# The Asymptotic Optimality of Spider-Web Networks 

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We determine the limiting behavior of the linking probability for large spider-web networks. The result confirms a conjecture made by Ikeno in 1959. We also show that no balanced crossbar network, in which the same components are interconnected according to a different pattern, can have an asymptotically larger linking probability.

## I. Introduction

We seek to analyze the performance of some switching networks under probabilistic traffic assumptions. The probabilistic model that we use is that of Lee [L] and Le Gall [L1, L2]. According to this model, every link (or wire) within the network is independently idle with some probability $q$ (called the "vacancy probability") and busy with the complementary probability $p=1-q$ (called the "occupancy probability"). For the networks that we study, it is natural to assume that these probabilities are the same for all links. We then ask for the probability $Q$ (called the "linking probability") that there is an idle path (that is, a path containing only idle links) from a given idle input to a given idle output, or equivalently, for the complementary probability $P$ that there is a busy cut (that is, a cut containing only busy links) between that input and that output. For the networks that we consider, these probabilities turn out to be independent of the particular input and output considered.

In 1959, Ikeno [I] used this probabilistic model to analyze a family of networks known as "series-parallel networks". For a sequence of such networks that is optimal (in the sense that the ratio of the number of switches to the amount of traffic carried is minimized), he proved that the linking probability tends to a limit and determined the value of that limit.

Ikeno also analyzed a family of networks with a randomized interconnection pattern, and showed that for an analogous sequence the linking probability tends to a larger limit. A randomized interconnection pattern has many disadvantages which prevent its practical application, so he sought an explicit interconnection pattern that would also achieve this larger limit. He defined a family of networks known as "spider-web networks" and conjectured that it had linking probability tending to the same limit as the randomized networks, but was unable to prove this.

In 1968, Takagi [ T ] showed that in a certain class of interconnection patterns (which includes series-parallel and spider-web patterns, but not randomized ones), the spider-web pattern is optimal, in the sense that it yields as large a linking probability as any other in the class. (A special case of this result had already been proved by Le Gall [L1].) If an analogous result could be proved for a broader class of networks, one that included randomized as well as spider-web interconnection patterns, Ikeno's conjecture would follow. This hope was dashed in 1980, however, when Chung and Hwang [CH] gave an example showing that the spider-web pattern is not optimal among all interconnection patterns.

In this paper we prove Ikeno's conjecture. Indeed, we show that spider-web networks are asymptotically optimal: other networks may have larger linking probability, but the margin of superiority tends to zero as the size of the networks increases. To do this we
first obtain an upper bound to the linking probability that applies to all interconnection patterns. The main ingredients of this proof are (1) an analysis of the survival probability for a certain branching process, and (2) some correlation inequalities of the "FKG" type. We then obtain an asymptotically matching lower bound to the linking probability for spider-web networks. The ingredients of this proof are (1) a "large deviations" estimate for the lower tail of the branching process mentioned above, and (2) a "second moment" or "Chebyshev" type of argument.

A few words are in order about the term "spider-web". Neither Ikeno nor Takagi gave a name to the connection pattern we call "spider-web", though Le Gall used the term "enchevêtre". The term "spiderweb" first appears in Feiner and Kappel [FK], though without a precise definition. Pippenger $[\mathrm{P}]$ used it in the sense defined here, but Chung and Hwang [CH1] have used it in the broader sense of "non-series-parallel".

## II. Networks

We consider switching networks that are constructed by the interconnection of crossbars. A d-crossbar is a component comprising $d$ terminals called inlets, $d$ other terminals called outlets, and $d^{2}$ switches, one for establishing a path between each of the inlets and each of the outlets. We assume that all crossbars in a network are $d$-crossbars for a common value of $d$ called the base of the network. Typical values of $d$ range from 2 through 16; the value 5 has a certain optimality property described by Ikeno [I].

We consider networks that provide paths between $n$ terminals called inputs and $n$ other terminals called outputs. We assume for simplicity that $n=d^{k}$ for some natural number $k \geq 1$ called the scale of the network. We regard $k$ as tending to $\infty$ while $d$ remains fixed.

We consider networks in which the crossbars are arranged in $l$ classes called stages, each containing $d^{k-1}$ crossbars, where $l$ is a natural number in the range $k \leq l \leq 2 k$ called the depth of the network. The essence of our results appears in the case in which $k$ and $l$ tend to $\infty$ in such a way that their ratio tends to a limit, though we do not need to make so specific an assumption.

We consider networks in which the inputs are connected in a one-to-one fashion with the inlets of the crossbars in the 1-st stage by means of wires (also called, by abuse of language, inputs), the outlets of the crossbars in the $l$-th stage are connected in a one-to-one fashion with the outputs by means of wires (also called outputs) and, for each $j$ in the range $1 \leq j \leq l-1$, the outlets of the crossbars in the $j$-th stage are connected
in a one-to-one fashion with the inlets of the crossbars in the $(j+1)$-st stage by means of $d^{k}$ wires called links. Inputs, outputs and links are collectively called vertices, and the vertices are partitioned into $l+1$ classes called ranks. The inputs form the 0 -th rank, the outputs form the $l$-th rank, and the links connecting the $j$-th stage to the $(j+1)$-st stage form the $j$-th rank.

Each switch in a crossbar provides a path from a vertex in the $j$-th rank, for some $j$ in the range $0 \leq j \leq l-1$, to some vertex in the $(j+1)$-st rank, and thus may be regarded as an edge directed from the former to the latter. In this way a netwrok may be regarded as an acyclic directed graph possessing the following properties.
(II.1) Every vertex that is not an input has in-degree $d$.
(II.2) Every vertex that is not an output has out-degree $d$.
(II.3) The existence of edges $(v, w),\left(v^{\prime}, w\right)$ and $\left(v, w^{\prime}\right)$ implies the existence of the edge $\left(v^{\prime}, w^{\prime}\right)$.
(II.4) Every path from an input to an output has length $l$ (that is, contains $l$ edges).

Conversely, every acyclic directed graph with these four properties may be regarded as a network of the type considered here.

In a network with an input $v$ and an output $w$, the channel graph from $v$ to $w$ is the acyclic directed graph comprising those vertices and edges lying on paths from $v$ to $w$. A network is balanced if any two channel graphs are isomorphic. In a balanced network, any two channel graphs contain the same number of paths between the input and the output; this number is called the diversity of the network. Since there are $d^{k+l}$ paths from an input to an output, and there are $d^{2 k}$ channel graphs, it follows that the diversity is $d^{l-k}$.

All of the networks we ahll consider in this paper will be balanced. We shall often frame definitions and arguments in terms of an input $v$ and an output $w$, without saying explicitly that the result does not depend on the choice of $v$ or $w$.

## III. Probabilities

A state of an acyclic directed graph is an assignment of one two conditions, idle or busy, to each of its vertices.

We consider a balanced network $G$ with base $d$, scale $k$ and depth $l$. We consider a random state of $G$ in which the input $v$ and the output $w$ are idle, and every other vertex is independently idle with a common probability $q$ (called the vacancy probability; the
complementary probability $p=1-q$ is called the occupancy probability). Typical values of $q$ are about $1 / 2$; the solution in the interval $(0,1)$ of the equation $1-q=q \ln (d q)$ (which is $0.514 \ldots$ for $d=5$ ) has a certain optimality property described by Ikeno [I].

In a random state of $G$, a path is idle if every vertex on it is idle; otherwise it is busy. The input $v$ and output $w$ are linked if there is a free path from $v$ to $w$; otherwise they are blocked.

The probability $Q$ of the event " $v$ and $w$ are linked" is called the linking probability; the complementary probability $P=1-Q$ is called the blocking probability.

Following Lotze [Lo], we define the transparency $J$ to be the expected number of idle paths from $v$ to $w$. There are $d^{l-k}$ paths from $v$ to $w$, and each of them contains $l-1$ links, each of which is independently idle with probability $q$. Thus

$$
J=d^{l-k} q^{l-1}
$$

By Markov's inequality, the probability that there is an idle path from $v$ to $w$ is at most the expected number of such paths:

$$
Q \leq J
$$

Thus when $J \rightarrow 0$, so does $Q \rightarrow 0$. Our goal in this paper is to obtain bounds for $Q$ when $J \rightarrow \infty$. We shall see that $Q$ is asymptotically at most $(1-\xi)^{2}$ in this case, where $\xi$ is a number (the solution in the interval $(0,1)$ of the equation $\left.(p+q \xi)^{d}=\xi\right)$ that depends only on $d$ and $q$, and not on $k$ or $l$, or on the interconnection pattern used between stages. We shall also see that there is a particular interconnection pattern, the "spider-web" pattern, for which $Q$ is indeed asymptotic to $(1-\xi)^{2}$. This will establish the "asymptotic optimality" of the spider-web pattern.

## IV. Infinite Trees

An infinite $d$-tree is an infinite acyclic directed graph, with a distinguished vertex called the root, possessing the following properties:
(IV.1) Every vertex has out-degree $d$.
(IV.2) For every vertex $x$, there is a unique path from the root to $x$.

The length of the path from the root to $x$ is called the level of $x$.

We consider an infinite $d$-tree $G$. We consider a random state of $G$ in which the root is idle and every other vertex is independently idle with probability $q$. We say that a vertex $x$ is live if the path from the root to $x$ is idle; otherwise we say it is dead. We say that $G$ survives if there are infinitely many live vertices (and thus an infinite path of live vertices from the root); otherwise we say $G$ is extinct.

If $Z_{r}$ denotes the number of live vertices on level $r$, then the sequence $Z_{0}=$ $1, Z_{1}, \ldots, Z_{r}, \ldots$ forms a branching process (see Harris [H2] or Athreya and Ney [AN]) for which the generating function for the offspring distribution is

$$
f(x)=(p+q x)^{d}
$$

the generating function for the number of successes among $d$ trials, each of which independently succeeds with probability $q$. If $J \rightarrow \infty$ with $d$ and $q$ fixed, then we must have $d q>1$, since $(d q)^{t} \geq d^{t} q^{l-1}=J$. Thus we assume that $f^{\prime}(1)=d q>1$.

The generating function for $Z_{r}$ is $r$-th iterate $f^{(r)}(x)$ of $f(x)$, defined by $f^{(0)}(x)=x$ and $f^{(r+1)}(x)=f\left(f^{(r)}(x)\right)$. The probability that $Z_{r}=0$ is the constant term $f^{(r)}(0)$ in this generating function. The event of extinction is the union as $r \rightarrow \infty$ of the nondecreasing sequence of events " $Z_{r}=0$ ". Thus probability of extinction is the limit as $r \rightarrow \infty$ of the non-decreasing sequence of probabilities $f^{(r)}(0)$. It is well known that this limit is the unique solution $\xi$ in the interval $(0,1)$ of the equation $f(\xi)=\xi$. Indeed, this follows as a corollary of the following more precise result (see Athreya and Ney [AN], Chapter I, Section 11, Corollary 1).

Lemma IV.1: If $d q>1$, then there is a unique $\xi<1$ such that $f(\xi)=\xi$, and $f^{\prime}(\xi)<1$. Furthermore, for any fixed $\zeta$ in the range $0 \leq \zeta<1$ we have

$$
f^{(r)}(\zeta)=\xi+O\left(\eta^{r}\right)
$$

as $r \rightarrow \infty$, where $\eta=f^{\prime}(\xi)$.
Corollary IV.2: We have

$$
f^{(r)}(0)=\xi+O\left(\eta^{r}\right)
$$

as $r \rightarrow \infty$.

## V. Jumbles

A $d$-jumble of depth $r$ is an acyclic directed graph with a distinguished vertex called the source, a set of other distinguished vertices called targets, possessing the following properties:
(V.1) Every vertex that is not a target has out-degree $d$.
(V.2) Every path from the source to a target has length $r$.

We consider a $d$-jumble $G$ of depth $r$. We consider a random state of $G$ in which the source is idle and every other vertex is independently idle with probability $q$. We say that $G$ flourishes if there is an idle path from the source to a target; otherwise we say $G$ withers. Lemma $V$ : The probability $W$ that $G$ withers satisfies

$$
W \geq f^{(r)}(0)
$$

Proof: We proceed by induction on $r$. If $r=1$, there are at most $d$ targets, each of which is independently busy with probability $p=1-q$. If they are all busy, $G$ withers, so $W \geq p^{d}=f(0)=f^{(1)}(0)$.

Suppose then that $r \geq 2$. Let the $d$ edges directed out of the root be directed into the vertices $u_{1}, \ldots, u_{d}$ (not necessarily all distinct). For $1 \leq c \leq d$ let $G_{c}$ be the subgraph of $G$ induced by the vertices on paths from $u_{c}$ to a target. Clearly $G_{c}$ is a $d$-jumble of depth $r-1$, with $u_{c}$ as root. Thus, if $u_{c}$ is idle, then $G_{c}$ withers with probability at least $f^{(r-1)}(0)$. Let $E_{c}$ denote the event "either $u_{c}$ is busy, or $u_{c}$ is idle and $G_{c}$ withers". Then

$$
\operatorname{Pr}\left(E_{c}\right) \geq p+q f^{(r-1)}(0)
$$

Since the occupancies of the vertices (that is, the events of their being busy) are independent, and since the events $E_{1}, \ldots, E_{d}$ are increasing Boolean functions of the occupancies of the vertices, we have

$$
\operatorname{Pr}\left(E_{1}, \ldots, E_{d}\right) \geq \operatorname{Pr}\left(E_{1}\right) \cdots \operatorname{Pr}\left(E_{d}\right)
$$

by Harris's inequality ([H1], Appendix 1). If $E_{1}, \ldots, E_{d}$ all occur, then $G$ withers, so we have

$$
\begin{aligned}
W & \geq \operatorname{Pr}\left(E_{1}, \ldots, E_{d}\right) \\
& \geq \operatorname{Pr}\left(E_{1}\right) \cdots \operatorname{Pr}\left(E_{d}\right) \\
& \geq\left(p+q f^{(r-1)}(0)\right)^{d}=f\left(f^{(r-1)}(0)\right)=f^{(r)}(0)
\end{aligned}
$$

$\triangle$

## VI. The Upper Bound

We consider a balanced network $G$ with base $d$, scale $k$ and depth $l$. For simplicity we shall assume that $l$ is even and set $l=2 r$. Let $G(v, w)$ be the channel graph from the
input $v$ to the output $w$. Let $G_{\mathrm{I}}$ and $G_{\text {II }}$ denote the subgraphs of $G(v, w)$ induced by the union of ranks 0 through $r-1$ and the union of ranks $r+1$ through $l$, respectively. The following properties are immediate consequences of the definitions.
(VI.1) $G_{\mathrm{I}}$ is a $d$-jumble of depth $r-1$.
(VI.2) $G_{\text {II }}$ is a $d$-jumble of depth $r-1$ with the directions of its edges reversed.

We consider a random state of $G(v, w)$ in which $v$ and $w$ are idle and each link is independently idle with probability $q$. This gives rise to random states of the jumbles $G_{\mathrm{I}}$ and $G_{\mathrm{II}}$ in which the sources are idle and each other vertex is independently idle with probability $q$. If $v$ and $w$ are linked, then $G_{\mathrm{I}}$ and $G_{\mathrm{II}}$ flourish. Since $G_{\mathrm{I}}$ and $G_{\mathrm{II}}$ are disjoint, the events " $G_{\mathrm{I}}$ flourishes" and $G_{\text {II }}$ flourishes" are independent. Thus, using Corollary IV. 2 and Lemma V, we have

$$
\begin{aligned}
Q & \leq \operatorname{Pr}\left(G_{\mathrm{I}} \text { and } G_{\mathrm{II}} \text { flourish }\right) \\
& =\operatorname{Pr}\left(G_{\mathrm{I}} \text { flourishes }\right) \operatorname{Pr}\left(G_{\mathrm{II}} \text { flourishes }\right) \\
& \leq\left(1-f^{(r-1)}(0)\right)^{2} \\
& =(1-\xi)^{2}+O\left(\eta^{r-1}\right)
\end{aligned}
$$

Since $J=d^{l-k} q^{l-1} \leq(d q)^{r}$, we have $O\left(\eta^{r-1}\right)=O\left(J^{-\alpha}\right)$, where $\alpha=-\log \eta / \log (d q)>0$. Thus we have proved the following.

Theorem VI: Let $d$ and $q$ be fixed. For any sequence of balanced crossbar networks for which $J \rightarrow \infty$, we have

$$
Q \leq(1-\xi)^{2}+O\left(J^{-\alpha}\right)
$$

## VII. Spider-Web Networks

A network of base $d$, scale $k$ and depth $l$ has $l+1$ ranks of vertices, each containing $d^{k}$ vertices. Let us agree to label the vertices in each rank with the strings of length $k$ over the alphabet $\{0, \ldots, d-1\}$. The positions in each string will be referred to as the 1 -st (leftmost) through the $k$-th (rightmost).

The $j$-th stage in such a network, for $1 \leq j \leq l$, contains $d^{k+1}$ edges directed out of the vertices in the $(j-1)$-st rank and directed into the vertices in the $j$-th rank. We shall say that the $j$-th stage is of type $i$, for $1 \leq i \leq k$, if there is an edge between vertex $x$ in rank $j-1$ and vertex $y$ in rank $j$ if and only if the labels of these vertices differ at most in
the $i$-th position. It is clear that the edges in such a stage for $d^{k-1}$ disjoint crossbars. A network in which each stage is of type $i$ for some $1 \leq i \leq k$ is called a rhyming network.

A rhyme scheme is a string of length $l$ over the alphabet $\{1, \ldots, k\}$. The positions in a rhyme scheme will be referred to as the 1 -st (leftmost) through the $l$-th (rightmost). For each rhyme scheme, we may construct a rhyming network in which, for each $1 \leq j \leq l$, the type of the $j$-th stage is as specified by the $j$-th position of the rhyme scheme.

The spider-web network with base $d$, scale $k$ and depth $l$ is the network constructed according to the rhyme scheme $12 \cdots k 12 \cdots(l-k)$. The essential feature of this rhyme scheme is that the symbols in any $k$ consecutive positions are distinct. Two alternate characterizations of spider-web networks are given in the Appendix.

Spider-web networks were introduced by Ikeno [I] in 1959. Takagi [T] proved in 1968 that, for any occupancy probability, spider-web networks have the smallest blocking probability among rhyming networks with the same base, scale and depth. (The special case of $k=2$ and $l=4$, where the spider-web scheme 1212 has smaller blocking probability than the "series-parallel" scheme 1221, was given by Le Gall [L1].) Chung and Hwang [CH] gave in 1980 an example of an interconnection pattern for a crossbar network with base 2 , scale 5 and depth 8 that has lower blocking probability (for any occupancy probability) than the spider-web network with the same parameters (it is not, of course, a rhyming network).

Consider a spider-web network $G$, an input $v$ and an output $w$ in $G$, and two distinct paths $\pi$ and $\pi^{\prime}$ from $v$ to $w$ in $G$. Let $\varrho$ denote the longest common initial segment of $\pi$ and $\pi^{\prime}$. Since $\pi$ and $\pi^{\prime}$ both originate at $v, \varrho$ contains at least $v$. Since $\pi$ and $\pi^{\prime}$ are distinct, $\varrho$ does not continue as far as $w$. Suppose the last vertex of $\varrho$ is in the $(a-1)$-st rank. Then $\pi$ and $\pi^{\prime}$ must differ in the $a$-th rank, and the labels of their vertices in the $a$-th rank must differ in the $a$-th position (if $1 \leq a \leq k$ ) or in the ( $a-k$ )-th position (if $k+1 \leq a \leq l$ ). The labels of the vertices of $\pi$ and $\pi^{\prime}$ in a given rank cannot again coincide until $a$ appears again in the rhyme scheme. Since $\pi$ and $\pi^{\prime}$ eventually coincide at $w$, it follows that $1 \leq a \leq k$ (so that $a$ does appear again in the rhyme scheme) and that $\pi$ and $\pi^{\prime}$ differ at all ranks from the $a$-th through the $(a+k-1)$-st (the rank preceding the second stage of type $a$ ).

A similar argument working backward from the common output $w$ leads to the following conclusion: if the first vertex in the longest common final segment of $\pi$ and $\pi^{\prime}$ is in the $b$-th rank, then $l-k+1 \leq b \leq l$, and $\pi$ and $\pi^{\prime}$ differ at all ranks from the $(b-k)$-th through the $(b-1)$-st. Thus we conclude that a spider-web network with scale $k$ possesses the following property.
(VII.1) Any two paths from a common input to a common output consist of a common initial segment, followed by disjoint intermediate segments of extending over at least $k$ ranks, followed by a common final segment.

Let $G(v, w)$ denote the channel graph between input $v$ and output $w$ in the spider-web network $G$. The channel graph $G\left(v^{\prime}, w^{\prime}\right)$ between any other input $v^{\prime}$ and any other output $w^{\prime}$ is isomorphic to $G(v, w)$ : the map that changes the last $k-j$ positions in the label of each vertex in rank $j$ (for $0 \leq j \leq k-1$ ) from the last $k-j$ positions of $v$ to the last $k-j$ positions of $v^{\prime}$, and changes the first $j+k-l$ positions in the label of each vertex in rank $j$ (for $l-k+1 \leq j \leq l$ ) from the first $j+k-l$ positions of $w$ to the first $j+k-l$ positions of $w^{\prime}$, is an isomorphism. Thus a spider-web network is balanced.

## VIII. Finite Trees

A finite $d$-tree of depth $r$ is the subgraph of an infinite $d$-tree induced by the vertices with level at most $r$. The vertices with level exactly $r$ are called leaves.

We consider a finite $d$-tree $G$ with depth $r$. We consider a random state of $G$ in which the root is idle and every other vertex is independently idle with probability $q$. We shall be concernd with the distribution of the number of live leaves. This number is the random variable $Z_{r}$ in the branching process $Z_{0}=1, Z_{1}, \ldots, Z_{r}, \ldots$ defined in Section IV. By the chain rule, we have $\operatorname{Ex}\left(Z_{r}\right)=f^{(r)}(1)=(d q)^{r}$. We see from Corollary IV. 2 that

$$
\operatorname{Pr}\left(Z_{r}=0\right)=\xi+O\left(\eta^{r}\right)
$$

Our goal in this section is to show that the probability that $Z_{r}$ is much smaller than its expected value is not much larger than the probability that it is zero.

Let $u$ be an integer in the range $1 \leq u \leq r$. Let $U=(d q)^{u}$ and suppose that $U \geq 2$. We have

$$
\operatorname{Pr}\left(Z_{r} \leq U\right)=\operatorname{Pr}\left(Z_{r}=0\right)+\sum_{1 \leq z \leq U} \operatorname{Pr}\left(Z_{r}=z\right) .
$$

The first term is $f^{(r)}(0)$. If $x \leq 1$ we have

$$
\begin{aligned}
\sum_{1 \leq z \leq U} \operatorname{Pr}\left(Z_{r}=z\right) & \leq x^{-U} \sum_{1 \leq z \leq U} \operatorname{Pr}\left(Z_{r}=z\right) x^{z} \\
& \leq x^{-U}\left(f^{(r)}(x)-f^{(r)}(0)\right)
\end{aligned}
$$

Since $U \geq 2$ implies $(1-1 / U)^{-U} \leq 4$, taking $x=1-1 / U$ yields

$$
\operatorname{Pr}\left(Z_{r} \leq U\right) \leq f^{(r)}(0)+4\left(f^{(r)}(1-1 / U)-f^{(r)}(0)\right)
$$

By Equation (7) of Ikeno [I], we have

$$
f^{(u)}(1-1 / U)=f^{(u)}\left(1-(d q)^{-u}\right) \leq(1+d q) / 2 d q
$$

Since $f^{(r)}(x)=f^{(r-u)}\left(f^{(u)}(x)\right)$ and $f^{(r-u)}$ is a non-decreasing function, this yields

$$
\operatorname{Pr}\left(Z_{r} \leq U\right) \leq f^{(r)}(0)+4\left(f^{(r-u)}((1+d q) / 2 d q)-f^{(r)}(0)\right)
$$

By Lemma IV. 1 with $\zeta=(1+d q) / 2 d q$ we have

$$
f^{(r-u)}((1+d q) / 2 d q)=\xi+O\left(\eta^{r-u}\right)
$$

for $r-u \rightarrow \infty$ with $d$ and $q$ fixed. Using this with Corollary IV. 2 yields

$$
\operatorname{Pr}\left(Z_{r} \leq U\right) \leq \xi+O\left(\eta^{r-u}\right)
$$

Taking $u=\lceil\log H / \log (d q)\rceil$, we have proved the following. Lemma VIII: Suppose $H \rightarrow \infty$ and $R / H \rightarrow \infty$, where $R=(d q)^{r}$. Then we have

$$
\operatorname{Pr}\left(Z_{r} \leq H\right) \leq \xi+O\left((H / R)^{\alpha}\right)
$$

where $\alpha=-\log \eta / \log (d q)>0$.

## IX. Tangles

An acyclic directed graph is an $m$-tangle with depth $k$ and intersection $r$ if it possesses the following properties.
(IX.1) There are $m$ distinguished vertices called sources and $m$ distinguished vertices called targets.
(IX.2) From each source to each target, there is a unique path of length $k$.
(IX.3) Any two paths from a common source to distinct targets, or from distinct sources to a common target, have in common a segment of length at most $r$.
(IX.4) Any two paths from distinct sources to distinct targets are disjoint.

We consider a tangle in a random state for which every source and target is idle and every other vertex is independently idle with probability $q$. We say that the tangle is traversable if there is a path from a source to a target in which every vertex is idle.
Lemma IX: The probability that a tangle is not traversable is at most $1 / m^{2} q^{k-1}+2 / m q^{r-1}$.

Proof: Let $T$ denote the number of paths from a source to a target for which every vertex is idle. Clearly (IX.1) and (IX.2) imply that

$$
\operatorname{Ex}(T)=m^{2} q^{k-1}
$$

We shall show that

$$
\begin{equation*}
\operatorname{Var}(T) \leq m^{2} q^{k-1}+2 m^{3} q^{2 k-r-1} \tag{IX.5}
\end{equation*}
$$

The Lemma will then follow from Chebyshev's inequality: $\operatorname{Pr}(T=0) \leq \operatorname{Var}(T) / \operatorname{Ex}(T)^{2}$.
To prove (IX.5), we observe that

$$
\operatorname{Var}(T)=\sum_{\pi, \pi^{\prime}}\left(\operatorname{Pr}\left(A_{\pi}, A_{\pi^{\prime}}\right)-\operatorname{Pr}\left(A_{\pi}\right) \operatorname{Pr}\left(A_{\pi^{\prime}}\right)\right)
$$

where $\pi$ and $\pi^{\prime}$ each run over the set of paths from a source to a target, and $A_{\pi}$ denotes the event "all vertices on $\pi$ are idle". We classify the terms in the sum according to whether $\pi$ and $\pi^{\prime}$ have a common source, and according to whether they have a common target.

If $\pi$ and $\pi^{\prime}$ have both a common source and a common target, then by (IX.2) they are the equal, and their contribution to the sum is $\operatorname{Pr}\left(A_{\pi}\right)-\operatorname{Pr}\left(A_{\pi}\right)^{2} \leq \operatorname{Pr}\left(A_{\pi}\right)=q^{k-1}$. There are $m^{2}$ such pairs, so their total contribution to the sum is at most $m^{2} q^{k-1}$.

If $\pi$ and $\pi^{\prime}$ have a common source but distinct targets, then by (IX.3) they have in common a segment of length at most $r$, and their contribution to the sum is $\operatorname{Pr}\left(A_{\pi}, A_{\pi^{\prime}}\right)-$ $\operatorname{Pr}\left(A_{\pi}\right) \operatorname{Pr}\left(A_{\pi^{\prime}}\right) \leq \operatorname{Pr}\left(A_{\pi}, A_{\pi^{\prime}}\right) \leq q^{2 k-r-1}$. There are $m^{2}(m-1) \leq m^{3}$ such pairs, so their total contribution to the sum is at most $m^{3} q^{2 k-r-1}$. The $m^{2}(m-1)$ pairs with distinct sources but a common target similarly contribute at most $m^{3} q^{2 k-r-1}$ to the sum.

Finally, if $\pi$ and $\pi^{\prime}$ have both distinct sources and distinct targets, then by (IX.4) they are disjoint, so $A_{\pi}$ and $A_{\pi^{\prime}}$ are independent. Thus these $m^{2}(m-1)^{2}$ pairs make no contribution to the sum. Combining these estimates yields (IX.5) and completes the proof of the Lemma. $\triangle$

## X. The Lower Bound

We consider a spider-web network $G$ with base $d$, scale $k$ and depth $l$. We consider a random state of $G$ in which the input $v$ and the output $w$ are idle, and all other vertices are independently idle with probability $q$. We seek a lower bound to the probability $Q$ that $v$ and $w$ are linked. For simplicity we shall assume henceforth that $l-k$ is even and set $l-k=2 r$.

Let $G_{\mathrm{I}}, G_{\mathrm{II}}$ and $G_{\text {III }}$ denote the subgraphs of $G(v, w)$ induced by the union of ranks 0 through $r$, the union of ranks $l-r$ through $l$, and the union of ranks $r$ through $l-r$, respectively. The following properties of these subgraphs are immediate consequences of property (VII.1).
(X.1) $G_{\mathrm{I}}$ is a complete $d$-ary tree of depth $r$, with $v$ as root and $d^{r}$ links in rank $r$ as leaves.
(X.2) $G_{\text {II }}$ is a complete $d$-ary tree of depth $r$, with $w$ as root and $d^{r}$ links in rank $l-r$ as leaves, with the directions of its edges reversed.
(X.3) $G_{\mathrm{III}}$ is a $d^{r}$-tangle of depth $k$ and intersection $r$, with the leaves of $G_{\mathrm{I}}$ as sources and the leaves of $G_{\mathrm{II}}$ as targets.

Let $X$ denote the set of links $x$ in rank $r$ for which all vertices of the path from $v$ to $x$ are idle. Let $H$ be a parameter to be chosen later, and let $A$ denote the event " $X$ contains at least $H$ links". By Lemma VIII, we have

$$
\operatorname{Pr}(A) \geq 1-\xi+O\left((H / R)^{\alpha}\right)
$$

where $R=(d q)^{r}$ and $\alpha=-\log f^{\prime}(\xi) / \log (d q)>0$.
Let $Y$ denote the set of links $y$ in rank $l-r$ for which all vertices of the path from $y$ to $w$ are idle. Let $B$ denote the event " $Y$ contains at least $H$ links". By Lemma VIII, we have

$$
\operatorname{Pr}(B) \geq 1-\xi+O\left((H / R)^{\alpha}\right)
$$

Furthermore, $A$ and $B$ are independent, since they since they are determined by the conditions of disjoint sets of links, so

$$
\begin{equation*}
\operatorname{Pr}(A, B) \geq(1-\xi)^{2}+O\left((H / R)^{\alpha}\right) \tag{X.1}
\end{equation*}
$$

Let $C$ denote the event "there is a path from a link in $X$ to a link in $Y$ for which all links are idle". If $C$ occurs, then $v$ and $w$ are linked, so we have

$$
\begin{equation*}
Q \geq \operatorname{Pr}(C) \geq \operatorname{Pr}(C \mid A, B) \operatorname{Pr}(A, B) \tag{X.2}
\end{equation*}
$$

Thus we seek a lower bound for $\operatorname{Pr}(B \mid A, C)$.
Since $G_{\text {III }}$ is disjoint from $G_{\mathrm{I}}$ and $G_{\mathrm{II}}$ except for links in ranks $r$ and $l-r, C$ depends on $A$ and $B$ only through $X$ and $Y$. Thus we have

$$
\operatorname{Pr}(C \mid A, B) \geq \max _{X^{\prime}, Y^{\prime}} \operatorname{Pr}\left(C \mid X=X^{\prime}, Y=Y^{\prime}\right)
$$

where $X^{\prime}$ and $Y^{\prime}$ range over all sets of links in ranks $r$ and $l-r$, respectively, containing at least $H$ links. It will suffice, therefore, to obtain a lower bound for $\operatorname{Pr}\left(B \mid X=X^{\prime}, Y=Y^{\prime}\right)$ that holds when $X^{\prime}$ and $Y^{\prime}$ each contain at least $H$ links. Since the event $C$ is a nondecreasing Boolean function of the vacancies (that is, the conditions of being idle) of the vertices, it will suffice to obtain a lower bound for $\operatorname{Pr}\left(B \mid X=X^{\prime}, Y=Y^{\prime}\right)$ that holds when $X^{\prime}$ and $Y^{\prime}$ each contain exactly $h=\lfloor H\rfloor$ links.

Let $G_{\text {III }}^{\prime}$ be the subgraph of $G_{\text {III }}$ induced by the set of links lying on paths from sources in $X^{\prime}$ to targets in $Y^{\prime}$. It follows from property (X.3) that $G_{\mathrm{III}}^{\prime}$ is an $h$-tangle of depth $k$ and intersection $r$. If this tangle is traversable, then $C$ occurs. Thus, by Lemma IX,

$$
\operatorname{Pr}(C \mid A, B) \geq 1-1 / h^{2} q^{k-1}-2 / h q^{r-1}
$$

Combining this inequality with (X.1) and (X.2) yields

$$
Q \geq(1-\xi)^{2}+O\left((H / R)^{\alpha}\right)+O\left(1 / H^{2} q^{k}\right)+O\left(1 / H q^{r}\right)
$$

It remains to choose a suitable value for $H$.
We first observe that $H q^{r}=\left(H^{2} q^{l-k}\right)^{1 / 2} \geq\left(H^{2} q^{k}\right)^{1 / 2}$, so the last term $O\left(1 / H q^{r}\right)$ may be replaced by $O\left(1 /\left(H^{2} q^{r}\right)^{1 / 2}\right)$. Of course, the term $O\left(1 / H^{2} q^{k}\right)$ can also be replaced by $O\left(1 /\left(H^{2} q^{r}\right)^{1 / 2}\right)$, whenever they tend to 0 . Thus we have

$$
Q \geq(1-\xi)^{2}+O\left((H / R)^{\alpha}\right)+O\left(1 /\left(H^{2} q^{r}\right)^{1 / 2}\right)
$$

We next observe that $q^{k}=q J / R^{2}$, so we have Thus we have

$$
Q \geq(1-\xi)^{2}+O\left((H / R)^{\alpha}\right)+O\left(R / H J^{1 / 2}\right)
$$

Taking $H=R / J^{1 / 2(\alpha+1)}$, we have proved the following.
Theorem X: Let $d$ and $q$ be fixed. For any sequence of spider-web networks for which $J \rightarrow \infty$, we have

$$
Q \geq(1-\xi)^{2}+O\left(J^{-\beta}\right)
$$

where $\beta=\alpha / 2(\alpha+1)>0$.

## XI. Conclusion

We have determined the limiting value of the linking probability for spider-web networks when the transparency tends to 0 or $\infty$ with the base and the vacancy probability
fixed. We have also shown that no other crossbar networks with the same scale and depth can have significantly larger linking probability.

One limitation of our lower bound is that the depth can be at most twice the scale. (For $l>2 k$, we define the spider-web network to be the rhyming network whose rhyme scheme is the initial segment of length $l$ of the sequence $12 \cdots k 12 \cdots k \cdots$.) While the optimal parameter values described by Ikeno [I] satisfy this constraint, the result of Takagi [ T ] does not require it, and it is natural to conjecture that the lower bound of this paper also holds without it. (The upper bound clearly does not require it.) It does not appear that this conjecture can be established by arguments similar to those used here.

Another natural direction for extension of our results is to more realistic probability measures than that of Lee [L] and Le Gall [L1, L2]. For the measure introduced by Pippenger $[\mathrm{P}]$ and Koverninskǐ̆ $[\mathrm{K}]$, it appears that this extension can be accomplished by a similar but much more elaborate version of the argument used here. (The behavior to be expected has been predicted on the basis of simulations by Neiman and Vvedenskaya [NV].) We plan to present this in a later paper.

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## Appendix

The description of spider-web networks given in Section VII is based on the representation of Takagi [ T ], which is most convenient for the purposes of this paper. In this appendix we shall give two alternate descriptions of spider-web networks, based on the representation used by Ikeno [I].

If we let $\mathrm{Cr}_{j}$ denote a crossbar stage of type $j$, then the spider-web network may be represented as $\mathrm{Cr}_{1} \cdots \mathrm{Cr}_{k} \mathrm{Cr}_{1} \cdots \mathrm{Cr}_{l-k}$.

Let $\operatorname{Sh}$ ("shuffle") denote the permutation of the set $\{0, \ldots, d-1\}^{k}$ of strings of length $k$ over the alphabet $\{0, \ldots, d-1\}$ that takes the string $c_{1} c_{2} \cdots c_{k}$ to the string $c_{2} \cdots c_{k} c_{1}$ (cyclically shifting the string one position to the left). The permutation Sh generates a cyclic group of order $k$. If we let $\mathrm{Sh}^{j}$ denote the $j$-th power of Sh , then we have $\mathrm{Cr}_{j}=\mathrm{Sh}^{j-1} \mathrm{Cr}_{1} \mathrm{Sh}^{-(j-1)}$. Since $\mathrm{Sh}^{k}=\mathrm{Id}$ (the identity permutation), we have $\mathrm{Cr}_{1} \cdots \mathrm{Cr}_{k}=\mathrm{Cr}_{1} \cdots \mathrm{Cr}_{k} \mathrm{Sh}^{k}=\left(\mathrm{Cr}_{1} \mathrm{Sh}\right)^{k}$. Similarly, $\mathrm{Cr}_{1} \cdots \mathrm{Cr}_{l-k} \mathrm{Sh}^{l-k}=\left(\mathrm{Cr}_{1} \mathrm{Sh}\right)^{l-k}$. If we abbreviate $\mathrm{Cr}_{1}$ to Cr , we have $\mathrm{Cr}_{1} \cdots \mathrm{Cr}_{k} \mathrm{Cr}_{1} \cdots \mathrm{Cr}_{l-k} \mathrm{Sh}^{l-k}=(\mathrm{CrSh})^{l}$; that is, the spider-web network is the same, to within a permutation of the outputs, as the crossbar
network in which each consecutive pair of stages "Cr" is interconnected according to the pattern "Sh".

Now let Tw ("twiddle") denote the permutation of the set $\{0, \ldots, d-1\}^{k}$ of strings of length $k$ over the alphabet $\{0, \ldots, d-1\}$ that takes the string $c_{2} c_{2} \cdots c_{k}$ to the string $c_{k} \cdots c_{2} c_{1}$ (reversing the order of the positions). The permutations Sh and Tw generate a dihedral group of order $2 k$. We have $\mathrm{Sh} \mathrm{Tw}=\mathrm{Tw} \mathrm{Sh}^{-1}$ and (since this permutation leaves the 1-st position of strings unchanged) we have $\mathrm{CrSh} \mathrm{Tw}=\mathrm{Tw} \mathrm{Sh}^{-1} \mathrm{Cr}$. It follows that $(\mathrm{CrSh})\left(\mathrm{TwSh}^{-1}\right)=\left(\mathrm{TwSh}^{-1}\right)\left(\mathrm{CrSh}^{-1}\right)$, and thus that $(\mathrm{CrSh})^{r}\left(\mathrm{Tw} \mathrm{Sh}^{-1}\right)=$ $\left(\mathrm{Tw} \mathrm{Sh}{ }^{-1}\right)\left(\mathrm{CrSh}^{-1}\right)^{r}$. Thus, if $l$ is even and $l=2 r$, we have $(\mathrm{CrSh})^{l}\left(\mathrm{Tw} \mathrm{Sh}^{-1}\right)=$ $(\mathrm{CrSh})^{r}\left(\mathrm{Tw} \mathrm{Sh}{ }^{-1}\right)(\mathrm{CrSh})^{r}=(\mathrm{CrSh})^{r} \mathrm{Tw}\left(\mathrm{Sh}^{-1} \mathrm{Cr}\right)^{r} \mathrm{Sh}^{-1}$; that is, the spiderweb network is the same, to within a permutation of the outputs, as the network obtained by interconnecting the network $(\mathrm{CrSh})^{r}$ with its "mirror image" $\left(\mathrm{Sh}^{-1} \mathrm{Cr}\right)^{r}$ according to the pattern "Tw".

