma)$ and $Z(u, \sigma)$ are even. It follows that all powers of $u$ in $\dot{X}(u), \dddot{X}(u), \ddot{Y}(u)$ and $\ddot{Z}(u)$ are even and all powers of $u$ in $\ddot{X}(u), \dot{Y}(u), \dddot{Y}(u), \dot{Z}(u)$ and $\dddot{Z}(u)$ are odd. Note also that those terms in which all powers of $u$ are odd, are equal to zero at $u_{0}$. Therefore torsion is zero at $u_{0}$ on $\Gamma_{\sigma+\delta}$. As $u$ grows, $u^{m}, u^{n}$ and $u^{p}$, that is the terms in $X(u, \sigma), Y(u, \sigma)$ and $Z(u, \sigma)$ with the largest powers of $u$, become dominant and
torsion is again given by equation (16). It follows that torsion is positive for positive $u$ and negative for negative $u$ on $\Gamma_{\sigma+\delta}$ in a neighborhood of $u_{0}$. Hence no new torsion zero-crossings have been created.

Case 2. $m$ is odd, $n$ is even and $p$ is odd.
Torsion on $\Gamma_{\sigma}$ is again given by (16). Since $p+n+m-6$ is even, torsion is positive for positive and negative $u$ on $\Gamma_{\sigma}$. We now investigate torsion on $\Gamma_{\sigma+\delta}$. Note that all powers of $u$ in $X(u, \sigma)$ are odd, all powers of $u$ in $Y(u, \sigma)$ are even and all powers of $u$ in $Z(u, \sigma)$ are odd. It follows that all powers of $u$ in $\dot{X}(u), \dddot{X}(u), \ddot{Y}(u)$, $\dot{Z}(u)$ and $\dddot{Z}(u)$ are even and all powers of $u$ in $\ddot{X}(u), \dot{Y}(u), \dddot{Y}(u)$ and $\ddot{Z}(u)$ are odd. Note also that those terms in which all powers of $u$ are odd, are equal to zero at $u_{0}$. It follows that torsion on $\Gamma_{\sigma+\delta}$ at $u_{0}$ is given by:

$$
\tau(u)=\frac{\dddot{Z} \dot{X} \ddot{Y}-\dddot{X} \dot{Z} \ddot{Y}}{(\dot{Z} \ddot{Y})^{2}+(\dot{X} \ddot{Y})^{2}}=\frac{\ddot{Y}(\ddot{Z} \dot{X}-\ddot{X} \dot{Z})}{(\dot{Z} \ddot{Y})^{2}+(\dot{X} \ddot{Y})^{2}}
$$

Since the denominator of the expression above is positive and $\ddot{Y}$ is positive, the sign of $\tau(u)$ is the same as the sign of the expression: $\dddot{Z} \dot{X}-\dddot{X} \dot{Z}$. At $u_{0}$, using (15) it can be shown that:

$$
\dddot{Z} \dot{X}-\dddot{X} \dot{Z}=(2 t)^{\frac{p+m-4}{2}}(1.3 .5 \cdots p)(1.3 .5 \cdots m)(p-m)
$$

which is positive at $u_{0}$. As $u$ grows larger, torsion is again given by (16) in a neighborhood of $u_{0}$ and is therefore positive in that neighborhood. Again no new torsion zero-crossings have been created.

Case 3. $m$ and $n$ are odd and $p$ is even.
Torsion is again given by (16) on $\Gamma_{\sigma}$. Since $p+n+m-6$ is even, torsion is positive for positive and negative $u$ on $\Gamma_{\sigma}$. We now investigate torsion on $\Gamma_{\sigma+\delta}$. Note that all powers of $u$ in $X(u, \sigma)$ and $Y(u, \sigma)$ are odd and all powers of $u$ in $Z(u, \sigma)$ are even. Hence all powers of $u$ in $\dot{X}(u), \ddot{X}(u), \dot{Y}(u), \dddot{Y}(u)$ and $\ddot{Z}(u)$ are even and all powers of $u$ in $\ddot{X}(u), \ddot{Y}(u), \dot{Z}(u)$ and $\dddot{Z}(u)$ are odd. Note also that those terms in which all powers of $u$ are odd, are equal to zero at $u_{0}$. Therefore torsion on $\Gamma_{\sigma+\delta}$ at $u_{0}$ is given by:

$$
\tau(u)=\frac{\dddot{X} \dot{Y} \ddot{Z}-\dddot{Y} \dot{X} \ddot{Z}}{(\dot{Y} \ddot{Z})^{2}+(\dot{X} \ddot{Z})^{2}}=\frac{\ddot{Z}(\ddot{X} \dot{Y}-\dddot{Y} \dot{X})}{(\dot{Y} \ddot{Z})^{2}+(\dot{X} \ddot{Z})^{2}}
$$

Since the denominator of the expression above is positive and $\ddot{Z}$ is positive, the sign of $\tau(u)$ is the same as the sign of the expression: $\ddot{X} \dot{Y}-\dddot{Y} \dot{X}$. At $u_{0}$, using (15) it can be shown that:

$$
\dddot{X} \dot{Y}-\dddot{Y} \dot{X}=(2 t)^{\frac{n+m-4}{2}}(1.3 .5 \cdots n)(1.3 .5 \cdots m)(m-n)
$$

which is negative since $n>m$. Therefore torsion is negative at $u_{0}$ on $M 6 A_{\sigma+\delta}$. As $u$ grows larger, torsion is again given by (16) in a neighborhood of $u_{0}$ and is therefore positive for positive and negative $u$. It follows that there are two new torsion zero-crossings in a neighborhood of $u_{0}$ on $\Gamma_{\sigma+\delta}$.

Case 4. $m, n$ and $p$ are odd.
Torsion on $\Gamma_{\sigma}$ is again given by equation (16). Since $p+n+m-6$ is odd, torsion is positive for positive $u$ and negative for negative $u$ on $\Gamma_{\sigma}$. We now investigate torsion on $\Gamma_{\sigma+\delta}$. Note that all powers of $u$ in $X(u, \sigma), Y(u, \sigma)$ and $Z(u, \sigma)$ are odd. Hence all powers of $u$ in $\dot{X}(u), \dot{Y}(u), \dot{Z}(u), \dddot{X}(u), \dddot{Y}(u)$ and $\dddot{Z}(u)$ are even and all powers of $u$ in $\ddot{X}(u), \ddot{Y}(u)$ and $\ddot{Z}(u)$ are odd. Note also that those terms in which all powers of $u$ are odd, are equal to zero at $u_{0}$. It follows that torsion is unbounded at $u_{0}$ on $\Gamma_{\sigma+\delta}$. As $u$ grows larger, torsion is again given by (16) in a neighborhood of $u_{0}$ and is therefore positive for positive $u$ and negative for negative $u$. Hence there are no new torsion zero-crossings in a neighborhood of $u_{0}$ on $\Gamma_{\sigma+\delta}$.


Figure 1. The Frenet trihedron for a space curve


Figure 2. A space curve depicting a fork


Figure 3. The fork during evolution


Figure 4. The torsion scale space image of the fork


Figure 5. A space curve depicting a bottle opener


Figure 6. The bottle opener during evolution


Figure 7. The torsion scale space image of the bottle opener


Figure 8. A space curve depicting an armchair


Figure 9. The armchair during evolution


Figure 10. The torsion scale space image of the armchair


Figure 11. The curvature scale space image of the armchair


Figure 12. The renormalized torsion scale space image of the armchair


Figure 13. The armchair with random noise

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Figure 14. The torsion scale space image of armchair with noise


Figure 15. The armchair with severe random noise


Figure 16. The torsion scale space image of armchair with severe noise


Figure 17. The resampled torsion scale space image of the armchair


Figure 18. The resampled torsion scale space image of armchair with severe noise

