HOW MANY REAL NUMBERS ARE THERE?

by

Paul C Gilmore

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6536 Agricultural Road, Vancouver, B.C., V6T 1W5
E-mail address: <gilmore@ean.ubc.ca>

ERRATA

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Notation: m,n

designates the m'th page, and the n'th line on that page. 'n' may be a positive or negative integer indicating the line count respectively from the top or the bottom of the page.

4,-9, -8	replace 'real numbers' and 'real number' by 'sequences of 0's and 1's' and 'sequence of 0's and 1's'
4,-6	add to the line 'That there are more real numbers than there are natural can be concluded from the lemma'
5	replace 'reals' and 'real number' by 'sequences', or 'sequence', or 'sequences of 0's and 1's', as appropriate
6,-10	'Tsikinis' should be 'Tsiknis'
10,-7	'$\in$ SEQ[n,$\mathfrak{A}$,$\mathfrak{V}$]' should be 'is a sequence with'
19,13	interchange '0' and '1' following '$\wedge$'
25,15	replace 'principal' with 'principle'
25,-10	replace 'principle' with 'principal'
26,11	replace 'principal' with 'principle'
26,12	insert 'consistent' between 'define' and 'third'
28,-9	'i' should be 'n' and 'b' should be 'u'
28,-5	'real numbers' should be 'sequences'
29,4	'real number' should be 'sequence'
32,6,7	drop these two lines
32,-5	'A derivation' should begin a new paragraph
34, -4,-5	drop these two lines
44,11	replace '1' by '2' throughout the formula
46,-7	'P' should be 'PR'

ABSTRACT

The question posed in the title of this paper is raised by a reexamination of Cantor's diagonal argument. Cantor used the argument in its most general form to prove that no mapping of the natural numbers into the reals could have all the reals as its range. It has subsequently been used in a more specific form to prove, for example, that the computable reals cannot be enumerated by a Turing machine. The distinction between these two forms of the argument can be expressed within a formal logic as a distinction between using the argument with a parameter F, denoting an arbitrary map from the natural numbers to the reals, and with a defined term **F**, representing a particular defined map.

The setting for the reexamination is a natural deduction based set theory, NaDSet, presented within a Gentzen sequent calculus. The form of NaDSet employed generalizes an earlier form of the theory by replacing its first and second order quantifiers by a single quantifier. The elementary and logical syntax of NaDSet, as well as its semantics, is described in the paper.

Within NaDSet, Cantor's diagonal argument for **F** can be formalized as a derived rule of deduction with two premisses $\langle i,C[\mathbf{F}]\rangle{:}\mathbf{F} \rightarrow \langle i,C[\mathbf{F}]\rangle{:}\mathbf{F}$ and $\langle i,r\rangle{:}\mathbf{F} \rightarrow \langle i,r\rangle{:}\mathbf{F}$. Here C[**F**] is a term denoting the real number constructed by the diagonal argument. The two sequents express that certain ordered pairs necessarily are or are not members of **F**. These two premisses are derivable when **F** is, for example, a mapping of the natural numbers into the computable reals provided by a Turing machine. The two premisses are not derivable, however, when **F** is a parameter. It is for this reason that Cantor's general diagonal argument is said to be unsound within NaDSet.

Very general forms of argument can, however, be expressed within NaDSet. The potential for NaDSet to provide logical foundations for category theory is demonstrated by proving a theorem suggested by Feferman: The set of structures $\langle A, \circ, =_A \rangle$ for which $\circ$ is a binary, commutative, and associative operation on A with identity $=_A$, is itself such a structure under cartesian product and isomorphism.

To provide a basis for a discussion of the question posed in the title, a formalization of Gödel-Bernays set theory is provided within NaDSet.

"So the logicians entered the picture in their usual style, as spoilers." [Moschovakis80]

1 INTRODUCTION

The question in the title of this paper is raised by a reexamination of Cantor's diagonal argument. Cantor used the argument in its most general form to prove that no mapping of the natural numbers into the reals could have all the reals as its range. It has subsequently been used in a more specific form to prove, for example, that the computable reals cannot be enumerated by a Turing machine. The distinction between these two forms of the argument can be expressed within a formal logic as a distinction between using the argument with a parameter F, denoting an arbitrary map from the natural numbers to the reals, and with a defined term **F**, representing a particular defined map.

This paper could be subtitled "Logicism Revisited". The Frege-Russell thesis of logicism is simply that number is a logical concept and that the theorems concerning natural and real numbers are tautologies of logic. This is in contrast to the axiomatic view of Hilbert that the theorems are logical consequences of assumed nonlogical axioms.

But what is logic? For the axiomatic view, it is sufficient to identify logic with first order logic. From the axioms of a set theory, such as those of Zermelo-Fraenkel [Shoenfield67] or Gödel-Bernays [Gödel40], a theory of natural and real numbers may be developed. The axioms of the set theory may be justified as expressing truths about a pre-existing or "constructed" universe, as is done in [Schoenfield67], for example.

To defend the thesis of logicism, however, first order logic is inadequate. Logic must admit abstraction as a primitive concept, along with logical connectives and quantifiers. Like connectives and quantifiers, abstraction is a sentence constructor. However, abstraction constructs sentences by constructing abstraction terms from sentences. The effect of this is the inclusion, in the domain of discourse, of the sets obtained by abstraction from properties of objects in the domain of discourse. But it was this inclusion that Russell exploited to show that Frege's logic was inconsistent. Thus a revisitation of logicism requires a reexamination of the paradoxes of set theory.

The semantics for first order logic described in [Tarski56] has become standard: An interpretation of the atomic sentences assign truth values to them; nonatomic sentences are assigned truth values by semantic rules for the logical connectives and quantifiers. A proof that every sentence receives one and only one truth value can be completed by finite induction on the definition of sentence. Since the truth values of nonatomic sentences depend upon the truth values of simpler sentences, the semantics can be said to be reductionist.

Treating abstraction as a sentence constructor, like the logical connectives and quantifiers, means that semantic rules for abstraction must be introduced that assign truth values to nonatomic sentences dependent upon the truth values of simpler sentences. In the papers [Gilmore71,80,86] a resolution of the paradoxes has been proposed that is based upon such a reductionist semantics for logic; it was first proposed for an earlier logic [Gilmore67]. The semantics for these logics differs from the standard semantics for first order logic in only two respects: First, transfinite induction must be used instead of finite induction; and second, although each sentence receives at most one truth value, some sentences receive none. A similar proposal for a resolution of the liar paradox has been proposed in [Kripke75].

That this semantics provides a successful resolution of the paradoxes is demonstrated in [Gilmore71,80,86]. A set theory with a logical syntax presented in the calculus of sequents [Gentzen34,35], was shown to be consistent. Further, the theory in its second order form is not trivial since it is as strong as second order arithmetic. However, one weakness of the logic was noted in [Gilmore86]; namely, the diagonal argument, used by Cantor to prove that there are more real numbers than there are natural numbers, cannot be formalized. This paper undertakes a reexamination of this argument.

The setting for the reexamination is a natural deduction based set theory, NaDSet, presented within a Gentzen sequent calculus. The form of NaDSet employed generalizes the earlier form of the theory, described in [Gilmore71,80,86], by replacing its first and second order quantifiers by a single bounded quantifier in which the bound expresses the "type" of the quantified variable. The bound is expressed by the abstraction terms of the logic; these generalize conventional abstraction terms. The elementary syntax for NaDSet is described in section 2, and the logical syntax in section 3. After some preliminary definitions and derived rules provided in section 4, the semantics for the logic is described in section 5. Unlike the earlier form of the set theory, a consistency proof for NaDSet is not yet known, although the consistency of the logic is shown in section 5 to follow from a proof that the cut rule is redundant.

In section 5 an answer to the question "why second order logic?" is also provided. Briefly the answer is that the first order theory of [Gilmore86] is inadequate for expressing the semantics of recursive definitions, while third and higher order theories using a reductionist semantics similar to that of NaDSet, are inconsistent.

In section 6, the main theme of the paper is addressed.

Consider the diagonal argument of Cantor. A real number in the closed interval [0,1] can be represented as a sequence $b_1, b_2, \dots, b_j, \dots$, where each b_j is 0 or 1. The real number represented by the sequence is

$$b_1 * 1/2 + b_2 * (1/2)^2 + \dots + b_j * (1/2)^j + \dots$$

Such a sequence can also represent a subset of a denumerable set, with $b_j = 0$ if the j'th element of the set is not a member of the subset, and $b_j = 1$ otherwise.

Let F be an enumeration of sequences of 0's and 1's; that is, for each i, where i=1, 2, ... , F[i] is a sequence ${}^i b_j$ of 0's and 1's. Define C[F] to be the sequent c_j, where

$c_j = 0$, if ${}^j b_j = 1$, and
$c_j = 1$, if ${}^j b_j = 0$.

Cantor's diagonal argument uses the "diagonal" sequence C[F] to prove:

Cantor's Lemma: For each enumeration F of real numbers, there is a real number not enumerated by F.

To prove the lemma, C[F] is shown to be a real number, and then shown to be not enumerated by F; the latter follows since for each j, $c_j \neq {}^j b_j$.

A simple instance of the lemma is helpful in clarifying the argument. Define FB[i] to be ${}^i bb_j$, where ${}^i bb_j$ is 1 if $i \leq j$, and is 0 if $i > j$. Thus

${}^1 bb$ is 1 0 ... 0 ...
${}^2 bb$ is 1 1 0 ... 0 ...
...

and C[FB] is 0 0 ... 0 ... , confirming the lemma in this case.

For any particular defined enumeration FB of sequences, the diagonal argument demonstrates that C[FB] is not in the range of FB. But the diagonal argument is used in a more general form to prove the lemma: For any enumeration F of the reals, C[F] is shown to be a real number not enumerated by F.

In a formal logic such as NaDSet, the distinction between these two forms of Cantor's diagonal argument can be expressed as a distinction between carrying out the argument for any defined term **F**, such as FB, and for a parameter F of the logic representing an arbirary map. In section 6, the more restricted use of the argument is formalized as a derived rule of deduction, called Cantor's rule.

Let **F** be a term representing the given mapping of the natural numbers into the reals. Cantor's rule has two premisses ‹ i,C[**F**]›:**F** → ‹ i,C[**F**]›:**F** and ‹ i,r›:**F** → ‹ i,r›:**F**; they express that certain ordered pairs necessarily are, or are not, members of **F**. These two premisses are derivable when **F** is FB or, for example, a mapping of the natural numbers into the computable reals provided by a Turing machine. The two premisses are not derivable, however, when **F** is a parameter needed to express the general form of Cantor's diagonal argument used to prove Cantor's lemma.

Although Cantor's general diagonal argument cannot be formalized within NaDSet, in section 7 it is demonstrated that one example of a common argument of modern algebra can be.

Let a B structure be a set on which a binary, commutative, and associative operation is defined. The set of B structures, with cartesian product and isomorphism, is a B structure. The proof of this theorem in section 7 makes particular use of the generalized abstraction and quantification of NaDSet. The theorem was suggested in [Feferman84] as an example of why type-free logics are needed to provide foundations for category theory. That the theorem can be proved within NaDSet, suggests that it has the potential for providing such foundations. This is briefly discussed in section 7.4.

To provide additional background for answering the question posed in the title of the paper, a formalization of Gödel-Bernays set theory is provided within NaDSet in section 8. As with the B structures of section 7, the formalization is accomplished by defining the set of structures that satisfy the axioms of the set theory. Unlike an axiomatization of the set theory within first order logic, however, such a formalization has no existential implications for NaDSet; that is, the formalization does not postulate the existence of any set that is not already postulated by NaDSet.

In section 8.3 the concern of classical set theories with the question "what sets exist ?", is contrasted with the concern of natural deduction based set theories, such as NaDSet, with the question "what arguments are sound?". The question posed in the title is returned to in section 8.4.

It may be argued that a fundamental weakness of NaDSet is exposed by its inability to formalize Cantor's diagonal argument in its most general form. But that remains to be seen. That NaDSet does not require the explicit assumption of an axiom of infinity, as do the Zermelo-Fraenkel or Gödel-Bernays set theories, for example, demonstrates the thesis of logicism at least for the natural numbers. That the logic should not at the same time support the full theory of transfinite ordinals and cardinals may say more about the interpretation of that theory than it does about the adequacy of the logic. The paradox of the greatest cardinal requires that the diagonal argument fail at some point. That the diagonal argument fails for NaDSet precisely at the point of introduction of the notion of nondenumerability, it may be argued, is a point in its favour.

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2 ELEMENTARY SYNTAX

The essentials of the elementary syntax of NaDSet are provided in definition 2.1 below. There are differences from the definitions of [Gilmore86]. These differences will be discussed as the presentation of the new logic proceeds.

One minor syntactic change in NaDSet is the replacement of 'ϵ' by ':'. The main reason for the change is the preemption of the 'ϵ' notation by set theories in which an axiom of extensionality is admitted; since NaDSet is an intensional logic, as demonstrated in section 4.5, a change of notation is appropriate. The choice of ':' as a replacement for 'ϵ' has been suggested by the conforming usage in category theory and in some programming languages.

To simplify the description of NaDSet, only a single logical connective '$\downarrow$' and only a universal quantifier are taken to be primitive. Other logical connectives and the existential quantifier will be freely used, however, with their usual definitions assumed given. The connective is joint denial, so that (**Sta**$\downarrow$**Stb**) has the same truth table as ($\sim$**Sta**$\wedge\sim$**Stb**).

Definition 2.1 allows for only one type of quantifiable variable in NaDSet, unlike the first order and second order variables appearing in the earlier version. However, first and second order parameters are introduced to play the role of unquantified quantifiable variables, and first and second order constants are introduced as well. The manner in which they are used in the logical syntax, and interpreted in the semantics, ensures that NaDSet is a second order, not first order logic.

In addition to quantifiable variables, the logic NaDSet admits abstraction variables as well. Abstraction variables are bound in abstraction terms. This is also a change from [Gilmore86], where no distinction was made between first order variables and abstraction variables. The change provides a more readable syntax.

The particular syntax used for quantifiable and abstraction variables, and for first and second order parameters and constants is unimportant. In the examples offered as illustrations in this section, strings of lower case Latin letters and numerals beginning with a letter 'u', 'v', 'w', 'x', 'y' or 'z' will be used as quantifiable variables, and other such strings will be used as first order parameters and constants. Second order parameters and constants will be strings of upper and lower case Latin letters, beginning with an upper case letter that is not an initial letter of a variable. Strings of lower case Greek and Latin letters and numerals, beginning with a Greek letter, will be used as abstraction variables. However, in the interests of readability, a less restrictive syntax will be used in later sections.

Definition 2.1. **Elementary Syntax**

1.1. A variable is a term. The single occurrence of the variable in the term is a free occurrence in the term.
1.2. A first or second order parameter or constant is a term of the same order. No variable has a free occurrence in the term.
1.3. A term in which no second order parameter occurs is a first order term.

2.1. If **ta** and **tb** are any terms, then **ta**:**tb** is a formula. A free occurrence of a variable in **ta** or in **tb**, is a free occurrence of the variable in the formula. If **ta** is a first order term, and **tb** is a second order constant or parameter, then **ta**:**tb** is an atomic formula.
2.2. If **F1a** and **F1b** are formulas then (**F1a**↓**F1b**) is a formula. A free occurrence of a variable in **F1a** or in **F1b** is a free occurrence in (**F1a**↓**F1b**).
2.3. Let **F1** be a formula, **vr** a quantifiable variable, and T a term in which **vr** has no free occurrence. Then [∀**vr**:T]**F1** is a formula. A free occurrence of a variable in T or in **F1**, other than **vr** in **F1**, is a free occurrence of the variable in [∀**vr**:T]**F1**; no occurrence of **vr** in [∀**vr**:T]**F1** is a free occurrence in the formula.

3. Let **ta** be any term in which there is at least one free occurrence of an abstraction variable, and in which there is no occurrence of a parameter, or no free occurrence of a quantifiable variable. Let **F1** be any formula. Then {**ta**|**F1**} is an abstraction term and a second order term. A free occurrence of a quantifiable variable in **F1** is a free occurrence in {**ta**|**F1**}. A free occurrence of an abstraction variable in **F1**, which does not also have a free occurrence in **ta**, is a free occurrence in {**ta**|**F1**}. An abstraction variable with a free occurrence in **ta** has no free occurrence in {**ta**|**F1**}; such an abstraction variable is called an abstracted variable of {**ta**|**F1**}. **ta** is called the abstracted term and **F1** the abstracted formula of the abstraction term {**ta**|**F1**}.

4. A term in which no variable has a free occurrence is a constant term. A formula in which no variable has a free occurrence is a sentence.

By clause 3, no abstracted variable of {**ta**|**F1**} has a free occurrence in that term, although it does have a free occurrence in **ta**. For example, let '[x,B,y/α,β,δ]' be a substitution operator that replaces free occurrences of 'α',

'β' and 'δ', respectively, by 'x', 'B' and 'y'. Then

$$[x,B,y/\alpha,\beta,\delta](\langle\alpha,\beta\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta \wedge \langle\beta,\delta\rangle{:}B\})$$

is the formula

$$\langle x,B\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta \wedge \langle\beta,y\rangle{:}B\}$$

since the occurrence of 'δ' in

$$\langle\alpha,\beta\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta \wedge \langle\beta,\delta\rangle{:}B\}$$

is free, while only the first occurrence of 'α' and 'β' is free. Here $\langle$**ta**,**tb**$\rangle$ is the ordered pair of **ta** and **tb** as it is defined in 4.1 below. The quantifiable variables 'x' and 'y' are the only variables with free occurrences in $\langle x,B\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta \wedge \langle\beta,y\rangle{:}B\}$.

Clause 1.3 applies to second order constants and to the abstraction terms defined in clause 3; for example, the second order constant 'B' is a first order, as well as a second order term, as is also the abstraction term $\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta \wedge \langle\beta,y\rangle{:}B\}$. The justification for clause 1.3 is provided in section 5.

The abstraction terms admitted into NaDSet are a generalization of the lamba abstractions of the lambda calculus of [Church41]. Indeed, as was described in the introduction to [Gilmore86], NaDSet can be seen as a solution to the problem posed by Church in section 21 of [Church41]: In NaDSet, unrestricted abstraction is combined with universal quantification. That problem provided the title for [Gilmore80]. The expansion rule of the lambda calculus that permits the conversion

$$[N/x]M \Rightarrow ((\lambda xM)N)$$

is generalized to the abstraction rule described in the next section that allows the conversion

$$[\underline{\mathbf{tm}}/\underline{\alpha}]\mathbf{F1} \Rightarrow [\underline{\mathbf{tm}}/\underline{\alpha}]\mathbf{ta}{:}\{\mathbf{ta} \mid \mathbf{F1}\}$$

Here, $\underline{\alpha}$ is the sequence of distinct abstraction variables with a free occurrence in **ta**, and $\underline{\mathbf{tm}}$ is a sequence of terms with the same length as $\underline{\alpha}$. An instance of the generalized abstraction rule is:

$$a{:}B \wedge B{:}P \Rightarrow \langle a,B\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta \wedge \beta{:}P\},$$

where 'a' is a first order constant, 'B' a second order constant that by 1.3 is a first order term, and 'P' a second order parameter. The conversion is correct because $a{:}B \wedge B{:}P$ is the formula $[a,B/\alpha,\beta](\alpha{:}\beta \wedge \beta{:}P)$, and $[a,B/\alpha,\beta]\langle\alpha,\beta\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta \wedge \beta{:}P\}$ is the formula $\langle a,B\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta \wedge \beta{:}P\}$.

In clause 2.1, **ta** need not be a first order term in order for **ta**:**tb** to be a formula. This is an important way in which NaDSet differs from its earlier version, since it has the effect of generalizing the abstraction rules. For example, under the previous assumptions on 'a', 'B', and 'P', 'a:P' and '{α| a:α}:P' are atomic sentences, but 'P:P', 'P:B', or 'P:{α| a:α}' are not. However, 'P:{α| a:α}' is a formula; note that

a:P ⇒ P:{α| a:α}

so that 'P:{α| a:α}' can be reduced to the atomic sentence 'a:P'. Such an application of generalized abstraction was not allowed in the original NaDSet.

3 LOGICAL SYNTAX

Familiarity with the Gentzen sequent calculus as described in [Gentzen34], [Kleene52], or [Prawitz65] is presumed. This natural deduction calculus is chosen for the formalization of NaDSet because it is one of the least complicated to describe and justify. However, any natural deduction formalization of first order logic, such as those presented in [Beth55], [Prawitz65], or [Fitch52], can be simply extended to be a formalization of NaDSet.

Definition 3.1: The axioms and rules of deduction of NaDSet

1. The axioms are all sequents

   ASt → **ASt**

   for which **ASt** is an atomic sentence.

2. The rules of deduction for the introduction of the single logical connective take their expected forms:

$$\frac{\Delta, \mathbf{Sta} \rightarrow \Lambda \quad \Gamma, \mathbf{Stb} \rightarrow \Theta}{\Delta, \Gamma \rightarrow (\mathbf{Sta}\downarrow\mathbf{Stb}), \Lambda, \Theta} \qquad \frac{\Gamma \rightarrow \mathbf{Sta}, \mathbf{Stb}, \Theta}{\Gamma, (\mathbf{Sta}\downarrow\mathbf{Stb}) \rightarrow \Theta}$$

3. In each of the following rules for the introduction of the bounded quantifiers, **ta** is a single abstraction variable when T is a second order parameter or constant, and is the abstracted term of T when it is a constant abstraction term {**ta**|**Fa**}. In the first case, $\underline{\alpha}$ is **ta**; in the second case, $\underline{\alpha} \in$ SEQ[n,$\mathcal{AV}$] has elements that are the distinct abstracted variables of {**ta**|**Fa**}. **Fl** is a formula in which only the quantifiable variable **vr** has a free occurrence.

   In the first of the rules, $\underline{\mathbf{p}}$ is a sequence, with the same length as $\underline{\alpha}$, with elements that are distinct parameters not occurring in any sentence of the conclusion. In the second of the rules, $\underline{\mathbf{tm}}$ is any sequence, of the same length as $\underline{\alpha}$, with elements that are constant terms.

$$\frac{\Gamma, [\underline{p}/\underline{\alpha}]ta{:}T \rightarrow \Theta, [[\underline{p}/\underline{\alpha}]ta/vr]F1}{\Gamma \rightarrow \Theta, [\forall vr{:}T]F1}$$

$$\frac{\Gamma \rightarrow \Theta, [\underline{tm}/\underline{\alpha}]ta{:}T \qquad \Delta, [[\underline{tm}/\underline{\alpha}]ta/vr]F1 \rightarrow \Lambda}{\Gamma, \Delta, [\forall vr{:}T]F1 \rightarrow \Theta, \Lambda}$$

4. Let {ta|Fa} be any abstraction term, and let $\underline{\alpha}$ be the sequence, without repetitions, of all abstraction variables with a free occurrence in ta. Let $\underline{tm}$ be any sequence of constant terms of the same length as $\underline{\alpha}$. The two rules of deduction for abstraction are the following:

$$\frac{\Gamma \rightarrow [\underline{tm}/\underline{\alpha}]Fa, \Theta}{\Gamma \rightarrow [\underline{tm}/\underline{\alpha}]ta{:}\{ta|Fa\}, \Theta} \qquad \frac{\Gamma, [\underline{tm}/\underline{\alpha}]Fa \rightarrow \Theta}{\Gamma, [\underline{tm}/\underline{\alpha}]ta{:}\{ta|Fa\} \rightarrow \Theta}$$

5. The thinning rules and the cut rule are unchanged:

$$\frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow St, \Theta} \qquad \frac{\Gamma \rightarrow \Theta}{\Gamma, St \rightarrow \Theta}$$

$$\frac{\Gamma \rightarrow St, \Theta \qquad \Delta, St \rightarrow \Lambda}{\Delta, \Gamma \rightarrow \Lambda, \Theta}$$

   where St is any sentence.

The contraction and interchange structural rules are not presented; by regarding the sequences Γ and Θ as sets of sentences, it is possible to drop these rules.

The rules 3.1.2, 3.1.3, and 3.1.4, will be referred to, in the order in which they have been presented, as →↓, ↓→, →∀, ∀→, →{}, and {}→. The derived rules for the defined existential quantifier [∃vr:T] and the logical connectives ~, ∨, ∧, and ⊃ will be refered to by a similar notation. Cut and thinning will be referred to by name.

A rule permitting the changing of bound variables cannot be derived in the logic as presented. Should such a rule be desired, it can be added and a formula or term interpreted as a representative of the equivalent class of its bound variable variants. Alternatively, the method used in [Gilmore80,86] may be employed. However, because all applications of the rules of deduction are to sequents of sentences, no application of a rule in a proof requires the changing of any bound variable.

4 SOME DEFINITIONS AND DERIVED RULES

In this section some definitions and derived rules needed in the remainder of the paper will be provided; they will, at the same time, illustrate applications of the rules of deduction.

4.1. Ordered Pairs and Identity

4.1.1 $\langle$**ta**,**tb**$\rangle$ **for** $\{\alpha|($**ta**$:C\downarrow$**tb**$:C)\}$,
where 'C' is a given second order constant. Any second order constant may be used since the constant need not satisfy any assumptions apart from the assumptions made for every second order constant.

This unusually simple definition of ordered pair is satisfactory in NaDSet because NaDSet is an intensional logic. Two rules of deduction will be derived for ordered pair in 4.1.7 and 4.1.8 below. As will be evident from the derivations of the rules, the particular form of the term chosen as ordered pair is unimportant: The rules can be derived as long as $\langle$**ta**,**tb**$\rangle$ is a first order term when **ta** and **tb** are first order terms. However, the statement of the rules must wait upon the definition and development of identity.

Triples and other tuples can be similarly defined directly, or can be defined by nesting pairs.

4.1.2. $=$ **for** $\{\langle\alpha,\beta\rangle|\,[\forall z:\{\gamma|\alpha:\gamma\}]\,\beta:z\}$
Since $\langle\alpha,\beta\rangle$, where α and β are abstraction variables, satisfies the conditions on **ta** in clause 3 of definition 2.1, $\{\langle\alpha,\beta\rangle|\,[\forall z:\{\gamma|\alpha:\gamma\}]\,\beta:z\}$ is an abstraction term. The definition provides an abbreviation for it.

Members of = are ordered pairs $\langle$**ta**,**tb**$\rangle$. The conventional infix notation **ta**=**tb** will be used instead of expressing membership by the formula $\langle$**ta**,**tb**$\rangle$:=.

4.1.3 ($\rightarrow=$)

For any constant first order term **ta**,

$\rightarrow \mathbf{ta}=\mathbf{ta}$

is a derivable sequent.

Here follows a derivation:

$\mathbf{ta}{:}P \rightarrow \mathbf{ta}{:}P$	axiom
$P{:}\{\gamma \mid \mathbf{ta}{:}\gamma\} \rightarrow \mathbf{ta}{:}P$	$\{\}\rightarrow$
$\rightarrow [\forall z{:}\{\gamma \mid \mathbf{ta}{:}\gamma\}]\ \mathbf{ta}{:}z$	$\rightarrow\forall$
$\rightarrow \langle \mathbf{ta},\mathbf{ta}\rangle{:}\{\langle\alpha,\beta\rangle \mid [\forall z{:}\{\gamma \mid \alpha{:}\gamma\}]\ \beta{:}z\}$	$\rightarrow\{\}$

The application of $\rightarrow\{\}$ in this derivation illustrates an important feature of the abstraction rules. Note that the ordered pair term $\langle \mathbf{ta},\mathbf{ta}\rangle$ does not appear in the premiss of the rule. That particular term is introduced in the conclusion of the rule as a member of $\{\langle\alpha,\beta\rangle \mid [\forall z{:}\{\gamma \mid \alpha{:}\gamma\}]\ \beta{:}z\}$, because $\langle\alpha,\beta\rangle$ is the abstracted term of that abstraction term.

The sequents of 4.1.3 can be regarded as instances of a derived rule of deduction which introduces formulas $\mathbf{ta}=\mathbf{ta}$ into the succedent of a sequence; uses of 4.1.3 will therefore be justified as an application of $\rightarrow=$. The following derived rule then is the dual rule $=\rightarrow$, that introduces identity into the antecent:

4.1.4. ($=\rightarrow$)

Let **F1** by any formula in which only the abstraction variable α has a free occurrence, and let **ta** and **tb** be any constant terms. Then the following is a derivable rule:

$$\frac{\Gamma \rightarrow \Theta, [\mathbf{ta}/\alpha]\mathbf{F1} \qquad [\mathbf{tb}/\alpha]\mathbf{F1}, \Delta \rightarrow \Lambda}{\Gamma, \Delta, \mathbf{ta}=\mathbf{tb} \rightarrow \Theta, \Lambda}$$

From the first premiss, $\Gamma \rightarrow \Theta, \mathbf{ta}{:}\{\alpha \mid \mathbf{F1}\}$ can be concluded by $\rightarrow\{\}$; from the second premiss $\mathbf{tb}{:}\{\alpha \mid \mathbf{F1}\}, \Delta \rightarrow \Lambda$ can be concluded by $\{\}\rightarrow$. The conclusion of the derived rule follows by $\forall\rightarrow$ from these two sequents.

4.1.5. Under the assumptions of 4.1.4, the following is a derivable rule:

$$\frac{[\mathbf{ta}/\alpha]\mathbf{F1} \rightarrow [\mathbf{ta}/\alpha]\mathbf{F1} \qquad [\mathbf{tb}/\alpha]\mathbf{F1} \rightarrow [\mathbf{tb}/\alpha]\mathbf{F1}}{\mathbf{ta}=\mathbf{tb},\ [\mathbf{ta}/\alpha]\mathbf{F1} \rightarrow [\mathbf{tb}/\alpha]\mathbf{F1}}$$

The conclusion of the rule follows by one application of =→.

4.1.6. (=axioms)
For constant first order terms **ta** and **tb**,

ta=**tb** → **ta**=**tb**

is a derivable sequent.

A derivation follows:

ta:P → **ta**:P **tb**:P → **tb**:P	axioms (P 2nd order parameter)
ta=**tb**, **ta**:P → **tb**:P	=→
ta=**tb**, P:{γ|**ta**:γ} → **tb**:P	{}→
ta=**tb** → **ta**=**tb**	→∀, →{}

With these results for =, it is now possible to return to the ordered pair definition 4.1.1 and demonstrate that it has the desired properties of ordered pair.

4.1.7. (→‹›)
For constant first order terms **ta1**, **tb1**, **ta2**, and **tb2**, the following is a derivable rule:

$$\frac{\Gamma \rightarrow \Theta,\ \mathbf{ta1}=\mathbf{tb1} \qquad \Delta \rightarrow \Lambda,\ \mathbf{ta2}=\mathbf{tb2}}{\Gamma,\ \Delta \rightarrow \Theta,\ \Lambda,\ \langle \mathbf{ta1},\mathbf{ta2}\rangle=\langle \mathbf{tb1},\mathbf{tb2}\rangle}$$

Since **ta1**, **tb1**, **ta2**, and **tb2** are first order terms, so are ‹**ta1**,**ta2**›, ‹**tb1**,**ta2**›, and ‹**tb1**,**tb2**›. Therefore from =axioms and 4.1.4, the following two sequents are derivable

ta1=**tb1**, ‹**ta1**,**ta2**›=‹**ta1**,**ta2**› → ‹**ta1**,**ta2**›=‹**tb1**,**ta2**›

ta2=**tb2**, ‹**ta1**,**ta2**›=‹**tb1**,**ta2**› → ‹**ta1**,**ta2**›=‹**tb1**,**tb2**›

The conclusion of the →‹› rule follows from its premisses and →= by four applications of cut.

4.1.8. $(\langle\rangle\rightarrow)$

For constant first order terms **ta1**, **tb1**, **ta2**, and **tb2**, the following are derivable rules:

$$\frac{\Gamma,\ \mathbf{ta1}=\mathbf{tb1} \rightarrow \Theta}{\Gamma,\ \langle\mathbf{ta1},\mathbf{ta2}\rangle=\langle\mathbf{tb1},\mathbf{tb2}\rangle \rightarrow \Theta}$$

$$\frac{\Gamma,\ \mathbf{ta2}=\mathbf{tb2} \rightarrow \Theta}{\Gamma,\ \langle\mathbf{ta1},\mathbf{ta2}\rangle=\langle\mathbf{tb1},\mathbf{tb2}\rangle \rightarrow \Theta}$$

Here is a derivation of the first rule:

$\rightarrow \mathbf{ta1}=\mathbf{ta1}$	$(\rightarrow=)$	$\Gamma,\ \mathbf{ta1}=\mathbf{tb1} \rightarrow \Theta$
$\rightarrow \langle\mathbf{ta1},\mathbf{ta2}\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha=\mathbf{ta1}\}$		$\Gamma,\ \langle\mathbf{tb1},\mathbf{tb2}\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha=\mathbf{ta1}\} \rightarrow \Theta$
$\Gamma,\ \langle\mathbf{ta1},\mathbf{ta2}\rangle=\langle\mathbf{tb1},\mathbf{tb2} \rightarrow \Theta$ $(=\rightarrow)$		

A similar derivation of the second rule can be provided.

4.2. Extensional Identity

4.2.1 $=_e$ **for** $\{\langle\alpha,\beta\rangle \mid [\forall u{:}\alpha]u{:}\beta \wedge [\forall u{:}\beta]u{:}\alpha\}$

Extensional identity provides a means for illustrating important aspects of bounded quantification within NaDSet.

4.2.2. Four Universal Sets

Consider the following defined 'universal' sets:

V1 **for** $\{\alpha \mid \alpha=\alpha\}$

V2 **for** $\{\gamma \mid \gamma=_e\gamma\}$

V22 **for** $\{\gamma \mid \{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}\gamma\}=_e\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}\gamma\}\}$

Cl **for** $\{\langle\alpha,\beta\rangle \mid \alpha{:}\beta\}$

For each of these terms **tm**, it is possible to derive the sequent

$\rightarrow [\forall u{:}\mathbf{tm}]u{:}\mathbf{tm}$

and therefore derive the sequent

$\rightarrow \mathbf{tm}=_e\mathbf{tm}$

But the derivations take on a very different character in each case. In the following derivations, p and q are first order parameters, and P is a second order parameter:

For V1:

$\rightarrow p=p$	$\rightarrow =$
$p=p \rightarrow p=p$	thinning
$p{:}V1 \rightarrow p{:}V1$	$\{\}\rightarrow, \rightarrow\{\}$
$\rightarrow [\forall u{:}V1]\, u{:}V1$	$\rightarrow\forall$

For V2:

$p{:}P \rightarrow p{:}P$	axiom
$\rightarrow [\forall u{:}P]u{:}P$	$\rightarrow\forall$
$\rightarrow [\forall u{:}P]u{:}P \wedge [\forall u{:}P]u{:}P$	$\rightarrow\wedge$
$\rightarrow \langle P,P\rangle{:}\{\langle\alpha,\beta\rangle \mid [\forall u{:}\alpha]u{:}\beta \wedge [\forall u{:}\beta]u{:}\alpha\}$	$\rightarrow\{\}$
$\rightarrow P=_e P$	defn
$\rightarrow P{:}V2$	$\rightarrow\{\}$
$P{:}V2 \rightarrow P{:}V2$	thinning
$\rightarrow [\forall u{:}V2]u{:}V2$	$\rightarrow\forall$

For V22:

$\langle p,q\rangle{:}P \rightarrow \langle p,q\rangle{:}P$	axiom
$\langle p,q\rangle{:}\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\} \rightarrow \langle p,q\rangle{:}\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}$	$\{\}\rightarrow,\rightarrow\{\}$
$\rightarrow [\forall u{:}\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}]u{:}\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}$	$\rightarrow\forall$
$\rightarrow [\forall u{:}\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}]u{:}\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\} \wedge [\forall u{:}\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}]u{:}\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}$	$\rightarrow\wedge$
$\rightarrow \langle\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\},\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}\rangle{:}\{\langle\alpha,\beta\rangle \mid [\forall u{:}\alpha]u{:}\beta \wedge [\forall u{:}\beta]u{:}\alpha\}$	$\rightarrow\{\}$
$\rightarrow \{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}=_e\{\langle\alpha,\beta\rangle \mid \langle\alpha,\beta\rangle{:}P\}$	defn
$\rightarrow P{:}V22$	$\rightarrow\{\}$
$P{:}V22 \rightarrow P{:}V22$	thinning
$\rightarrow [\forall u{:}V22]u{:}V22$	$\rightarrow\forall$

For C1:

$p{:}P \rightarrow p{:}P$ axiom

$\langle p,P\rangle{:}\{\langle\alpha,\beta\rangle|\ \alpha{:}\beta\} \rightarrow \langle p,P\rangle{:}\{\langle\alpha,\beta\rangle|\ \alpha{:}\beta\}$ $\{\}\rightarrow,\rightarrow\{\}$

$\rightarrow [\forall u{:}C1]\ u{:}C1$ $\rightarrow\forall$

4.2.3. A Class of Second Order Universal Sets

The term, $\{\langle\alpha,\beta\rangle|\langle\alpha,\beta\rangle{:}\gamma\}$, in the definition of V22, provides the bound on the quantified variable 'u'. It specifies its "type" to be $\langle\alpha,\beta\rangle$. Consider a more general example, where **ta** is assumed to satisfy the conditions of clause 3 of definition 2.1:

V2(**ta**) for $\{\gamma|\{\mathbf{ta}|\mathbf{ta}{:}\gamma\}=_e\{\mathbf{ta}|\mathbf{ta}{:}\gamma\}\}$

Here '(' and ')' are used instead of '[' and ']' because of the conditions placed on **ta**; only terms satisfying those conditions may replace it.

A derivation for the sequent $\rightarrow$ [∀u:V2(**ta**)]u:V2(**ta**), similar to the derivation for the sequent $\rightarrow$ [∀u:V22]u:V22, can be provided.

That V2(**ta**) is the "type" for abstraction terms {**ta**|**Fa**}, can be seen from the following derived rule:

$$\frac{[\underline{p}/\underline{\alpha}]\mathbf{Fa} \rightarrow [\underline{p}/\underline{\alpha}]\mathbf{Fa}}{\rightarrow \{\mathbf{ta}|\mathbf{Fa}\}{:}V2(\mathbf{ta})}$$

V2, V22, and V2(**ta**), are examples of domains for second order quantifiers.

4.3. Quantification and First and Second Order Logic

First order quantification in NaDSet is represented by the quantifier [∀**vr**:V1]. The following are derived rules of deduction:

$$\frac{\Gamma \rightarrow \Theta,\ [p/\mathbf{vr}]\mathbf{F1}}{\Gamma \rightarrow \Theta,\ [\forall \mathbf{vr}{:}V1]\mathbf{F1}} \qquad \frac{\Gamma,\ [\mathbf{tm}/\mathbf{vr}]\mathbf{F1} \rightarrow \Theta \qquad \rightarrow \mathbf{tm}{:}V1}{\Gamma,\ [\forall \mathbf{vr}{:}V1]\mathbf{F1} \rightarrow \Theta}$$

Here p is a first order parameter not occuring in any sentence of the conclusion of the →∀1 rule.

Apart from the additional premiss for the ∀→ rule, these are the rules for first order quantification in the sequent calculus.

Quantification in the monadic second order logic can similarly be represented by [∀**vr**:V2]. The following are derived rules of deduction:

$$\frac{\Gamma \rightarrow \Theta, [P/\mathbf{vr}]\mathbf{F1}}{\Gamma \rightarrow \Theta, [\forall \mathbf{vr}:V2]\mathbf{F1}} \qquad \frac{\Gamma, [\mathbf{tm}/\mathbf{vr}]\mathbf{F1} \rightarrow \Theta \qquad \rightarrow \mathbf{tm}:V2}{\Gamma, [\forall \mathbf{vr}:V2]\mathbf{F1} \rightarrow \Theta}$$

Here P is a second order parameters not occuring in any sentence of the conclusion of the →∀2 rule. Again, apart from the additional premiss, these are the usual rules.

The following are derived rules of deduction for the second order "type" V2(**ta**):

$$\frac{\Gamma \rightarrow \Theta, [P/\mathbf{vr}]\mathbf{F1}}{\Gamma \rightarrow \Theta, [\forall \mathbf{vr}:V2(\mathbf{ta})]\mathbf{F1}} \qquad \frac{\Gamma, [\mathbf{tm}/\mathbf{vr}]\mathbf{F1} \rightarrow \Theta \qquad \rightarrow \mathbf{tm}:V2(\mathbf{ta})}{\Gamma, [\forall \mathbf{vr}:V2(\mathbf{ta})]\mathbf{F1} \rightarrow \Theta}$$

Thus quantification over any of the types of the second order domains of second order logic, as well as many domains that are not of second order logic, can be represented in NaDSet.

4.4. Natural Numbers

Some definitions are provided here for the development of arithmetic within NaDSet:

0 **for** {α|~α=α}
Succ[**tm**] **for** {α|α:**tm**}
ClSucc **for** {γ|[∀u:γ]Succ[u]:γ}
N **for** {α|[∀z:ClSucc](0:z ⊃ α:z)}
1 **for** Succ[0]
N1 **for** {α|[∀z:ClSucc](1:z ⊃ α:z)}
≤ **for** {‹α,β›|[∀z:ClSucc](α:z ⊃ β:z)}

Explicit definitions for 1, N1, and ≤ are introduced here for use later in the paper. It is unnecessary to repeat here the development of arithmetic, since a sketch is provided in [Gilmore86], and since it follows closely the usual development within second order logic.

4.5. The Inconsistency of Extensionality

As was remarked earlier, NaDSet is an intensional logic. This will be demonstrated by showing that both $\rightarrow \langle 0,1\rangle =_e \langle 1,0\rangle$ and $\langle 0,1\rangle = \langle 1,0\rangle \rightarrow$ are derivable in NaDSet. A derivation of the first sequent follows:

$0{:}C \rightarrow 0{:}C \qquad 1{:}C \rightarrow 1{:}C$	axioms
$\rightarrow (0{:}C \downarrow 1{:}C),\ 0{:}C,\ 1{:}C$	$\rightarrow\downarrow$
$\rightarrow (0{:}C \downarrow 1{:}C),\ 1{:}C,\ 0{:}C$	interchange structural rule
$(1{:}C \downarrow 0{:}C) \rightarrow (0{:}C \downarrow 1{:}C)$	$\downarrow\rightarrow$
$p{:}\langle 1,0\rangle \rightarrow p{:}\langle 0,1\rangle$	$\{\}\rightarrow,\ \rightarrow\{\}$
(a) $\rightarrow [\forall u{:}\langle 1,0\rangle]\ u{:}\langle 0,1\rangle$	$\rightarrow\forall$
(b) $\rightarrow [\forall u{:}\langle 0,1\rangle]\ u{:}\langle 1,0\rangle$	derivation similar to (a)
$\rightarrow [\forall u{:}\langle 1,0\rangle]\ u{:}\langle 0,1\rangle \wedge [\forall u{:}\langle 1,0\rangle]\ u{:}\langle 0,1\rangle$	$\rightarrow\wedge$
$\rightarrow \langle 1,0\rangle =_e \langle 0,1\rangle$	$\rightarrow\{\}$

A derivation of the second sequent follows:

$\rightarrow 1=1$	$0=1 \rightarrow$	arithmetic
$\rightarrow \langle 1,0\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha=1\}$	$\langle 0,1\rangle{:}\{\langle\alpha,\beta\rangle \mid \alpha=1\} \rightarrow$	$\rightarrow\{\},\ \{\}\rightarrow$
$\langle 1,0\rangle = \langle 0,1\rangle \rightarrow$		$=\rightarrow$

The failure of extensionality within NaDSet is not to be regretted. Such a principal would be undesirable for some of the proposed applications for the logic. The models considered in applications of logic to mathematics are static; that is the extension of a set is not expected to change over time. But in the applications of logic to data modelling [Gilmore87a,87b,88], the extension of a set such as, for example, the set of employees of a particular corporation, is expected to change over time, although its intension remains fixed. As a consequence two sets with distinct intensions may have the same extension at one time, and different extensions at another time.

4.6. On Parameter Occurrences in a Derivation

There are three ways in which a parameter may be introduced into a derivation:

(i) An axiom **tm**:P → **tm**:P introduces the second order parameter 'P', and any first order parameter occurring in **tm**. Several examples of such axioms are given in the derivations provided in 4.1.

(ii) Parameters may appear in a sentence introduced by thinning. For example, an occurrence of 'P' is introduced by thinning in the derivation of → [∀u:V2] u:V2 in 4.2. This introduction of 'P' is typical of such introductions: 'P' already occurs in a sentence of the premiss of the thinning rule.

(iii) Applications of →{} and {}→ can introduce occurrences of parameters. For example, in the derivation of the sequent $\rightarrow\langle 0,1\rangle =_e \langle 1,0\rangle$ in 4.5, the parameter 'p' appears in the sequent p:‹1,0› → p:‹0,1› as a consequence of an application of →{} and of {}→.

Instances of parameters introduced by thinning or abstraction do not have an order determined by the derivation; that is, a correct application of thinning or abstraction, which introduces instances of a parameter, remains correct when those instances are replaced by a parameter of the opposite order. An instance of a parameter introduced in an axiom, on the other hand, may not have its order switched: In an atomic sentence, a parameter appearing to the right of ':' must be second order, and a parameter occurring in the term to the left of ':' must be first order.

Three rules of deduction can remove instances of parameters, →∀, ∀→, or cut. Instances of parameters introduced into a derivation can be traced through a derivation until they are removed by an application of one of these rules.

In an application of →{} and of {}→ two or more instances of a single parameter in the premiss of the rule may be reduced to a single instance of the parameter in the conclusion. The multiple instances of the parameter in the premiss may be traced back to different origins; that is, one may have been introduced by an axiom, and another by thinning or abstraction. In this case the single instance of the parameter in the conclusion has multiple origins. However, if one of its origins is an axiom, then its order cannot be switched.

It is this property of derivations that preserves the second order character of NaDSet.

4.6.1. Changing parameters in a derivation

Consider a derivation, and consider all occurrences of a given parameter within the derivation. All such occurrences may be replaced by another parameter of the same order that is new to the derivation.

5 SEMANTICS

The traditional semantics for classical logics is described in [Tarski56]. It is a reductionist semantics in the following sense: An interpretation of the atomic sentences assign truth values to them; nonatomic sentences are assigned truth values by semantic rules for the logical connectives and quantifiers. These semantic rules express the truth value for a nonatomic sentence in terms of the truth values already assigned to its parts. A proof that every sentence receives a truth value is completed by finite induction on the number of occurrences of connectives and quantifiers in the sentences.

Here a traditional semantics will be provided for NaDSet. However, as with earlier logics, [Gilmore67,71,80,86], finite induction no longer suffices, and not all sentences receive a truth value.

An atomic sentence of NaDSet has the form

tm:**T**,

where **tm** is a constant first order term and **T** is a second order constant or parameter. Interpretations of atomic sentences are conventional: Given a domain of discourse $\mathfrak{D}$, an interpretation assigns an object in $\mathfrak{D}$ to **tm** and a subset of $\mathfrak{D}$ to T. The sentence is true in the interpretation, if the object assigned to **tm** is a member of the subset assigned to T.

The following notation is helpful for the remainder of this section:

Definition 5.0.1. $\mathfrak{D}$, $\mathbb{P}[\mathfrak{D}]$, $\mathfrak{AV}$, $\mathfrak{P}$, $\mathfrak{CT}$, SEQ[n,$\mathfrak{AV}$], SEQ[n,$\mathfrak{P}$] and SEQ[n,$\mathfrak{CT}$].

1. $\mathfrak{D}$ is defined to be the set of constant terms in which no parameter has an occurrence. $\mathbb{P}[\mathfrak{D}]$ is the set of all subsets of $\mathfrak{D}$.
2. $\mathfrak{AV}$ is the set of abstraction variables;
   $\mathfrak{P}$ is the set of parameters, both first and second order;
   $\mathfrak{CT}$ is the set of all constant terms, both first and second order.
3. SEQ[n,$\mathfrak{S}$], where $\mathfrak{S}$ is one of $\mathfrak{AV}$, $\mathfrak{P}$, or $\mathfrak{CT}$, is the set of all finite

sequences of length n of members of $\mathcal{S}$.

Definition 5.0.2: <u>Assignments</u>

1. An <u>assignment</u> $\mathfrak{a}$ is a mapping of the first order parameters and constants into members of $\mathcal{D}$, and the second order parameters and constants into members of $\mathbb{P}[\mathcal{D}]$. For a first order parameter or constant **p**, $\mathfrak{a}$[**p**] is the member of $\mathcal{D}$ to which **p** has been assigned. For a second order parameter or constant P, $\mathfrak{a}$[P] is the member of $\mathbb{P}[\mathcal{D}]$ to which P has been assigned.
2. Let **tm** be a constant first order term, and let **p** $\in$ SEQ[n,$\mathcal{P}$] be a sequence of the distinct first order parameters occurring in **tm**. Let $\mathfrak{a}$[**p**] $\in$ SEQ[n,$\mathcal{D}$] be the corresponding sequence of members of $\mathcal{D}$ assigned to the first order parameters **p**. Then $\mathfrak{a}$[**tm**] is [$\mathfrak{a}$[**p**]/**p**]**tm**, and is necessarily a member of $\mathcal{D}$.
3. Two assignments $\mathfrak{a}$ and $\mathfrak{a}'$ are said to differ for **p** $\in$ SEQ[n,$\mathcal{P}$], if $\mathfrak{a}$[**q**] is $\mathfrak{a}'$[q] for each parameter **q** that is not an element of **p**.

5.1. Truth Value Assignments for Atomic Sentences

5.1.1. Let $\mathfrak{a}$ be a given assignment. Then $\mathbb{T}_0[\mathfrak{a}]$ is the set of atomic sentences **tm**:T for which $\mathfrak{a}$[**tm**] $\in$ $\mathfrak{a}$[T], the "true" sentences of $\mathfrak{a}$; and $\mathbb{F}_0[\mathfrak{a}]$ is the set of atomic sentences **tm**:T for which $\mathfrak{a}$[**tm**] $\notin$ $\mathfrak{a}$[T], the "false" sentences of $\mathfrak{a}$.

A sentence **tm**:T, where T is an abstraction term, is not an atomic sentence and therefore is assigned a truth value, if one is assigned at all, by the semantic rule for abstraction stated in 5.2.3 below. Thus an abstraction term, unlike second order parameters and constants, is not assigned a subset of $\mathcal{D}$ directly by $\mathfrak{a}$.

5.2. The Semantic Rules

Assume that $\mathbb{T}_\mu[\mathfrak{a}]$ and $\mathbb{F}_\mu[\mathfrak{a}]$ have been defined for a given ordinal number μ.

5.2.1. The semantic rules for $\downarrow$.
Let **Sta** and **Stb** be any sentences. Then

Sta, **Stb** ε $\mathbb{F}_\mu[\mathfrak{a}] \Rightarrow$ (**Sta**$\downarrow$**Stb**) ε $\mathbb{T}_{\mu+1}[\mathfrak{a}]$,
Sta ε $\mathbb{T}_\mu[\mathfrak{a}] \Rightarrow$ (**Sta**$\downarrow$**Stb**) ε $\mathbb{F}_{\mu+1}[\mathfrak{a}]$, and

$\mathbf{Stb} \ \varepsilon\ \mathbb{T}_{\mu}[\mathcal{A}] \Rightarrow (\mathbf{Sta}{\downarrow}\mathbf{Stb})\ \varepsilon\ \mathbb{F}_{\mu+1}[\mathcal{A}]$.

5.2.2. The semantic rules for $\forall$.
In the following rules, **ta** is a single abstraction variable when T is a second order parameter or constant, and is the abstracted term of T when it is a constant abstraction term {**ta**|**Fa**}. In the first case, $\underline{\alpha}$ is **ta**; in the second case, $\underline{\alpha} \in$ SEQ[n,$\mathcal{AV}$] has elements that are the distinct abstracted variables of {**ta**|**Fa**}. For both rules, **F1** is a formula in which only the quantifiable variable **vr** has a free occurrence.

5.2.2.1. The first semantic rule for $\forall$ is: Let $\underline{\mathbf{p}} \in$ SEQ[n,$\mathcal{P}$] have elements that are distinct from each other and from all parameters occurring in T, or in F1.

For all assignments $\mathcal{A}'$ differing from $\mathcal{A}$ only for $\underline{\mathbf{p}}$,
$[\underline{\mathbf{p}}/\underline{\alpha}]\mathbf{ta}{:}\mathbf{T}\ \varepsilon\ \mathbb{F}_{\mu}[\mathcal{A}']$ or $[[\underline{\mathbf{p}}/\underline{\alpha}]\mathbf{ta}/\mathbf{vr}]\mathbf{F1}\ \varepsilon\ \mathbb{T}_{\mu}[\mathcal{A}']$
$\Rightarrow [\forall \mathbf{vr}{:}\mathbf{T}]\mathbf{F1}\ \varepsilon\ \mathbb{T}_{\mu+1}[\mathcal{A}]$.

5.2.2.2. The second semantic rule for $\forall$ is:

For some $\underline{\mathbf{tm}} \in$ SEQ[n,$\mathcal{CT}$],
$[\underline{\mathbf{tm}}/\underline{\alpha}]\mathbf{ta}{:}\mathbf{T}\ \varepsilon\ \mathbb{T}_{\mu}[\mathcal{A}]$ and $[[\underline{\mathbf{tm}}/\underline{\alpha}]\mathbf{ta}/\mathbf{vr}]\mathbf{F1}\ \varepsilon\ \mathbb{F}_{\mu}[\mathcal{A}]$
$\Rightarrow [\forall \mathbf{vr}{:}\mathbf{T}]\mathbf{F1}\ \varepsilon\ \mathbb{F}_{\mu+1}[\mathcal{A}]$.

5.2.3. The semantic rules for {}:
Let {**ta**|**Fa**} be any abstraction term, let $\underline{\alpha} \in$ SEQ[n,$\mathcal{AV}$] have elements that are the distinct abstracted variables of {**ta**|**Fa**}, and let $\underline{\mathbf{tm}} \in$ SEQ[n,$\mathcal{CT}$]. The semantic rules for {} are:

$[\underline{\mathbf{tm}}/\underline{\alpha}]\mathbf{Fa}\ \varepsilon\ \mathbb{T}_{\mu}[\mathcal{A}] \Rightarrow [\underline{\mathbf{tm}}/\underline{\alpha}]\mathbf{ta}{:}\{\mathbf{ta}|\mathbf{Fa}\}\ \varepsilon\ \mathbb{T}_{\mu+1}[\mathcal{A}]$, and
$[\underline{\mathbf{tm}}/\underline{\alpha}]\mathbf{Fa}\ \varepsilon\ \mathbb{F}_{\mu}[\mathcal{A}] \Rightarrow [\underline{\mathbf{tm}}/\underline{\alpha}]\mathbf{ta}{:}\{\mathbf{ta}|\mathbf{Fa}\}\ \varepsilon\ \mathbb{F}_{\mu+1}[\mathcal{A}]$.

5.2.4. For a limit ordinal υ,
$\mathbb{T}_{\upsilon}[\mathcal{A}]$ is $\cup\{\ \mathbb{T}_{\mu}[\mathcal{A}] \mid \mu < \upsilon\}$, and
$\mathbb{F}_{\upsilon}[\mathcal{A}]$ is $\cup\{\ \mathbb{F}_{\mu}[\mathcal{A}] \mid \mu < \upsilon\}$

5.2.5. $\mathbb{T}[\mathcal{A}]$ is $\cup\{\ \mathbb{T}_{\mu}[\mathcal{A}] \mid \mu \geq 0\}$, and
$\mathbb{F}[\mathcal{A}]$ is $\cup\{\ \mathbb{F}_{\mu}[\mathcal{A}] \mid \mu \geq 0\}$.

The sets $\mathbb{T}[\mathfrak{a}]$ and $\mathbb{F}[\mathfrak{a}]$ are well defined since

$\mathbb{T}_\mu[\mathfrak{a}] \subseteq \mathbb{T}_\upsilon[\mathfrak{a}]$, for $\mu \le \upsilon$, and

$\mathbb{F}_\mu[\mathfrak{a}] \subseteq \mathbb{F}_\upsilon[\mathfrak{a}]$, for $\mu \le \upsilon$,

while $\mathbb{T}_\upsilon[\mathfrak{a}]$ and $\mathbb{F}_\upsilon[\mathfrak{a}]$ are subsets of the denumerable set of all sentences of NaDSet.

5.3. Satisfiability and Validity

5.3.1. A sequent $\Gamma \rightarrow \Theta$ of sentences is <u>satisfied</u> by an assignment $\mathfrak{a}$ if at least one of $\Gamma \cap \mathbb{F}[\mathfrak{a}]$ and $\Theta \cap \mathbb{T}[\mathfrak{a}]$ is not empty. The sequent is <u>valid</u> if it is satisfied by every assignment $\mathfrak{a}$.

5.3.2. Theorem: Every sequent with a cut-free derivation is valid.

The proof proceeds quite simply by induction on the number of applications of rules of deduction in the cut-free derivation.

The consistency of NaDSet would follow either from a proof that cut preserves validity, or from a proof that cut is a redundant rule of deduction.

5.4. Why must NaDSet be second order?

5.4.1. First order is not adequate.
The original first order form of NaDSet described in [Gilmore86] cannot express the semantics of recursive definitions. For example, the definition of the set N of natural numbers given in section 4.4 takes the form:

N **for** $\{\alpha|[\forall z{:}ClSucc](0{:}z \supset \alpha{:}z)\}$,

where ClSucc is the set of sets closed under successor. The definition expresses that N is the smallest set with 0 as a member that is closed under successor. The quantifier $[\forall z{:}ClSucc]$ is second order.

The importance of such a recursive definition is that all properties of the set defined by it, can be derived from the definition. In a first order logic it is necessary, for example, to assume all of Peano's axioms as nonlogical assumptions in order to develop arithmetic.

Another example of importance to computer science arises from the recursive definitions of a programming language like Prolog that provides recursive definitions by means of collections of Horn clauses. A trivial variant of the above definition of N is:

NN for $\{\alpha|[\forall z{:}ClSucc](0{:}z \wedge 1{:}z \supset \alpha{:}z)\}$.

Without additional assumptions, it is not possible to prove within a first order logic that N and NN are extensionally identical.

That recursive definitions are entirely self contained is important for some of the intended applications of NaDSet. The recursive definitions needed for expressing the semantics of some computer programs can take complex forms, and the discovery and justification of induction axioms needed to prove the programs correct can be difficult. Further, for a theoretical result showing that a conjecture, such as P=NP, is not derivable in NaDSet, it is essential that the logic be self contained.

5.4.2. The third order form of NaDSet is inconsistent.

One of the ways in which NaDSet differs from the monadic second order logic is in allowing second order terms, in which no second order parameter appears, to be first order. It is natural to ask if third and higher order forms of NaDSet, based on a similar principal, are consistent. In a third order form of NaDSet, for example, third order terms in which no third order parameter occurs would be second or first order depending upon whether a second order parameter did, or did not, occur in it. But, as the following derivation shows, this third order form of the logic is inconsistent.

Define

R for $\{\alpha|[\forall u{:}\{\beta|\alpha=\beta\}]{\sim}\alpha{:}u\}$.

Derivations for both $\rightarrow$ R:R and R:R $\rightarrow$ are provided below. The rules used in each step of the derivation are not cited since they are obvious from the principle sentence introduced into the conclusion of the rules. Also, in some cases, several steps are expressed as one when the reconstruction of the missing steps is obvious.

In the following derivation, P is necessarily a second order, and Q a third order, parameter.

$$P{:}Q \rightarrow P{:}Q \qquad R{:}Q \rightarrow R{:}Q$$

$$P{:}Q,\ P=R \rightarrow R{:}Q$$

$$P=R \rightarrow P=R \qquad\qquad R{:}P \rightarrow R{:}P$$

$$P=R \rightarrow P{:}\{\beta|\beta=R\} \qquad\qquad {\sim}R{:}P \rightarrow {\sim}R{:}P$$

$$[\forall u{:}\{\beta|\beta=R\}]\ {\sim}R{:}u,\ P=R \rightarrow {\sim}R{:}P$$

$$R{:}R,\ P{:}\{\beta|\beta=R\} \rightarrow {\sim}R{:}P$$

$$R{:}R \rightarrow [\forall u{:}\{\beta|\beta=R\}]\ {\sim}R{:}u$$

$$\rightarrow R=R \qquad R{:}R \rightarrow R{:}R$$

$$\rightarrow R{:}\{\beta|\beta=R\} \qquad {\sim}R{:}R,\ R{:}R \rightarrow$$

$$[\forall u{:}\{\beta|\beta=R\}]\ {\sim}R{:}u,\ R{:}R \rightarrow$$

$$R{:}P \rightarrow R{:}P \qquad R{:}R \rightarrow$$

$$R{:}P,\ P=R \rightarrow$$

$$P=R \rightarrow {\sim}R{:}P$$

$$\rightarrow [\forall u{:}\{\beta|\beta=R\}]\ {\sim}R{:}u$$

$$\rightarrow R{:}R$$

Thus the principal that distinguishes NaDSet from the monadic second order logic cannot be used to define third and higher order forms of NaDSet.

Incidentally, this is the contradiction found in the earliest form of NaDSet presented in [Gilmore68]. In that form of the logic, "atomic" was not correctly defined, permitting the derivation of the sequent $P=R \rightarrow P=R$.

5.4.3. The atomic sentences of NaDSet have clear interpretations
Consider a given assignment $\mathfrak{A}$. In the following discussion, a sentence is said to be true if it is a member of $\mathbb{T}[\mathfrak{A}]$, and is said to be false if it is a member of $\mathbb{F}[\mathfrak{A}]$, as these sets have been defined in 5.2.5.

The atomic sentence

$$\{\alpha|\alpha=\alpha\}{:}P$$

is true or false according to whether '$\{\alpha|\alpha=\alpha\}$' is or is not a member of the set assigned to 'P'. In the displayed sentence '$\{\alpha|\alpha=\alpha\}$' is being mentioned, while 'P' is being used. The sentence could, therefore, equally well be written

$$\text{'}\{\alpha|\alpha=\alpha\}\text{'}{:}P.$$

The abstraction term '$\{\alpha|\alpha=\alpha\}$' is, however, also a second order term. The sentence

$$\{\alpha|\alpha=\alpha\}{:}\{\alpha|\alpha=\alpha\}$$

is true. For it receives the same truth value as the sentence

$\{\alpha|\alpha=\alpha\}=\{\alpha|\alpha=\alpha\}$,

which, by definition 4.1.2, is the sentence

$\langle\{\alpha|\alpha=\alpha\},\{\alpha|\alpha=\alpha\}\rangle:\{\langle\alpha,\beta\rangle|\ [\forall z:\{\gamma|\alpha:\gamma\}]\ \beta:z\}$.

This sentence receives the same truth value as

$[\forall z:\{\gamma|\ \{\alpha|\alpha=\alpha\}:\gamma\}]\ \{\alpha|\alpha=\alpha\}:z$.

This last sentence is true, because no matter what set is assigned to the second order parameter 'P', the sentence

$\{\alpha|\alpha=\alpha\}:P$,

which can be written

$'\{\alpha|\alpha=\alpha\}':P$,

is either true or false. Carrying the single quotes back through the applications of the semantic rules, the sentence $\{\alpha|\alpha=\alpha\}:\{\alpha|\alpha=\alpha\}$ can be written

$'\{\alpha|\alpha=\alpha\}':\{\alpha|\alpha=\alpha\}$

to emphasize that the first occurrence of $'\{\alpha|\alpha=\alpha\}'$ in the sentence is being mentioned, while the second occurrence is being used.

In NaDSet, an abstraction term occurring to the left of ':' in an atomic sentence is always interpreted as a name for itself and could therefore be enclosed in single quotes. But, because such occurrences of these terms are uniformly regarded as names for themselves, single quotes are not needed to avoid confusions of use and mention.

The systematic dropping of quotes must not be abused. Both Church and Tarski have warned of the possible abuse of the quote notation; see, for example, footnote 136 on page 62 of [Church56], or the discussion of quotes in the first section of [Tarski56]. Treating quotes as a function from subsets of $\mathcal{D}$ to names of $\mathcal{D}$ is an abuse of quotes.

Each second order constant **C** has a pair of subsets of $\mathcal{D}$ associated with it:

The set of terms **ta** $\in \mathcal{D}$, for which **ta**:C $\in \mathbb{T}[\mathfrak{A}]$, and

the set of terms **ta** $\in \mathcal{D}$, for which **ta**:**C** $\in \mathbb{F}[\mathfrak{A}]$.

The union of these two sets is $\mathcal{D}$. Similarly, each second order term **tm** that is a member of $\mathcal{D}$ has a pair of subsets of $\mathcal{D}$ associated with it:

The set of terms **ta** $\in \mathcal{D}$, for which **ta**:**tm** $\in \mathbb{T}[\mathfrak{A}]$, and

the set of terms **ta** $\in \mathcal{D}$, for which **ta**:**tm** $\in \mathbb{F}[\mathfrak{A}]$.

However, the union of these two sets may, or may not, be $\mathcal{D}$. Nevertheless, each such term **tm** may be used as the name of a pair of subsets of $\mathcal{D}$, just as a second order constant may be used.

No means has been provided for supplying an <u>arbitrary</u> pair of subsets of

$\mathcal{D}$ with a name that is a member of $\mathcal{D}$. Therefore, since second order parameters are variables that may be assigned any subset of $\mathcal{D}$, sequents of the form

$$P{:}Q \rightarrow P{:}Q$$

cannot be given an interpetation when 'P' is second order: A parameter 'P' cannot act as a function from subsets of $\mathcal{D}$ to members of $\mathcal{D}$.

The nominalist interpretation of second order terms to the left of ':' in atomic sentences as names for themselves, requires, therefore, that there be both first and second order parameters in NaDSet. For a parameter cannot be assigned an arbitrary subset of $\mathcal{D}$, and at the same time be interpreted as a name for the subset in the set $\mathcal{D}$. Further, it justifies allowing a second order term to be first order when no second order parameter occurs in it.

In the first description of the original form of NaDSet [Gilmore71], the underlying reason for the logic being second order was reinforced by calling first order variables, variables of mention, and second order variables, variables of use.

A consequence of accepting this nominalist interpretation is that NaDSet is necessarily an intensional logic, as was shown in section 4. Another consequence is the failure of the general Cantor diagonal argument, as will be shown next.

6 CANTOR'S DIAGONAL ARGUMENT

As described in the introduction, a real number γ in the interval [0,1] can be represented as a total single valued function with arguments from N1, and values that are 0 or 1:

$$R \textbf{ for } \{\gamma \mid [\forall n{:}N1][\exists u{:}B](\langle i,u\rangle{:}\gamma \wedge [\forall v{:}B](\langle n,v\rangle{:}\gamma \supset v=b))\},$$

where B is defined:

$$B \textbf{ for } \{\alpha \mid \alpha=0 \vee \alpha=1\}.$$

N1, 0, and 1 are defined in 4.4, and = in 4.1.2.

Identity $=_R$ between real numbers is necessarily extensional identity.

$$=_R \textbf{ for } \{\langle\gamma 1,\gamma 2\rangle \mid \{\langle\nu,\beta\rangle \mid \langle\nu,\beta\rangle{:}\gamma 1\} =_e \{\langle\nu,\beta\rangle \mid \langle\nu,\beta\rangle{:}\gamma 2\}\}$$

As with = and $=_e$, the usual infix notation will be used.

A single valued map ϕ of N1 into R is a member of the set

$$MapN1R \textbf{ for } \{\phi \mid M[\phi]\},$$

where, for any term **F**,

$M[\mathbf{F}]$ for $[\forall n{:}N1][\exists x{:}R](\langle n,x\rangle{:}\mathbf{F} \wedge [\forall y{:}R](\langle n,y\rangle{:}\mathbf{F} \supset y{=}_R x))$.

In the informal diagonal argument of section 1 used to prove Cantor's lemma, a real number C[**F**] is defined from a member **F** of MapN1R. That number is defined here for any term **F**:

$C[\mathbf{F}]$ for $\{\langle\nu,\beta\rangle | [\forall x{:}R](\langle\nu,x\rangle{:}\mathbf{F} \supset [\forall v{:}B](\langle\nu,v\rangle{:}x \supset \sim v{=}\beta))\}$.

The argument proceeds by proving that C[**F**] is indeed a real number, if **F** is a member of MapN1R, and then by proving $[\forall n{:}N1] \sim\langle n,C[\mathbf{F}]\rangle{:}\mathbf{F}$, providing an instantiation for the existential quantifier of Cantor's lemma.

A derivation of the following rule of deduction provides a full formalization of Cantor's diagonal argument within NaDSet:

6.1. Cantor's Rule of Deduction

$$\frac{\langle i,r\rangle{:}\mathbf{F} \rightarrow \langle i,r\rangle{:}\mathbf{F} \qquad \langle i,C[\mathbf{F}]\rangle{:}\mathbf{F} \rightarrow \langle i,C[\mathbf{F}]\rangle{:}\mathbf{F}}{\mathbf{F}{:}MAPN1R \rightarrow [\exists x{:}R][\forall n{:}N1] \sim\langle n,x\rangle{:}\mathbf{F}}$$

Here 'i' is any first order parameter, 'r' any second order parameter, and **F** a constant term in which there are no occurrences of 'i' or 'r'.

6.2. A Derivation of Cantor's Rule of Deduction

The following abbreviations will be used in derivations provided in this section:

$A[i,r]$ for $\langle i,r\rangle{:}\mathbf{F} \wedge [\forall y{:}R](\langle i,y\rangle{:}\mathbf{F} \supset y{=}_R r)$

$S[i,b,r]$ for $\langle i,b\rangle{:}r \wedge [\forall v{:}B](\langle i,v\rangle{:}r \supset v{=}b)$

$T[i,x,\beta]$ for $(\langle i,x\rangle{:}\mathbf{F} \supset [\forall v{:}B](\langle i,v\rangle{:}x \supset \sim v{=}\beta))$

With these abbreviations, the following abbreviations are possible:

$M[\mathbf{F}]$ for $[\forall n{:}N1][\exists x{:}R]A[n,x]$

R for $\{\gamma | [\forall n{:}N1][\exists u{:}B]S[n,u,\gamma]\}$

$C[\mathbf{F}]$ for $\{\langle\nu,\beta\rangle | [\forall x{:}R]T[\nu,x,\beta]\}$.

6.2.1. The following rule of deduction is derivable:

$$\frac{M[\mathbf{F}] \rightarrow C[\mathbf{F}]{:}R \qquad C[\mathbf{F}]{:}R,\ i{:}N1,\ \langle i,C[\mathbf{F}]\rangle{:}\mathbf{F} \rightarrow}{\mathbf{F}{:}MAPN1R \rightarrow [\exists x{:}R][\forall n{:}N1] \sim\langle n,x\rangle{:}\mathbf{F}}$$

where 'i' is a first order parameter, and **F** is a constant term in which 'i' does not occur.

A derivation of this rule follows:

C[**F**]:R, i:N 1, ‹i,C[**F**]›:**F** →

C[**F**]:R, i:N 1 → ~‹i,C[**F**]›:**F**

M[**F**] → C[**F**]:R   C[**F**]:R → [∀n:N 1] ~‹n,C[**F**]›:**F**

M[**F**] → C[**F**]:R   M[**F**] → [∀n:N 1] ~‹n,C[**F**]›:**F**   cut

M[**F**] → [∃x:R][∀i:N 1] ~‹n,x›:**F**   →∃

F:MAPN 1R → [∃x:R][∀n:N 1] ~‹n,x›:**F**

6.2.2. For any first order parameters 'b' and 'i', and second order parameter 'r', the following sequents are derivable:

b:B → b:B
i:N 1 → i:N 1
r:R → r:R
[∀v:B](‹i,v›:r ⊃ ~v=b) → [∀v:B](‹i,v›:r ⊃ ~v=b)

The need for derivations of sequents such as these is typical of NaDSet. The construction of a derivation for such a sequent is generally an elementary exercise in Gentzen's sequent logic. When that is the case, a derivation of the sequent will be omitted.

6.2.3. From the premisses of Cantor's rule, the following sequents can be derived:

‹i,b›:C[**F**] → ‹i,b›:C[**F**]
C[**F**]:R → C[**F**]:R

Derivations for these sequents follow. They, as well as all subsequent derivations, take the following form: The main branch of the derivation appears on the left. When a two premiss rule is applied, in which one of the premisses is an axiom or has been previously derived, that premiss appears on the right. When neither premiss is an axiom or has been previously derived, the two premisses are numbered or lettered, and separate derivations provided for them. Only occasionally will a rule of deduction be referenced since the derivations are only rarely abbreviated and the rule or rules being applied in any step should be apparent. To assist in identifying the rule being used, the principal sentence in the conclusion of the rule will sometimes be identified with a prefixed *.

Derivations of sequents judged to be elementary will be omitted wihout comment.

A derivation of ‹i,b›:C[**F**] → ‹i,b›:C[**F**] follows:

[∀v:B](‹i,v›:r ⊃ ~v=b) → [∀v:B](‹i,v›:r ⊃ ~v=b)   ‹i,r›:**F** → ‹i,r›:**F**

*(‹i,r›:**F** ⊃ [∀v:B](‹i,v›:r ⊃ ~v=b)), ‹i,r›:**F** → [∀v:B](‹i,v›:r ⊃ ~v=b)

(‹i,r›:**F** ⊃ [∀v:B](‹i,v›:r ⊃ ~v=b)) → *(‹i,r›:**F** ⊃ [∀v:B](‹i,v›:r ⊃ ~v=b))

T[i,r,b] → T[i,r,b]   r:R → r:R

[∀x:R]T[i,x,b], r:R → T[i,r,b]

[∀x:R]T[i,x,b] → [∀x:R]T[i,x,b]

‹i,b›:C[**F**] → ‹i,b›:C[**F**]

A derivation of C[**F**]:R → C[**F**]:R follows:

‹i,c›:C[**F**] → ‹i,c›:C[**F**]   c=b → c=b

(‹i,c›:C[**F**] ⊃ c=b) → (‹i,c›:C[**F**] ⊃ c=b)   c:B → c:B

*[∀v:B](‹i,v›:C[**F**] ⊃ v=b), c:B → (‹i,c›:C[**F**] ⊃ c=b)

[∀v:B](‹i,v›:C[**F**] ⊃ v=b) → *[∀v:B](‹i,v›:C[**F**] ⊃ v=b)   ‹i,b›:C[**F**] → ‹i,b›:C[**F**]

‹i,b›:C[**F**] ∧ [∀v:B](‹i,v›:C[**F**] ⊃ v=b) → ‹i,b›:C[**F**] ∧ [∀v:B](‹i,v›:C[**F**] ⊃ v=b)

S[i,b,C[**F**]] → S[i,b,C[**F**]]   b:B → b:B

b:B, S[i,b,C[**F**]] → *[∃u:B]S[i,u,C[**F**]]

*[∃u:B]S[i,u,C[**F**]] → [∃u:B]S[i,u,C[**F**]]   i:N1 → i:N1

*[∀n:N1][∃u:B]S[n,u,C[**F**]], i:N1 → [∃u:B]S[i,u,C[**F**]]

[∀n:N1][∃u:B]S[n,u,C[**F**]] → *[∀n:N1][∃u:B]S[n,u,C[**F**]]

C[**F**]:R → C[**F**]:R

6.2.4. Using the premisses of Cantor's rule, the following sequent is derivable:

C[**F**]:R, i:N1, ‹i,C[**F**]›:**F** →

A derivation follows:

→ b=b

~b=b →   ‹i,b›:C[**F**] → ‹i,b›:C[**F**] ‹i,b›:C[F], *(‹i,b›:C[F] ⊃ ~b=b) →

b:B → b:B

b:B, ‹i,b›:C[F], *[∀v:B](‹i,v›:C[F] ⊃ ~v=b) →

‹i,C[F]›:F → ‹i,C[F]›:F

b:B, ‹i,b›:C[F], *(‹i,C[F]›:F ⊃ [∀v:B](‹i,v›:C[F] ⊃ ~v=b)), ‹i,C[F]›:F →

b:B, ‹i,b›:C[F], *T[i,C[F],b], ‹i,C[F]›:F →

C[F]:R → C[F]:R

b:B, ‹i,b›:C[F], *[∀x:R]T[i,x,b], C[F]:R, ‹i,C[F]›:F →

b:B, ‹i,b›:C[F], *‹i,b›:C[F], C[F]:R, ‹i,C[F]›:F →

b:B, ‹i,b›:C[F], C[F]:R, ‹i,C[F]›:F →

b:B, ‹i,b›:C[F], *‹i,b›:C[F], C[F]:R, ‹i,C[F]›:F →

→ b=b

b:B, ‹i,b›:C[F], *(‹i,b›:C[F] ⊃ b=b), C[F]:R, ‹i,C[F]›:F →

b:B, ‹i,b›:C[F], *[∀v:B](‹i,v›:C[F] ⊃ v=b), C[F]:R, ‹i,C[F]›:F →

b:B, *‹i,b›:C[F] ∧ [∀v:B](‹i,v›:C[F] ⊃ v=b), C[F]:R, ‹i,C[F]›:F →

b:B, *S[i,b,C[F]], C[F]:R, ‹i,C[F]›:F →

*[∃u:B]S[i,u,C[F]], C[F]:R, ‹i,C[F]›:F →

i:N1 → i:N1

*[∀n:N1][∃u:B]S[n,u,C[F]], C[F]:R, i:N1, ‹i,C[F]›:F →

*C[F]:R, C[F]:R, i:N1, ‹i,C[F]›:F →

C[F]:R, i:N1, ‹i,C[F]›:F →

6.2.5. Using the premisses of Cantor's rule, the sequent
M[F] → C[F]:R
can be derived from the sequents
(1a) [∀v:B](‹i,v›:t ⊃ v=0), [∀x:R](‹i,x›:F ⊃ $x =_R t$) → ‹i,1›:C[F]
(1b) [∀v:B](‹i,v›:t ⊃ v=1), [∀x:R](‹i,x›:F ⊃ $x =_R t$) → ‹i,0›:C[F]
(2a) t:R, ‹i,0›:t, ‹i,t›:F → [∀v:B](‹i,v›:C[F] ⊃ 1=v)
(2b) t:R, ‹i,1›:t, ‹i,t›:F → [∀v:B](‹i,v›:C[F] ⊃ 0=v)
Note that (1b) is obtained from (1a), and (2b) from (2a), by replacing '0' by '1' and '1' by '0'. A derivation follows:

(1a) [∀v:B](‹i,v›:t ⊃ v=0), [∀x:R](‹i,x›:F ⊃ $x =_R t$) → ‹i,1›:C[F]

(2a) t:R, ‹i,0›:t, ‹i,t›:F → [∀v:B](‹i,v›:C[F] ⊃ 1=v)

---

t:R, [∀v:B](‹i,v›:t ⊃ v=0), [∀x:R](‹i,x›:F ⊃ $x =_R t$), ‹i,0›:t, ‹i,t›:F →

*‹i,1›:C[F] ∧ [∀v:B](‹i,v›:C[F] ⊃ 1=v)

t:R, *‹i,0›:t ∧ [∀v:B](‹i,v›:t ⊃ v=0), *‹i,t›:F ∧ [∀x:R](‹i,x›:F ⊃ $x=_R t$) → *S[i,1,C[F]]

t:R, *S[i,0,t], *A[i,t] → S[i,1,C[F]]   S[i,b,t] → S[i,b,t]

t:R, *b=0, *S[i,b,t], A[i,t] → S[i,1,C[F]]   → 1:B

(a) t:R, b=0, S[i,b,t], A[i,t] → *[∃u:B]S[i,u,C[F]]

Derivation of (b): Interchange '0' and '1' throughout derivation of (a)

(a) t:R, b=0, S[i,b,t], A[i,t] → [∃u:B]S[i,u,C[F]]

(b) t:R, b=1, S[i,b,t], A[i,t] → [∃u:B]S[i,u,C[F]]

---

t:R, *b=0 ∨ b=1, S[i,b,t], A[i,t] → [∃u:B]S[i,u,C[F]]

t:R, *b:B, S[i,b,t], A[i,t] → [∃u:B]S[i,u,C[F]]

t:R, *[∃u:B]S[i,u,t], A[i,t] → [∃u:B]S[i,u,C[F]]   i:N1 → i:N1

t:R, i:N1, *[∀n:N1][∃u:B]S[n,u,t], A[i,t] → [∃u:B]S[i,u,C[F]]

i:N1, t:R, *t:R, A[i,t] → [∃u:B]S[i,u,C[F]]

i:N1, *t:R, A[i,t] → [∃u:B]S[i,u,C[F]]

i:N1, *[∃x:R]A[i,x] → [∃u:B]S[i,u,C[F]]   i:N1 → i:N1

i:N1, i:N1, *[∀n:N1][∃x:R]A[i,x] → [∃u:B]S[i,u,C[F]]

[∀n:N1][∃x:R]A[i,x], *i:N1 → [∃u:B]S[i,u,C[F]]

*M[F], i:N1 → [∃u:B]S[i,u,C[F]]

M[F] → *[∀n:N1][∃u:B]S[n,u,C[F]]

M[F] → C[F]:R

6.2.6. Using the premisses of Cantor's rule, the sequents (1a), (1b), (2a) and (2b) of 6.2.5 can be derived.

Derivations of (1a) and (2a) follow. Derivations of (1b) and (2b) can be obtained from these by interchanging '0' and '1'.

A derivation of (1a) follows:

$b=0, b=1 \rightarrow \qquad\qquad \langle i,b\rangle{:}t \rightarrow \langle i,b\rangle{:}t$

$*(\langle i,b\rangle{:}t \supset b=0), \langle i,b\rangle{:}t, b=1 \rightarrow$

$(\langle i,b\rangle{:}t \supset b=0), *\langle i,b\rangle{:}\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}t\}, b=1 \rightarrow \qquad\qquad \langle i,b\rangle{:}r \rightarrow \langle i,b\rangle{:}\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}r\}$

$(\langle i,b\rangle{:}t \supset b=0), *[\forall x{:}\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}r\}]x{:}\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}t\}, \langle i,b\rangle{:}r, b=1 \rightarrow$

$(\langle i,b\rangle{:}t \supset b=0),$

$*[\forall x{:}\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}r\}]x{:}\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}t\} \wedge [\forall x{:}\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}t\}]x{:}\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}r\},$

$\langle i,b\rangle{:}r, b=1 \rightarrow$

$(\langle i,b\rangle{:}t \supset b=0), *\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}r\}=_e\{\langle\nu,\beta\rangle|\langle\nu,\beta\rangle{:}t\}, \langle i,b\rangle{:}r, b=1 \rightarrow$

$(\langle i,b\rangle{:}t \supset b=0), *r=_R t, \langle i,b\rangle{:}r, b=1 \rightarrow \qquad\qquad b{:}B \rightarrow b{:}B$

$*[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), r=_R t, b{:}B, \langle i,b\rangle{:}r, b=1 \rightarrow \qquad\qquad \langle i,r\rangle{:}\mathbf{F} \rightarrow \langle i,r\rangle{:}\mathbf{F}$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), *(\langle i,r\rangle{:}\mathbf{F} \supset r=_R t), \langle i,r\rangle{:}\mathbf{F}, b{:}B, \langle i,b\rangle{:}r, b=1 \rightarrow$

$r{:}R \rightarrow r{:}R$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), *[\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t), r{:}R, \langle i,r\rangle{:}\mathbf{F}, b{:}B, \langle i,b\rangle{:}r, b=1 \rightarrow$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), [\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t), r{:}R, \langle i,r\rangle{:}\mathbf{F}, b{:}B, \langle i,b\rangle{:}r \rightarrow *{\sim}b=1$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), [\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t), r{:}R, \langle i,r\rangle{:}\mathbf{F}, b{:}B \rightarrow *(\langle i,b\rangle{:}r \supset {\sim}b=1)$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), [\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t), r{:}R, \langle i,r\rangle{:}\mathbf{F} \rightarrow *[\forall v{:}B](\langle i,v\rangle{:}r \supset {\sim}v=1)$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), [\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t), r{:}R \rightarrow$

$*(\langle i,r\rangle{:}\mathbf{F} \supset [\forall v{:}B](\langle i,v\rangle{:}r \supset {\sim}v=1))$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), [\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t), r{:}R \rightarrow$

$*[\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset [\forall v{:}B](\langle i,v\rangle{:}x \supset {\sim}v=1))$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), [\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t), r{:}R \rightarrow *T[i,r,1]$

$[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0), [\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t) \rightarrow *[\forall x{:}R]T[i,x,1]$

(1a) $[\forall v{:}B](\langle i,v\rangle{:}t \supset v=0),\ [\forall x{:}R](\langle i,x\rangle{:}\mathbf{F} \supset x=_R t) \rightarrow {*}\langle i,1\rangle{:}C[\mathbf{F}]$

A derivation of (2a) follows:

$b=0 \rightarrow 0=b$	
$b=0 \rightarrow {*}1=b,\ 0=b$	
$b=0,\ {*}{\sim}0=b \rightarrow 1=b$	$\langle i,0\rangle{:}t \rightarrow \langle i,0\rangle{:}t$
$\langle i,0\rangle{:}t,\ b=0,\ {*}(\langle i,0\rangle{:}t \supset {\sim}0=b) \rightarrow 1=b$	$\rightarrow 0{:}B$
$\langle i,0\rangle{:}t,\ b=0,\ {*}[\forall v{:}B](\langle i,v\rangle{:}t \supset {\sim}v=b) \rightarrow 1=b$	
$\langle i,0\rangle{:}t,\ b=1,\ [\forall v{:}B](\langle i,v\rangle{:}t \supset {\sim}v=b) \rightarrow 1=b$	from $b=1 \rightarrow 1=b$ by thinning

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$\langle i,0\rangle{:}t,\ {*}b=0 \vee b=1,\ [\forall v{:}B](\langle i,v\rangle{:}t \supset {\sim}v=b) \rightarrow 1=b$	
$\langle i,0\rangle{:}t,\ {*}b{:}B,\ [\forall v{:}B](\langle i,v\rangle{:}t \supset {\sim}v=b) \rightarrow 1=b$	$\langle i,t\rangle{:}\mathbf{F} \rightarrow \langle i,t\rangle{:}\mathbf{F}$
$\langle i,0\rangle{:}t,\ \langle i,t\rangle{:}\mathbf{F},\ b{:}B,\ {*}(\langle i,t\rangle{:}\mathbf{F} \supset [\forall v{:}B](\langle i,v\rangle{:}t \supset {\sim}v=b)) \rightarrow 1=b$	
$t{:}R,\ \langle i,0\rangle{:}t,\ \langle i,t\rangle{:}\mathbf{F},\ b{:}B,\ {*}T[i,t,b] \rightarrow 1=b$	$t{:}R \rightarrow t{:}R$
$t{:}R,\ \langle i,0\rangle{:}t,\ \langle i,t\rangle{:}\mathbf{F},\ b{:}B,\ {*}[\forall x{:}R]T[i,x,b] \rightarrow 1=b$	
$t{:}R,\ \langle i,0\rangle{:}t,\ \langle i,t\rangle{:}\mathbf{F},\ b{:}B,\ {*}\langle i,b\rangle{:}C[\mathbf{F}] \rightarrow 1=b$	
$t{:}R,\ \langle i,0\rangle{:}t,\ \langle i,t\rangle{:}\mathbf{F},\ b{:}B \rightarrow {*}(\langle i,b\rangle{:}C[\mathbf{F}] \supset 1=b)$	
(2a) $t{:}R,\ \langle i,0\rangle{:}t,\ \langle i,t\rangle{:}\mathbf{F} \rightarrow {*}[\forall v{:}B](\langle i,v\rangle{:}C[\mathbf{F}] \supset 1=v)$	

6.3. An Application of Cantor's Rule

Consider the enumeration ${}^{i}b$ of real numbers defined in the introduction. It is the enumeration FB:

FB **for** $\{\langle \nu,\rho\rangle \mid \nu{:}N1 \wedge \rho =_R B[\nu]\}$, where

B[i] **for** $\{\langle \mu,\beta\rangle \mid \mu{:}N1 \wedge (\mu \le i \supset \beta=1) \wedge (\mu > i \supset \beta=0)\}$.

Consider the premisses of Cantor's rule when FB replaces **F**:

$\langle i,r\rangle{:}FB \rightarrow \langle i,r\rangle{:}FB$

$\langle i,C[FB]\rangle{:}FB \rightarrow \langle i,C[FB]\rangle{:}FB$

That each is derivable for any first order parameter 'i' and second order parameter 'r' follows quickly from the definition of FB. Therefore, by

Cantor's rule, the following sequent is derivable:

$FB{:}MAPN\,1R \rightarrow [\exists x{:}R][\forall n{:}N\,1] \sim\langle n,x\rangle{:}FB$

That $\rightarrow FB{:}MAPN\,1R$ is derivable is shown below, so that by cut,

$\rightarrow [\exists x{:}R][\forall n{:}N\,1] \sim\langle n,x\rangle{:}FB$

is derivable.

A derivation of $\rightarrow FB{:}MAPN\,1R$ from sequents (1) and (2) follows:

(1) $i{:}N\,1 \rightarrow A[i,B[i]]$

(2) $i{:}N\,1 \rightarrow B[i]{:}R$

$i{:}N\,1 \rightarrow {*}[\exists x{:}R]A[i,x]$

$\rightarrow {*}[\forall n{:}N\,1][\exists x{:}R]A[n,x]$

$\rightarrow {*}M[FB]$

$\rightarrow {*}FB{:}MAPN\,1R$

A derivation of (1) follows:

$r =_R B[i] \rightarrow r =_R B[i])$

$i{:}N\,1,\ r{:}R,\ i{:}N\,1 \wedge r =_R B[i] \rightarrow r =_R B[i])$ thin, $\wedge \rightarrow$

$i{:}N\,1,\ r{:}R,\ {*}\langle i,r\rangle{:}FB \rightarrow r =_R B[i])$

$i{:}N\,1,\ r{:}R \rightarrow {*}(\langle i,r\rangle{:}FB \supset r =_R B[i])$

(a) $i{:}N\,1 \rightarrow {*}[\forall y{:}R](\langle i,y\rangle{:}FB \supset y =_R B[i])$

(b) $i{:}N\,1 \rightarrow \langle i,B[i]\rangle{:}FB$ ((b) follows easily from $\rightarrow B[i] =_R B[i]$)

$i{:}N\,1 \rightarrow {*}\langle i,B[i]\rangle{:}FB \wedge [\forall y{:}R](\langle i,y\rangle{:}FB \supset y =_R B[i])$

(1) $i{:}N\,1 \rightarrow {*}A[i,B[i]]$

A derivation of (2) from two sequents (3) and (4) follows. The derivation makes use of the following derivable sequents of the arithmetic introduced in section 4:

$i{:}N1, j{:}N1 \rightarrow j \le i \vee j > i$, and $i{:}N1, j{:}N1, j \le i, j > i \rightarrow$

(3) $i{:}N1, j{:}N1, j \le i \rightarrow [\exists u{:}B]S[j,u,B[i]]$

(4) $i{:}N1, j{:}N1, j > i \rightarrow [\exists u{:}B]S[j,u,B[i]]$

$i{:}N1, j{:}N1, {*}j \le i \vee j > i \rightarrow [\exists u{:}B]S[j,u,B[i]]$ $\qquad$ $i{:}N1, j{:}N1 \rightarrow j \le i \vee j > i$

$i{:}N1, j{:}N1 \rightarrow [\exists u{:}B]S[j,u,B[i]]$ $\qquad$ cut

$i{:}N1 \rightarrow {*}[\forall n{:}N1][\exists u{:}B]S[n,u,B[i]]$

(2) $i{:}N1 \rightarrow {*}B[i]{:}R$

A derivation of (3) from sequents (5) and (6) follows:

(5) $i{:}N1, j{:}N1, j \le i \rightarrow \langle j,1\rangle{:}B[i]$

(6) $j \le i \rightarrow [\forall v{:}B](\langle j,v\rangle{:}B[i] \supset v=1)$

$i{:}N1, j{:}N1, j \le i \rightarrow {*}\langle j,1\rangle{:}B[i] \wedge [\forall v{:}B](\langle j,v\rangle{:}B[i] \supset v=1)$

$i{:}N1, j{:}N1, j \le i \rightarrow {*}S[j,1,B[i]]$ $\qquad$ $\rightarrow 1{:}B$

(3) $i{:}N1, j{:}N1, j \le i \rightarrow {*}[\exists u{:}B]S[j,u,B[i]]$

A derivation of (5) follows from the first three sequents:

$j{:}N1 \rightarrow j{:}N1$

$\rightarrow (j \le i \supset 1=1)$ $\qquad$ (from $\rightarrow 1=1$ by thinning and $\rightarrow\supset$)

$i{:}N1, j{:}N1, j \le i \rightarrow (j > i \supset 1=0)$ (from $i{:}N1, j{:}N1, j \le i, j > i \rightarrow$ by thinning and $\rightarrow\supset$)

$i{:}N1, j{:}N1, j \le i \rightarrow {*}j{:}N1 \wedge (j \le i \supset 1=1) \wedge (j > i \supset 1=0)$

(5) $i{:}N1, j{:}N1, j \le i \rightarrow {*}\langle j,1\rangle{:}B[i]$

A derivation of (6) follows:

$j \le i \rightarrow j \le i$ $\qquad$ $b=1 \rightarrow b=1$

$j \le i, {*}(j \le i \supset b=1) \rightarrow b=1$

$j \le i, (j \le i \supset b=1), {*}j{:}N1, {*}(j > i \supset b=0) \rightarrow b=1$ $\qquad$ thin

$j \le i, {*}j{:}N1 \wedge (j \le i \supset b=1) \wedge (j > i \supset b=0) \rightarrow b=1$

$j \le i, {*}\langle j,b\rangle{:}B[i] \rightarrow b=1$

$j \leq i \rightarrow *(\langle j,b\rangle{:}B[i] \supset b=1)$

$j \leq i, *b{:}B \rightarrow (\langle j,b\rangle{:}B[i] \supset b=1)$

(6) $j \leq i \rightarrow *[\forall v{:}B](\langle j,v\rangle{:}B[i] \supset v=1)$

A derivation of (4) can be adapted from the derivation of (3).

This application of Cantor's rule is typical of applications made in computer science. For example, the rule may be applied to prove that the computable real numbers, that is the sequences of 0's and 1's generated by Turing machines, cannot be enumerated by a Turing machine. In this case, as with FB, a particular enumeration of reals is defined for which the premisses of Cantor's rule can be derived.

6.4. Cantor's General Diagonal Argument Fails

A proof of Cantor's lemma within NaDSet requires a derivation of the sequent:

$\rightarrow [\forall w{:}\mathrm{MapN1R}][\exists x{:}R][\forall n{:}N1] \sim\langle n,x\rangle{:}w$

A derivation of this sequent could be provided with one application of $\rightarrow\forall$ from a derivation of the sequent:

$F{:}\mathrm{MapN1R} \rightarrow [\exists x{:}R][\forall n{:}N1] \sim\langle n,x\rangle{:}F$

where 'F' is a second order parameter. Note that this sequent differs from the sequent

$\mathbf{F}{:}\mathrm{MapN1R} \rightarrow [\exists x{:}R][\forall n{:}N1] \sim\langle n,x\rangle{:}\mathbf{F}$

that is the conclusion of Cantor's rule. In the latter, **F** is any constant term in which neither of the parameters 'i' and 'r' have occurrences, while in the former, 'F' is a second order parameter.

Cantor's rule of deduction is taken to be a formalization of his diagonal argument when applied to particular maps **F**. This rule of deduction cannot be applied to the more general case in which 'F' is a second order parameter, because neither $\langle i,r\rangle{:}F \rightarrow \langle i,r\rangle{:}F$ nor $\langle i,C[F]\rangle{:}F \rightarrow \langle i,C[F]\rangle{:}F$ is an axiom of NaDSet. The first is not an axiom because 'r' is a second order parameter so that '$\langle i,r\rangle$' is not a first order term. The second is not an axiom because 'F' is a second order parameter so that '$\langle i,C[F]\rangle$' is not a first order term. Therefore, Cantor's general diagonal argument cannot be formalized within NaDSet.

It does not follow, of course, that a derivation for Cantor's lemma cannot be found in NaDSet; it only follows that Cantor's diagonal argument cannot be used for the derivation.

7. ALGEBRAIC STRUCTURES

In section A1 of [Feferman84], motivation is provided for considering set theories other than the widely accepted Zermelo-Fraenkel and Gödel-Bernays set theories. The example $\langle A, \bullet, =_A \rangle$ of structures with a commutative and associative binary operation $\bullet$ and an identity $=_A$ over a set A, is discussed. Let B be the set of all such structures, PR the Cartesian product on B, and ISO isomorphism between members of B. Then the triple ‹B, PR, ISO› is itself a member of B.

Feferman suggested the set B as a simple example of a common argument in modern algebra. A set of structures is defined, and the set itself is shown to be one of the structures under appropriate definitions of operations. Category theory provides richer examples of the argument. As Feferman noted, however, the argument cannot be formalized within the Zermelo-Fraenkel or Gödel-Bernays set theories, since these theories do not permit the triple ‹B, PR, ISO› to be a member of B. NaDSet, however, faces no such difficulty:

7.1. **Theorem**: The sequent

$$\rightarrow \langle B, PR, ISO \rangle : B$$

is derivable in NaDSet.

The definitions of B, PR and ISO, together with a sketch of how the theorem may be proved, are given below. In the definitions, the allowable notations for abstraction and second order variables will be greatly expanded to include more conventional algebraic notations. At the same time, the same notation may be used in one context for abstraction variables, in another context for second order variables, and in bold in a third context to represent variables over second order terms. Although context will always make clear the meaning of the notation, explanations will be offered at the same time to ensure that there is no misunderstanding.

Although a functional notation for NaDSet was introduced in [Gilmore86], that notation will not be used to avoid an unecessary digression. Thus, for example, a binary function is represented by a ternary relation.

7.1.1. Definition of B Structures:

$$B \textbf{ for } \{\langle A, \circ, =_A \rangle \mid BStr[A, \circ, =_A]\}, \text{ where}$$

BStr[A,∘,$=_A$] for axioms.

Here axioms is the conjunction of all the sentences listed below in 7.1.2.

In the first definition A, ∘ and $=_A$ are used as abstraction variables, while in the second definition, **A**, **∘** and **$=_A$** are used as metavariables over second order terms; that is, as variables in the metatheory of NaDSet.

The definition of B is typical of the formalization of an axiomatic theory within NaDSet. The axioms of the theory are used to define the set of structures satisfying the axioms; theorems in the axiomatic theory are sentences of NaDSet stating properties of structures in the set, or in the example at hand, stating that a particular structure is a member of the set. Further details are provided in 8.2 below.

It is important to recognize that the definition of B, like all definitions presented within NaDSet, have no existential content. 'B' is offered as an abbreviation for a rather long string of symbols that form an abstraction term of NaDSet. The role of the definition is to focus attention on the abstraction term 'B' abbreviates, and to suggest an interpretation for it. Models of NaDSet are not affected in any way by the definitions, neither increased nor decreased in number. This is in sharp contrast to the standard formalizations of axiomatic theories within first order logic. The nonlogical axioms, necessary for the formalization of an axiomatic theory, restrict the models of the first order theory.

7.1.2. Axioms

In the axioms listed here, multiple bounded quantifiers are used with their usual meaning. For example, a sentence

[∀u,v,w:**A**]**F1**

is an abbreviation for

[∀u:**A**][∀v:**A**][∀w:**A**]**F1**.

Further, **A** is used as the name of unary set, **∘** as the name of a ternary set, and **$=_A$** as the name of a binary set. The customary infix notation is used for the latter, rather than the postfix notation of NaDSet.

The axioms are in two groups:

1. Axioms asserting **∘** is a binary, commutative and associative single valued function on **A**:
   [∀u,v:**A**][∃w:**A**] <u,v,w>:**∘**

$[\forall u,v,w{:}\mathbf{A}](\langle u,v,w\rangle{:}\circ \supset \langle v,u,w\rangle{:}\circ)$

$[\forall u,v,w,wa,wb,za,zb{:}\mathbf{A}]$

$\quad(\langle u,v,wa\rangle{:}\circ \wedge \langle wa,w,za\rangle{:}\circ \wedge \langle u,wb,zb\rangle{:}\circ \wedge \langle v,w,wb\rangle{:}\circ \supset za =_{\mathbf{A}} zb)$

$[\forall u,v,wa,wb{:}\mathbf{A}](\langle u,v,wa\rangle{:}\circ \wedge \langle u,v,wb\rangle{:}\circ \supset wa =_{\mathbf{A}} wb)$

2. Axioms asserting $=_{\mathbf{A}}$ is an identity with respect to **A** and $\circ$:

$[\forall u{:}\mathbf{A}]\; u =_{\mathbf{A}} u$

$[\forall u,v{:}\mathbf{A}](u =_{\mathbf{A}} v \supset v =_{\mathbf{A}} u)$

$[\forall u,v,w{:}\mathbf{A}](u =_{\mathbf{A}} v \wedge v =_{\mathbf{A}} w \supset u =_{\mathbf{A}} w)$

$[\forall ua,ub,v,w{:}\mathbf{A}](\langle ua,v,w\rangle{:}\circ \wedge ua =_{\mathbf{A}} ub \supset \langle ub,v,w\rangle{:}\circ)$

$[\forall u,v,wa,wb{:}\mathbf{A}](\langle u,v,wa\rangle{:}\circ \wedge wa =_{\mathbf{A}} wb \supset \langle u,v,wb\rangle{:}\circ)$

7.1.3. Definitions of PRA, PR∘, and PR=.

In the following definitions, **A1**, **A2**, $\circ 1$, $\circ 2$, $=_{\mathbf{A1}}$ and $=_{\mathbf{A2}}$ are being used as metavariables over terms. These definitions, together with those for ISO and PR below, provide rich examples of the use of the generalized abstraction terms of NaDSet.

PRA[A1,A2] **for** $\{\langle\alpha 1,\alpha 2\rangle \mid \alpha 1{:}\mathbf{A1} \wedge \alpha 2{:}\mathbf{A2}\}$

PR∘[∘1,∘2] **for** $\{\langle\langle\alpha 1,\alpha 2\rangle,\langle\beta 1,\beta 2\rangle,\langle\delta 1,\delta 2\rangle\rangle \mid$
$\quad\langle\alpha 1,\beta 1,\delta 1\rangle{:}\circ 1 \wedge \langle\alpha 2,\beta 2,\delta 2\rangle{:}\circ 2\}$

PR=$[=_{\mathbf{A1}},=_{\mathbf{A2}}]$ **for** $\{\langle\langle\alpha 1,\alpha 2\rangle,\langle\beta 1,\beta 2\rangle\rangle \mid \alpha 1 =_{\mathbf{A1}} \beta 1 \wedge \alpha 2 =_{\mathbf{A2}} \beta 2\}$

7.1.4. Functor and Isomorphism

'Functor' is used in the sense of [Barr,Wells85]. It is defined in two steps. The conjunction of the following three sentences expresses that **F** is a functor from $\langle\mathbf{A1},\circ 1,=_{\mathbf{A1}}\rangle$ to $\langle\mathbf{A2},\circ 2,=_{\mathbf{A2}}\rangle$. In these definitions, **F**, **A1**, **A2**, $\circ 1$, $\circ 2$, $=_{\mathbf{A1}}$ and $=_{\mathbf{A2}}$ are used as metavariables over second order terms.

FA[F,A1,A2] for $[\forall x1{:}\mathbf{A1}][\exists x2{:}\mathbf{A2}]\,\langle x1,x2\rangle{:}\mathbf{F}$

F∘[F,A1,∘1,A2,∘2] for

$\quad[\forall x1,y1,z1{:}\mathbf{A1}][\forall x2,y2,z2{:}\mathbf{A2}](\langle x1,y1,z1\rangle{:}\circ 1 \wedge$
$\quad\quad\langle x1,x2\rangle{:}\mathbf{F} \wedge \langle y1,y2\rangle{:}\mathbf{F} \wedge \langle z1,z2\rangle{:}\mathbf{F} \supset \langle x2,y2,z2\rangle{:}\circ 2)$

F=$[\mathbf{F},\mathbf{A1},=_{\mathbf{A1}},\mathbf{A2},=_{\mathbf{A2}}]$ for

$[\forall x1,y1{:}\mathbf{A1}][\forall x2,y2{:}\mathbf{A2}](x1 =_{\mathbf{A1}} y1 \wedge \langle x1,x2\rangle{:}\mathbf{F} \wedge \langle y1,y2\rangle{:}\mathbf{F} \supset x2 =_{\mathbf{A2}} y2)$

Using these definitions, the set of functors from $\langle\mathbf{A1},\circ 1,=_{\mathbf{A1}}\rangle$ to $\langle\mathbf{A2},\circ 2,=_{\mathbf{A2}}\rangle$ is defined. In this definition, **F** is an abstraction variable.

$$FNR[\langle A1,\circ 1,=_{A1}\rangle,\langle A2,\circ 2,=_{A2}\rangle] \text{ for }$$
$$\{F \mid FA[F,A1,A2] \wedge F\circ[F,A1,\circ 1,A2,\circ 2] \wedge F{=}[F,A1,=_{A1},A2,=_{A2}]\}$$

The composition of functors on a common domain is defined next:

$$Comp[F,G,A] \text{ for } \{\langle\alpha,\beta\rangle \mid [\exists u{:}A](\langle\alpha,u\rangle{:}F \wedge \langle u,\beta\rangle{:}G)\}$$

This definition is used in the proof of the theorem in 7.3 below.

In the next definition, A1, $\circ 1$, $=_{A1}$, A2, $\circ 2$ and $=_{A2}$ are being used as abstraction variables, while w is being used as a quantifiable variable.

$$ISO \text{ for } \{\langle\langle A1,\circ 1,=_{A1}\rangle,\langle A2,\circ 2,=_{A2}\rangle\rangle \mid$$
$$[\exists w{:}FNR[\langle A1,\circ 1,=_{A1}\rangle,\langle A2,\circ 2,=_{A2}\rangle]]$$
$$Inv[w]{:}FNR[\langle A2,\circ 2,=_{A2}\rangle,\langle A1,\circ 1,=_{A1}\rangle]\}$$

where, the inverse of functors, Inv, is defined:

$$Inv[F] \text{ for } \{\langle\beta,\alpha\rangle \mid \langle\alpha,\beta\rangle{:}F\}$$

7.1.5. The product PR defined on B

In this last definition, A1, $\circ 1$, $=_{A1}$, A2, $\circ 2$, $=_{A2}$, A3, $\circ 3$, and $=_{A3}$, are being used as abstraction variables:

$$PR \text{ for }$$
$$\{\langle\langle A1,\circ 1,=_{A1}\rangle,\langle A2,\circ 2,=_{A2}\rangle,\langle A3,\circ 3,=_{A3}\rangle\rangle \mid$$
$$\langle A1,\circ 1,=_{A1}\rangle{:}B \wedge \langle A2,\circ 2,=_{A2}\rangle{:}B \wedge \langle A3,\circ 3,=_{A3}\rangle{:}B$$
$$\langle\langle A3,\circ 3,=_{A3}\rangle,\langle PRA[A1,A2],PR\circ[\circ 1,\circ 2],PR{=}[=_{A1},=_{A2}]\rangle\rangle{:}ISO\}$$

7.2. Lemma for Theorem:

The following notation will be used in the statement and derivation of the lemma, and in the derivation of the theorem: A1, $\circ 1$, $=_{A1}$, A2, $\circ 2$, and $=_{A2}$, are used as second order parameters. '1' will abbreviate the sequence A1, $\circ 1$, $=_{A1}$, and '2' the sequence A2, $\circ 2$, $=_{A2}$; '1x2' will abbreviate the sequence PRA[A1,A2], PR$\circ$[$\circ 1$,$\circ 2$], PR=[$=_{A1}$,$=_{A2}$].

Lemma: The sequent

$$\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow \langle 1x2\rangle{:}B$$

is derivable.

Let Ax[A, $\circ$, $=_A$] be one of the axioms listed in 7.1.2, and let Ax[1] be the result of replacing **A**, $\circ$, and $=_A$, respectively by A1, $\circ 1$, and $=_{A1}$. Let Ax[2] and Ax[1x2] be similarly defined. From the definition of B, and elementary logic, it follows that from derivations of

Ax[1], Ax[2] → Ax[1x2],

for each of the axioms Ax, a proof of the lemma can be obtained. Example derivations for this sequent will be provided for two axioms only, since derivations for the other axioms are similar.

7.2.1. The sequent Ax[1], Ax[2] → Ax[1x2] is derivable when Ax is the first of the axioms of group (1).

Ax[1] is [∀u:A1] [∀v:A1] [∃w:A1]‹u,v,w›:∘1,
Ax[2] is [∀u:A2] [∀v:A2] [∃w:A2]‹u,v,w›:∘2 , and
Ax[1x2] is

[∀u:PRA[A1,A2]] [∀v:PRA[A1,A2]] [∃w:PRA[A1,A2]]‹u,v,w›:PR∘[∘1,∘2]

A condensed derivation of Ax[1], Ax[2] → Ax[1x2] follows. In this derivation several applications of the rules of deduction may be represented as one application. The practice of prefixing with an * the principal formula in the conclusion of a rule is continued here. When the application of more than one rule is represented as the application of one, more than one sentence may be prefixed.

‹p1,q1,r1›:∘1, ‹p2,q2,r2›:∘2 → ‹p1,q1,r1›:∘1 ∧ ‹p2,q2,r2›:∘2

‹p1,q1,r1›:∘1, ‹p2,q2,r2›:∘2 → *‹‹p1,p2›,‹q1,q2›,‹r1,r2››:PR∘[∘1,∘2]

r1:A1, r2:A2 → ‹r1,r2›:PRA[A1,A2]

---

r1:A1, ‹p1,q1,r1›:∘1, r2:A2, ‹p2,q2,r2›:∘2 →

*[∃w:PRA[A1,A2]]‹‹p1,p2›,‹q1,q2›,w›:PR∘[∘1,∘2]

*[∃w:A1]‹p1,q1,w›:∘1, *[∃w:A2]‹p2,q2,w›:∘2 →

[∃w:PRA[A1,A2]]‹‹p1,p2›,‹q1,q2›,w›:PR∘[∘1,∘2]

with axioms: p1:A1 → p1:A1 q1:A1 → q1:A1

p2:A2 → p2:A2 q2:A2 → q2:A2

---

p1:A1, q1:A1, *Ax[1], p2:A2, q2:A2, *Ax[2] →

[∃w:PRA[A1,A2]]‹‹p1,p2›,‹q1,q2›,w›:PR∘[∘1,∘2]

Ax[1], Ax[2], *<p1,p2>:PRA[A1,A2], *<q1,q2>:PRA[A1,A2] →

[∃w:PRA[A1,A2]]<<p1,p2>,<q1,q2>,w>:PR∘[∘1,∘2]

Ax[1], Ax[2] →

*[∀u:PRA[A1,A2]] [∀v:PRA[A1,A2]] [∃w:PRA[A1,A2]]<u,v,w>:PR∘[∘1,∘2]

Ax[1], Ax[2] → Ax[1x2]

7.2.2. The sequent Ax[1], Ax[2] → Ax[1x2] is derivable when Ax is the last of the axioms of group (2).

Ax[1] is

[∀u,v,wa,wb:A1](<u,v,wa>:∘1 ∧ wa $=_{A1}$ wb ⊃ <u,v,wb>:∘1)

Ax[2] is

[∀u,v,wa,wb:A1](<u,v,wa>:∘1 ∧ wa $=_{A1}$ wb ⊃ <u,v,wb>:∘1)

Ax[1x2] is

[∀u,v,wa,wb:PRA[A1,A2]](<u,v,wa>:PR∘[∘1,∘2] ∧ <wa,wb>:PR=[$=_{A1}$, $=_{A2}$]

⊃ <u,v,wb>:PR∘[∘1,∘2])

An abbreviated derivation follows:

(<p1,q1,ra1>:∘1 ∧ ra1$=_{A1}$rb1 ⊃ <p1,q1,rb1>:∘1), <p1,q1,ra1>:∘1,

ra1$=_{A1}$rb1 → <p1,q1,rb1>:∘1 (with axioms)

(a) p1:A1, q1:A1, ra1:A1, rb1:A1, *Ax(1), <p1,q1,ra1>:∘1, ra1$=_{A1}$rb1

→ <p1,q1,rb1>:∘1

(b) p2:A2, q2:A2, ra2:A2, rb2:A2, Ax(2), <p2,q2,ra2>:∘2, ra2$=_{A2}$rb2

→ <p2,q2,rb2>:∘2

(derivation of (b) similar to that of (a))

Ax[1], Ax[2], p1:A1, p2:A2, q1:A1, q2:A2, ra1:A1, ra2:A2, rb1:A1, rb2:A2,

<p1,q1,ra1>:∘1, <p2,q2,ra2>:∘2,

ra1$=_{A1}$rb1, ra2$=_{A2}$rb2,

→ *<p1,q1,rb1>:∘1 ∧ <p2,q2,rb2>:∘2

$Ax[1], Ax[2], p1{:}A1, p2{:}A2, q1{:}A1, q2{:}A2, ra1{:}A1, ra2{:}A2, rb1{:}A1, rb2{:}A2,$

$\quad \langle p1,q1,ra1\rangle{:}\circ 1, \langle p2,q2,ra2\rangle{:}\circ 2,$

$\quad {*}\langle\langle ra1,ra2\rangle,\langle rb1,rb2\rangle\rangle{:}PR{=}[=_{A1}, =_{A2}]$

$\qquad \rightarrow \langle p1,q1,rb1\rangle{:}\circ 1 \wedge \langle p2,q2,rb2\rangle{:}\circ 2$

$Ax[1], Ax[2], p1{:}A1, p2{:}A2, q1{:}A1, q2{:}A2, ra1{:}A1, ra2{:}A2, rb1{:}A1, rb2{:}A2,$

$\quad {*}\langle\langle p1,p2\rangle,\langle q1,q2\rangle,\langle ra1,ra2\rangle\rangle{:}PR\circ[\circ 1,\circ 2]$

$\quad \langle\langle ra1,ra2\rangle,\langle rb1,rb2\rangle\rangle{:}PR{=}[=_{A1}, =_{A2}]$

$\qquad \rightarrow \langle\langle p1,p2\rangle,\langle q1,q2\rangle,\langle rb1,rb2\rangle\rangle{:}PR\circ[\circ 1,\circ 2]$

$Ax[1], Ax[2],$

$\quad {*}\langle p1,p2\rangle{:}PRA[A1,A2], {*}\langle q1,q2\rangle{:}PRA[A1,A2],$

$\quad {*}\langle ra1,ra2\rangle{:}PRA[A1,A2], {*}\langle rb1,rb2\rangle{:}PRA[A1,A2],$

$\quad \langle\langle p1,p2\rangle,\langle q1,q2\rangle,\langle ra1,ra2\rangle\rangle{:}PR\circ[\circ 1,\circ 2]$

$\quad \langle\langle ra1,ra2\rangle,\langle rb1,rb2\rangle\rangle{:}PR{=}[=_{A1}, =_{A2}]$

$\qquad \rightarrow \langle\langle p1,p2\rangle,\langle q1,q2\rangle,\langle rb1,rb2\rangle\rangle{:}PR\circ[\circ 1,\circ 2]$

$Ax[1], Ax[2] \rightarrow$

${*}[\forall u,v,wa,wb{:}PRA[A1,A2]](\langle u,v,wa\rangle{:}PR\circ[\circ 1,\circ 2] \wedge \langle wa,wb\rangle{:}PR{=}[=_{A1}, =_{A2}]$

$\qquad \supset \langle u,v,wb\rangle{:}PR\circ[\circ 1,\circ 2])$

$Ax[1], Ax[2] \rightarrow {*}Ax[1x2]$

7.3 Proof of Theorem

A derivation of $\rightarrow \langle B,PR,ISO\rangle{:}B$ can be obtained from a derivation of $\rightarrow BStr[B,PR,ISO]$ by one application of $\rightarrow\{\}$, using the definition of B. To provide a derivation for the latter sequent, it is necessary to provide a derivation for each sequent of the form

$\rightarrow Ax[B,PR,ISO]$,

where $Ax[A,\circ,=_A]$ is one of the axioms listed in 7.1.2. Derivations will be provided for two axioms in 7.3.1 and 7.3.2.

7.3.1. The sequent $\rightarrow Ax[B,PR,ISO]$ is derivable when Ax is the first of the axioms of group (1); Ax[B,PR,ISO] is $[\forall u{:}B][\forall v{:}B][\exists w{:}B]\langle u,v,w\rangle{:}PR$.

In the following derivation, $\langle 1\rangle$, $\langle 2\rangle$, and $\langle 1x2\rangle$, have the meaning given to them in section 7.2; $=_{1x2}$ will abbreviate $PR{=}[=_{A1},=_{A2}]$.

The first sequent in the derivation follows from the axioms for B structures, since $\langle 1x2\rangle$ is a B structure by lemma 7.2.

$\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow [\forall x1,y1,z1{:}PR[\langle 1\rangle,\langle 2\rangle]][\forall x2,y2,z2{:}PR[\langle 1\rangle,\langle 2\rangle]]$

$(\langle x1,y1,z1\rangle{:}PR\circ[\circ 1,\circ 2] \wedge$

$\langle x1,x2\rangle{:}=_{1x2} \wedge \langle y1,y2\rangle{:}=_{1x2} \wedge \langle z1,z2\rangle{:}=_{1x2} \supset \langle x2,y2,z2\rangle{:}PR\circ[\circ 1,\circ 2])$

(a) $\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow F\circ[=_{1x2},\langle 1x2\rangle,\langle 1x2\rangle]$

(b) $\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow FA[=_{1x2},\langle 1x2\rangle,\langle 1x2\rangle]$ similar to (a)

(c) $\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow F{=}[=_{1x2},\langle 1x2\rangle,\langle 1x2\rangle]$ similar to (a)

(d) $\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow\ =_{1x2}{:}FNR[\langle 1x2\rangle,\langle 1x2\rangle]$

(e) $\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow Inv[=_{1x2}]{:}FNR[\langle 1x2\rangle,\langle 1x2\rangle]$ similar to (d)

$\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow {*}[\exists w{:}FNR[\langle 1x2\rangle,\langle 1x2\rangle]]Inv[w]{:}FNR[\langle 1x2\rangle,\langle 1x2\rangle]$

$\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow {*}\langle\langle 1x2\rangle,\langle 1x2\rangle\rangle{:}ISO$ $\quad \langle 1\rangle{:}B \rightarrow \langle 1\rangle{:}B$ $\quad \langle 2\rangle{:}B \rightarrow \langle 2\rangle{:}B$

$\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow {*}\langle 1\rangle{:}B \wedge \langle 2\rangle{:}B \wedge \langle\langle 1x2\rangle,\langle 1x2\rangle\rangle{:}ISO$

$\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow {*}\langle\langle 1\rangle,\langle 2\rangle,\langle 1x2\rangle\rangle{:}P$ $\quad \langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow \langle 1x2\rangle{:}B$ (lemma 7.2)

$\langle 1\rangle{:}B,\ \langle 2\rangle{:}B \rightarrow {*}[\exists w{:}B]\langle\langle 1\rangle,\langle 2\rangle,w\rangle{:}PR$

$\rightarrow {*}[\forall u{:}B][\forall v{:}B][\exists w{:}B]\langle u,v,w\rangle{:}PR$

7.3.2. The sequent $\rightarrow$ Ax[B,PR,ISO] is derivable when Ax is the last of the axioms of group (1); Ax[B,PR,ISO] is

$[\forall u,v,wa,wb{:}B](\langle u,v,wa\rangle{:}PR \wedge \langle u,v,wb\rangle{:}PR \supset \langle wa,wb\rangle{:}ISO)$.

A derivation follows:

‹p3,‹p1,p2››:F, ‹‹p1,p2›,p4›:Inv[G] → (‹p3,‹p1,p2››:F ∧ ‹‹p1,p2›,p4›:Inv[G])

‹p1,p2›:PRA[A1,A2] → ‹p1,p2›:PRA[A1,A2]

‹p1,p2›:PRA[A1,A2], ‹p3,‹p1,p2››:F, ‹‹p1,p2›,p4›:Inv[G] →
*[∃u:PRA[A1,A2]](‹p3,u›:F ∧ ‹u,p4›:Inv[G])

‹p1,p2›:PRA[A1,A2], ‹p3,‹p1,p2››:F, ‹‹p1,p2›,p4›:Inv[G] →
*‹p3,p4›:Comp[F,Inv[G],PRA[A1,A2]]

p4:A4 → p4:A4

‹p1,p2›:PRA[A1,A2], ‹p3,‹p1,p2››:F, p4:A4, ‹‹p1,p2›,p4›:Inv[G] →
*[∃x2:A4]‹p3,x2›:Comp[F,Inv[G],PRA[A1,A2]]

‹p1,p2›:PRA[A1,A2], ‹p3,‹p1,p2››:F, *[∃x2:A4]‹‹p1,p2›,x2›:Inv[G] →
[∃x2:A4]‹p3,x2›:Comp[F,Inv[G],PRA[A1,A2]]

‹p1,p2›:PRA[A1,A2] → ‹p1,p2›:PRA[A1,A2]

‹p1,p2›:PRA[A1,A2], ‹p3,‹p1,p2››:F,
*[∀x1:PRA[A1,A2]][∃x2:A4]‹x1,x2›:Inv[G] →
[∃x2:A4]‹p3,x2›:Comp[F,Inv[G],PRA[A1,A2]]

*[∃x2:PRA[A1,A2]]‹p3,x2›:F,
[∀x1:PRA[A1,A2]][∃x2:A4]‹x1,x2›:Inv[G] →
[∃x2:A4]‹p3,x2›:Comp[F,Inv[G],PRA[A1,A2]]

P3:A3 → P3:A3

---

p3:A3, *[∀x1:A3][∃x2:PRA[A1,A2]]‹x1,x2›:F,
[∀x1:PRA[A1,A2]][∃x2:A4]‹x1,x2›:Inv[G] →
[∃x2:A4]‹p3,x2›:Comp[F,Inv[G],PRA[A1,A2]]

[∀x1:A3][∃x2:PRA[A1,A2]]‹x1,x2›:F,
[∀x1:PRA[A1,A2]][∃x2:A4]‹x1,x2›:Inv[G] →
*[∀x1:A3][∃x2:A4]‹x1,x2›:Comp[F,Inv[G],PRA[A1,A2]]

(a) *FA[F,<3>,<1x2>], *FA[Inv[G],<1x2>,<4>] →

*FA[Comp[F,Inv[G],PRA[A1,A2]],<3>,<4>]

(b) F∘[F,<3>,<1x2>], F∘[Inv[G],<1x2>,<4>] →

F∘[Comp[F,Inv[G],PRA[A1,A2]],<3>,<4>] similar to (a)

(c) F=[F,<3>,<1x2>], F=[Inv[G],<1x2>,<4>] →

F=[Comp[F,Inv[G],PRA[A1,A2]],<3>,<4>] similar to (a)

(d) F:FNR[<3>,<1x2>], Inv[G]:FNR[<1x2>,<4>] →

Comp[F,Inv[G],<1x2>]:FNR[<3>,<4>]

(e) G:FNR[<4>,<1x2>], Inv[F]:FNR[<1x2>,<3>] →

Inv[Comp[F,Inv[G],<1x2>]]:FNR[<4>,<3>] similar to (d)

F:FNR[<3>,<1x2>], Inv[F]:FNR[<1x2>,<3>],

G:FNR[<4>,<1x2>], Inv[G]:FNR[<1x2>,<4>] →

*[∃w:FNR[<3>,<4>]Inv[w]:FNR[<4>,<3>]

*[∃w:FNR[<3>,<1x2>]Inv[w]:FNR[<1x2>,<3>],

*[∃w:FNR[<4>,<1x2>]Inv[w]:FNR[<1x2>,<4>] →

[∃w:FNR[<3>,<4>]Inv[w]:FNR[<4>,<3>]

*<<3>,<1x2>,>:ISO, *<<4>,<1x2>>:ISO → *<<3>,<4>>:ISO

<1>:B, <2>:B, <3>:B, <4>:B, <<3>,<1x2>>:ISO, <<4>,<1x2>>:ISO → <<3>,<4>>:ISO (thin)

<1>:B, <2>:B, <3>:B, <4>:B, <<1>,<2>,<3>>:PR*, <<1>,<2>,<4>>:PR* → <<3>,<4>>:ISO

→ *[∀u,v,wa,wb:B](<u,v,wa>:PR ∧ <u,v,wb>:PR ⊃ <wa,wb>:ISO)

7.4. Category Theory

No matter how category theory is regarded, either as a theory with its foundations in set theory, or as an axiomatic theory that provides an alternative to set theory, a logic must be provided for drawing conclusions from its axioms. As the theory of B structures developed above has demonstrated, there are advantages to using NaDSet as the logic.

Category theory involves many more primitive concepts than does the theory of B structures. Only very preliminary research has been undertaken of its formalization within NaDSet. This work suggests that it may be possible to prove within NaDSet that the category of categories is itself a category. Other tentative conclusions can also be made.

The formalization of a theory, like category theory, within NaDSet has no existential implications for NaDSet. This fact may help to provide an answer to the question posed in [Blass84]: Does category theory necessarily involve existential principles that go beyond those of other mathematical disciplines? When Zermelo-Fraenkel, or Gödel-Bernays, set theory is used as a foundation for category theory, it is necessary to distinguish between small and large categories.[Mac Lane71] That may no longer be necessary when category theory is formalized within NaDSet.

8 SET THEORY

Zermelo-Fraenkel and Gödel-Bernays have come to be regarded as the standard axiomatizations of set theory. They have been shown to be equivalent, the main difference between them being that Gödel-Bernays set theory has only finitely many axioms. In this section an axiomatization of Gödel-Bernays set theory is provided within NaDSet, based on the axioms provided in [Gödel40]. The main purpose of the exercise is to provide a basis for a continuation of the discussion of the question posed in the title of the paper.

To maintain a close adherence to the notation of [Gödel40], 'Cls' and 'M' will be used as abstraction variables. Also, 'ϵ' will be used as an abstraction variable for a binary relation, and the usual infix notation for it will be employed.

Here is the definition of the set of structures satisfying the axioms of Gödel-Bernays set theory:

GBST **for** {<Cls,M ϵ> | <u>axioms</u>}

Here <u>axioms</u> are the sentences corresponding to the axioms of the four groups A, B, C and D of [Gödel40] described next.

8.1. The axioms of Gödel-Bernays set theory

Several changes of notation from [Gödel40] are necessary. The practice followed there of having upper case variables range over Cls, and lower case variables range over M will not be followed, since all the axioms involve first order quantification only. However, quantifiable variables za, zb, and zc will be used where Gödel has used A, B, and C.

The axioms of the theory use first order bounded quantifiers over M and Cls. These quantifiers are represented in NaDSet as follows:

$[\forall vr\in M]F1$ **for** $[\forall vr{:}V1](vr\in M \supset F1)$,

with $[\forall vr\in Cls]$, $[\exists vr\in M]$, and $[\exists vr\in Cls]$ similarly defined.

The identity used in Gödel-Bernays set theory is not the identity of NaDSet, but is defined here in terms of $\in$:

$x=_\epsilon y$ **for** $[\forall z\in Cls](x\in z \supset y\in z)$

Finally, to avoid confusion with the use of the ordered pair notation of NaDSet, '[x,y]' will be used where Gödel uses '‹xy›'.

<u>Group A</u>

1. $[\forall x\in M]x\in Cls$
2. $[\forall x,y\in Cls](x\in y \supset x\in M)$
3. $[\forall x,y\in Cls]([\forall u\in M](u\in x \equiv u\in y) \supset x=_\epsilon y)$
4. $[\forall x,y\in M][\exists z\in M][\forall u\in M](u\in z \equiv (u=_\epsilon x \vee u=_\epsilon y))$

<u>Group B</u>

1. $[\exists za\in Cls][\forall x,y\in M]([x,y]\in za \equiv x\in y)$
2. $[\forall za,zb\in Cls][\exists zc\in Cls][\forall u\in M](u\in zc \equiv u\in za \wedge u\in zb)$
3. $[\forall za\in Cls][\exists zb\in Cls][\forall u\in M](u\in zb \equiv \sim u\in za)$
4. $[\forall za\in Cls][\exists zb\in Cls][\forall x\in M](x\in zb \equiv [\exists y{:}M]\ [y,x]\in za)$
5. $[\forall za\in Cls][\exists zb\in Cls][\forall x,y\in M]([y,x]\in zb \equiv x\in za)$
6. $[\forall za\in Cls][\exists zb\in Cls][\forall x,y\in M]([x,y]\in zb \equiv [y,x]\in za)$
7. $[\forall za\in Cls][\exists zb\in Cls][\forall x,y,z\in M]([x,y,z]\in zb \equiv [y,z,x]\in za)$
8. $[\forall za\in Cls][\exists zb\in Cls][\forall x,y,z\in M]([x,y,z]\in zb \equiv [x,z,y]\in za)$

<u>Group C</u>

1. $[\exists z\in M]([\exists u\in M]u\in z \wedge [\forall x\in M](x\in z \supset [\exists y\in M](y\in z \wedge x\subset y)))$
2. $[\forall x\in M][\exists y\in M][\forall u,v\in M](u\in v \wedge v\in x \supset u\in y)$
3. $[\forall x\in M][\exists y\in M](u \subseteq x \supset u\in y)$
4. $[\forall x\in M][\forall za\in Cls](Un[za] \supset [\exists y\in M][\forall u\in M](u\in y \equiv [\exists v\in M](v\in x \wedge [u,v]\in za)))$

In 4 of group C, the following abbreviation was used:

Un[za] for [∀u,v,w∈M]([v,u]∈za ∧ [w,u]∈za ⊃ v=$_\epsilon$w)

Axiom D

[∀za∈Cls]([∃u∈M]u∈za ⊃ [∃u∈M](u∈za ∧ [∀v∈M] ~(v∈u ∧ v∈za)))

The axiom of choice may be added to the set theory:

Axiom E

[∃za∈Cls](Un[za] ∧ [∀x∈M]([∃u∈M]u∈x ⊃ [∃y∈M](y∈x ∧ [y,x]∈za)))

8.2. The theorems of Gödel-Bernays set theory

Let now 'Cls', 'M', and '∈', be second order parameters. Using these parameters, the first two as unary, and the third as binary, a subset of the sentences of NaDSet can be defined from the logical connective ↓, and the bounded quantifiers [∀vr∈M]**F1**, [∀vr∈Cls], [∃vr∈M], and [∃vr∈Cls]. Call these sentences Gödel-Bernays sentences. The theorems of the set theory are then those Gödel-Bernays sentences **gbst** for which

⟨Cls,M,∈⟩:GBST → **gbst**

is a derivable sequent of NaDSet. But such a sequent is derivable if and only if the sequent

→ axioms ⊃ **gbst**

is derivable. The bounded quantifiers of the **gbst** sentences are first order quantifiers, with derived rules of deduction that are standard, as demonstrated in 4.3. Therefore, if **gbst** is a theorem of Gödel-Bernays in its conventional first order formalization, then it is also a theorem in the NaDSet formalization. The converse is not necessarily true.

8.3. Existence of Sets vs Correctness of Arguments

Set theories such as Gödel-Bernays were developed in response to the fact that simple instances of the naive comprehension axiom scheme

[∃y][∀x](x∈y ≡ **F1**)

are contradictions in first order logic; here **F1** is a formula in which 'x', but not 'y', occurs free. The existential quantifier [∃y] postulates the existence of the set {α|[α/x]F1} in the domain of the variables. The construction of such set theories involves a compromise between adequacy and consistency. Enough instances of the axiom scheme must be theorems to meet the goals of the theory, but not so many as to threaten consistency. As a consequence, the axioms of such set theories take on an ad hoc character.

Perhaps this is the reason Gray wrote the following in the introduction to [Gray84]: "The paradoxes of naive set theory showed that the Cantorian version was inadequate, but the various axiomatizations that soon were devised, while serving their purpose, have never been of particular interest to mathematicians. They now function mainly as talismen to ward off evil."

Instances of the naive comprehension scheme can be derived in NaDSet using the following derivable rule of deduction:

$$\frac{[p/x]\mathbf{F1} \rightarrow [p/x]\mathbf{F1}}{\rightarrow [\exists y{:}V1][\forall x{:}V1](x{:}y \equiv \mathbf{F1})}$$

Here **F1** is a formula in which 'x' is the only variable with free occurrences, and in which no second order parameter occurs; 'p' is a first order parameter not occurring in the conclusion. But the primary concern of natural deduction based set theories is not what sets exist; rather it is the characterization of correct arguments involving sets.

Consider the Russell set, defined as $\{\alpha|{\sim}\alpha{:}\alpha\}$. A correct argument can be provided to prove that 0, the null set, is a member of the Russell set, since the null set is not a member of itself. Similarly, a correct argument can be provided to prove that the universal set V1 is not a member of Russell set since it is a member of itself. Thus correct arguments involving the Russell set do exist. However, incorrect arguments involving the set also exist; for example, the arguments used to prove that the Russell set is both a member of itself and not a member of itself are not correct. But this is no reason to discard the Russell set; only the incorrect arguments need be discarded. A natural deduction based set theory such as NaDSet provides the means for determining whether or not an argument is correct.

8.4. How many real numbers are there?

In NaDSet, it can be demonstrated that there are as many real numbers as there are natural numbers, for the natural numbers can be mapped one-to-one onto the sequences ${}^{i}bb$ defined in the introduction. But Cantor's diagonal argument, as it is applied to prove Cantor's lemma, is not a correct argument. Therefore in NaDSet, it is not possible to prove, in that way, that there are more real numbers than there are natural, as these numbers are defined within NaDSet; although another way might be found.

The real numbers defined within Gödel-Bernays set theory can be shown, in that theory, to exceed the number of natural numbers defined within the theory. The diagonal argument used to prove this result can be shown to be correct within NaDSet, as can any argument deriving a theorem from the first order axioms of the theory. But the argument is only correct as it is applied within Gödel-Bernays set theory to the real numbers and natural numbers defined in that theory. The concept of real number is relative to the theory in which it is defined; the real numbers of NaDSet are not the real numbers of Gödel-Bernays set theory. Skolem's comments on his "paradox" are relevant here.

Skolem demonstrated in [Skolem22] that Zermelo-Fraenkel set theory must have a denumerable model, if it has a model at all. He then drew attention to the paradoxical nature of the result: The theory is intended as a foundation for transfinite number theory, among other concepts, yet these numbers must have a denumerable representation in a model for the theory. But he concluded:

> The explanation [for this paradoxical result] is not difficult to find. In the axiomatization, 'set' does not mean an arbitrarily defined collection; the sets are nothing but objects that are connected with one another through certain relations expressed by the axioms. Hence there is no contradiction at all if a set M of the domain B [of the set theory] is nondenumerable in the sense of the axiomatization; for this means merely that *within* B there occurs no one-to-one mapping Φ of M onto Z_0 (Zermelo's number sequence). Nevertheless, there exists the possibility of numbering all objects in B, and therefore also the elements of M, by means of the positive integers; of course, such an enumeration too is a collection of certain pairs, but this collection is not a 'set' (that is, it does not occur in the domain B).

Since the concepts of real number and enumeration are relative to the theory within which they are formalized, Skolem's "paradox" is not a contradiction, but a property of formal systems. This paper reinforces Skolem's relativistic view. It is possible to prove within the NaDSet formalization of Gödel-Bernays set theory, that no Gödel-Bernays enumeration of the Gödel-Bernays real numbers can be defined within the theory. But within NaDSet itself, it is not possible to prove by means of Cantor's general diagonal argument, that a NaDSet enumeration of the NaDSet reals cannot exist.

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