Arc Length Evolution and the Resampled Scale Space Representations

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Abstract

The curvature scale space representations of planar curves are computed by combining information about the curvature of those curves at multiple levels of detail. Similarly, curvature and torsion scale space representations of space curves are computed by combining information about the curvature and torsion of those curves at varying levels of detail.

Curvature and torsion scale space representations satisfy a number of criteria such as efficiency, invariance, detail, sensitivity, robustness and uniqueness [Mokhtarian & Mackworth 1986] which makes them suitable for recognizing a noisy curve at any scale or orientation.

The renormalized curvature and torsion scale space representations [Mackworth & Mokhtarian 1988] are more suitable for recognition of curves with non-uniform noise added to them but can only be computed for closed curves.

The resampled curvature and torsion scale space representations introduced in this paper are shown to be more suitable than the renormalized curvature and torsion scale space representations for recognition of curves with non-uniform noise added to them. Furthermore, these representations can also be computed for open curves.

A number of properties of the representation are also investigated and described. An important new property presented in this paper is that no new curvature zero-crossing points can be created in the resampled curvature scale space representation of simple planar curves.

A. Introduction

A multi-scale representation for one-dimensional functions was first proposed by Stansfield [1980] and later developed by Witkin [1983]. The function $f(x)$ is convolved with a Gaussian function as its variance σ^2 varies from a small to a large value. The zero-crossings of the second derivative of each convolved function are extracted and marked in the x–σ plane. The result is the scale space image of the function.

The curvature scale space image was introduced in [Mokhtarian & Mackworth 1986] as a new shape representation for planar curves. The representation is computed by convolving a path-based parametric representation of the curve with a Gaussian function, as the standard deviation of the Gaussian varies from a small to a large value, and extracting the curvature zero-crossing points of the resulting curves. The representation is essentially invariant under rotation, uniform scaling and translation of the curve. This and a number of other properties makes it suitable for recognizing a noisy curve at any scale or orientation.

Mackworth and Mokhtarian [1988] introduced a modification of the curvature scale space image referred to as the renormalized curvature scale space image. This representation is computed in a similar fashion but the curve is reparametrized by arc length after convolution. As was demonstrated in [Mackworth & Mokhtarian 1988], the renormalized curvature scale space image is more suitable for recognizing a curve with non-uniform noise added to it. However, unlike the regular curvature scale space representation, the renormalized curvature scale space applies only to closed curves.

In this paper, we introduce a further refinement of the curvature scale space representation to which we refer as the *resampled* curvature scale space representation. It is shown that the resampled curvature scale space is even more suitable than the renormalized curvature scale space for recognition of curves with non-uniform noise added to them. Furthermore, the resampled curvature scale space can be computed for open as well as closed curves.

The properties of the new representation are also explored in this paper. It is shown that all the properties previously shown to be true about the regular and renormalized curvature scale space representations are also true about the resampled curvature scale space representation. A new property of the representation is also described. It is shown that no new curvature zero-crossing points can exist in the resampled curvature scale space image of simple curves.

In the rest of this paper, sections starting with **B** are devoted to planar curves and those starting with **C** are devoted to space curves. Section **B.I** reviews multi-scale representations of planar curves already proposed. Section **B.II** introduces the resampled curvature scale space representation for planar curves. Section **B.III** describes the arc length evolution properties of planar curves and section **B.IV** discusses the significance of the results described in section **B.III**. Similarly, section **C.I** reviews multi-scale representations of space curves already proposed, section **C.II** introduces the resampled curvature and torsion scale space representation for space curves, section **C.III** describes the arc length evolution properties of space curves and section **C.IV** discusses the importance of the results described in section **C.III**. Section **D** presents the conclusions of this paper.

B.I. Multi-Scale Representations of Planar Curves

A planar curve is the set of points whose position vectors are the values of a continuous vector-valued and locally one-to-one function. It can be represented by the parametric vector equation

$$\mathbf{r}(u) = (x(u), y(u)). \tag{1}$$

The function $\mathbf{r}(u)$ is a parametric representation of the curve. A planar curve has an infinite number of distinct parametric representations. A parametric representation in which the parameter is the arc length s is called a *natural* parametrization of the curve. A natural parametrization can be computed from an arbitrary parametrization using the following equation

$$s = \int_0^u |\dot{\mathbf{r}}(v)| dv.$$

It can be shown that the curvature $\kappa(u)$ of a planar curve is given by:

$$\kappa(u) = \frac{\dot{x}(u)\ddot{y}(u) - \dot{y}(u)\ddot{x}(u)}{(\dot{x}(u)^2 + \dot{y}(u)^2)^{3/2}}.$$

Therefore it is possible to compute the curvature of a planar curve from its parametric representation.

Given a planar curve

$$\Gamma = \{(x(w), y(w)) | w \in [0,1]\}$$

where w is the normalized arc length parameter, an *evolved* version of that curve is defined by

$$\Gamma_\sigma = \{(X(u,\sigma), Y(u,\sigma)) | u \in [0,1]\}$$

where

$$X(u,\sigma) = x(u) \circledast g(u,\sigma)$$

$$Y(u,\sigma) = y(u) \circledast g(u,\sigma)$$

and

$$g(u,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-u^2}{2\sigma^2}}.$$

The curvature of Γ_σ is:

$$\kappa(u,\sigma) = \frac{X_u(u,\sigma)\, Y_{uu}(u,\sigma) - X_{uu}(u,\sigma)\, Y_u(u,\sigma)}{(X_u(u,\sigma)^2 + Y_u(u,\sigma)^2)^{3/2}}.$$

The function defined implicitly by

$$\kappa(u,\sigma) = 0$$

is the *curvature scale space image* of Γ [Mokhtarian & Mackworth 1986]. Figure **1**(a) shows a planar curve depicting the shoreline of Africa. Figure **2**(a) shows the curvature scale space of that curve.

Mackworth and Mokhtarian [1988] observed that although w is the normalized arc length parameter on the original curve Γ, the parameter u is **not**, in general, the normalized arc length parameter on the evolved curve Γ_σ. Figure **1**(b) shows the shoreline of Africa with noise added to its lower half. Figure **2**(b) shows the curvature scale space of that curve. A comparison of figures **2**(a) and **2**(b) shows that there does not exist a good match of one curvature scale space image to the other. To overcome this problem, Mackworth and Mokhtarian [1988] proposed the *renormalized* curvature scale space image.

Let

$$\mathbf{R}(u,\sigma) = (X(u,\sigma), Y(u,\sigma))$$

and

$$w = \Phi_\sigma(u)$$

where

$$\Phi_\sigma(u) = \frac{\int_0^u |\mathbf{R}_v(v,\sigma)|\,dv}{\int_0^1 |\mathbf{R}_v(v,\sigma)|\,dv}.$$

Now define

$$\hat{X}(w,\sigma) = X(\Phi_\sigma^{-1}(w),\sigma)$$

$$\hat{Y}(w,\sigma) = Y(\Phi_\sigma^{-1}(w),\sigma)$$

That is, each evolved curve Γ_σ is reparametrized by its normalized arc length parameter w.

The function defined implicitly by

$$\kappa(w,\sigma) = 0$$

is the *renormalized* curvature scale space image of Γ. Figure **3**(**a**) shows the renormalized curvature scale space of Africa and figure **3**(**b**) shows the renormalized curvature scale space of noisy Africa. It can be seen that the degree of match of figure **3**(a) to figure **3**(**b**) is much better than the degree of match of figure **2**(a) to figure **2**(b).

B.II. The resampled curvature scale space of planar curves

Note that as a planar curve evolves according to the process defined in section **B.I**, the parametrization of its coordinate functions $x(u)$ and $y(u)$ does not change. In other words, the function mapping values of the parameter u of the original coordinate functions $x(u)$ and $y(u)$ to the values of the parameter u of the smoothed coordinate functions $X(u,\sigma)$ and $Y(u,\sigma)$ is the identity function.

For both theoretical and practical reasons, it is interesting to generalize the definition of evolution so that the mapping function can be different from the identity function. Again let Γ be defined by:

$$\Gamma = \{(x(w), y(w)) | w \in [0,1]\}.$$

The generalized evolution which maps Γ to Γ_σ is now defined by:

$$\Gamma \longrightarrow \Gamma_\sigma = \{(X(W,\sigma), Y(W,\sigma)) | W \in [0,1]\}$$

where

$$X(W,\sigma) = x(W) \circledast g(W,\sigma)$$

$$Y(W,\sigma) = y(W) \circledast g(W,\sigma)$$

Note that

$$W = W(w,\sigma)$$

and $W(w,\sigma_0)$ where σ_0 is any value of σ, is a continuous and monotonic function of w. This condition is necessary to ensure physical plausibility since W is the parameter of the evolved curve Γ_σ.

A specially interesting case is when W always remains the arc length parameter as the curve evolves. When this criterion is satisfied, the evolution of Γ is reffered to as *arc length* evolution. An explicit formula for W can be derived [Gage & Hamilton 1986].

Recall equation (**1**)

$$\mathbf{r}(u) = (x(u), y(u)).$$

The Frenet equations for a planar curve are given by

$$\frac{\partial \mathbf{t}}{\partial u} = |\frac{\partial \mathbf{r}}{\partial u}|\kappa\,\mathbf{n}$$

$$\frac{\partial \mathbf{n}}{\partial u} = -|\frac{\partial \mathbf{r}}{\partial u}|\kappa\,\mathbf{t}.$$

Let $t = \sigma^2/2$. Observe that

$$\frac{\partial}{\partial t}(|\frac{\partial \mathbf{r}}{\partial u}|^2) = \frac{\partial}{\partial t}(\frac{\partial \mathbf{r}}{\partial u}.\frac{\partial \mathbf{r}}{\partial u}) = 2(\frac{\partial \mathbf{r}}{\partial u}.\frac{\partial^2 \mathbf{r}}{\partial u \partial t}).$$

Note that

$$\frac{\partial \mathbf{r}}{\partial u} = |\frac{\partial \mathbf{r}}{\partial u}|\,\mathbf{t}$$

and

$$\frac{\partial \mathbf{r}}{\partial t} = \kappa\,\mathbf{n}$$

since the Gaussian function satisfies the heat equation. It follows that

$$\frac{\partial}{\partial t}(|\frac{\partial \mathbf{r}}{\partial u}|^2) = 2(|\frac{\partial \mathbf{r}}{\partial u}|\,\mathbf{t}.\frac{\partial}{\partial u}(\kappa\mathbf{n})) = 2(|\frac{\partial \mathbf{r}}{\partial u}|\,\mathbf{t}.(\frac{\partial \kappa}{\partial u}\mathbf{n} - |\frac{\partial \mathbf{r}}{\partial u}|\kappa^2\mathbf{t})) = -2|\frac{\partial \mathbf{r}}{\partial u}|^2\kappa^2.$$

Therefore

$$2|\frac{\partial \mathbf{r}}{\partial u}|\;\frac{\partial}{\partial t}|\frac{\partial \mathbf{r}}{\partial u}| = -2|\frac{\partial \mathbf{r}}{\partial u}|^2\kappa^2$$

or

$$\frac{\partial}{\partial t}|\frac{\partial \mathbf{r}}{\partial u}| = -|\frac{\partial \mathbf{r}}{\partial u}|\kappa^2.$$

Let L denote the length of the curve. Now observe that

$$\frac{\partial L}{\partial t} = \int_0^L \frac{\partial}{\partial t}|\frac{\partial \mathbf{r}}{\partial u}|\,du = -\int_0^L |\frac{\partial \mathbf{r}}{\partial u}|\kappa^2\,du = -\int_0^1 \kappa^2\,dw.$$

Since the value w_0 of the normalized arc length parameter w at a point P measures the length of the curve from the starting point to point P, it follows that

$$\frac{\partial W}{\partial t} = -\int_0^W \kappa^2(W,t)\, dW$$

and therefore

$$W(w,t) = -\int_0^t \int_0^W \kappa^2(W,t)\, dW\, dt. \tag{2}$$

Note that

$$W(w,0) = w.$$

The function defined implicitly by

$$\kappa(W,\sigma) = 0$$

is the *resampled* curvature scale space of Γ.

Since the function $\kappa(W,t)$ in (**2**) is unknown, $W(w,t)$ can not be computed directly from (**2**). However, the resampled curvature scale space can be computed in a simple way: A Gaussian filter based on a small value of the standard deviation is computed. The curve Γ is parametrized by the normalized arc length parameter and convolved with the filter. The resulting curve is reparametrized by the normalized arc length parameter and convolved again with the same filter. This process is repeated until the curve is convex and no longer has any curvature zero-crossing points. The curvature zero-crossings of each curve are marked in the resampled curvature scale space image.

Figure **4(a)** shows the resampled curvature scale space of Africa and figure **4(b)** shows the resampled curvature scale space of noisy Africa. Note that a very close match can be observed when matching figure **4(a)** to figure **4(b)**.

B.III. Arc length evolution properties of planar curves

This section contains a number of results on the arc length evolution of planar curves as defined in section **B.II.** Some of the results are generalizations of the results obtained for an earlier formulation of evolution of planar curves [Mackworth & Mokhtarian 1988] and others are new results.

The first five lemmas express a number of fundamental properties of arc length evolution.

Lemma 1. Arc length evolution of a planar curve is invariant under rotation, uniform scaling and translation of the curve.

Proof: It will be shown that arc length evolution is invariant under a general affine transform. Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma))$ be an arc length evolved version of $\Gamma = (x(w), y(w))$. If Γ_σ is transformed according to an affine transform, then its new coordinates, X_1 and Y_1, are given by

$$X_1(W,\sigma) = aX(W,\sigma) + b\,Y(W,\sigma) + c$$

$$Y_1(W,\sigma) = dX(W,\sigma) + e\,Y(W,\sigma) + f$$

Now suppose Γ is transformed according to an affine transform and then evolved. The coordinates X_2 and Y_2 of the new curve are

$$X_2(W,\sigma) = (ax(W) + b\,y(W) + c) \circledast g(W,\sigma)$$

$$Y_2(W,\sigma) = (dx(W) + e\,y(W) + f) \circledast g(W,\sigma)$$

Since the convolution operator is distributive [Kecs 1982], it follows that

$$X_2(W,\sigma) = X_1(W,\sigma)$$

$$Y_2(W,\sigma) = Y_1(W,\sigma)$$

and the lemma follows. □

Lemma 2. A closed planar curve remains closed during arc length evolution.

Proof: Let $\Gamma = (x(w),y(w))$ be a closed curve and let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma)$ be an arc length evolved version of Γ. On Γ:

$$(x(0), y(0)) = (x(1), y(1))$$

therefore on Γ_σ:

$$(X(0,\sigma), Y(0,\sigma)) = (X(1,\sigma), Y(1,\sigma))$$

and the lemma follows. □

Lemma 3. A connected planar curve remains connected during arc length evolution.

Proof: Let $\Gamma = (x(w),y(w))$ be a connected planar curve and $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma))$ be an arc length evolved version of that curve. Since Γ is connected, $x(w)$ and $y(w)$ are continuous functions and therefore $X(W,\sigma)$ and $Y(W,\sigma)$ are also continuous. The rest of the proof is similar to that of lemma **2** in [Mackworth & Mokhtarian 1988]. □

Lemma 4. The center of mass of a planar curve is invariant during arc length evolution.

Proof: Let M be the center of mass of $\Gamma = (x(w),y(w))$ with coordinates (x_M, y_M). Then

$$x_M = \frac{\int_0^1 x(w)\,dw}{\int_0^1 dw} = \int_0^1 x(w)\,dw$$

$$y_M = \frac{\int_0^1 y(w)\,dw}{\int_0^1 dw} = \int_0^1 y(w)\,dw$$

Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma))$ be an arc length evolved version of Γ with $N = (X_N, Y_N)$ as its center of mass. Observe that

$$X_N = \int_0^1 X(W,\sigma)\,dW = \int_0^1 \int_{-\infty}^{\infty} g(v,\sigma)\,x(W-v)\,dv\,dW = \int_{-\infty}^{\infty} g(v,\sigma)\Big(\int_0^1 x(W-v)\,dW\Big)dv$$

W covers Γ_σ exactly once, therefore

$$\int_0^1 x(W-v)\,dW = x_M.$$

So

$$X_N = x_M.$$

Similarly

$$Y_N = y_M.$$

It follows that M and N are the same point. □

Lemma 5. Let Γ be a closed planar curve and let G be its convex hull. Γ remains inside G during arc length evolution.

Proof: Since G is simple and convex, every line L tangent to G contains that curve in the left (or right) half-plane it creates. Since Γ is inside G, Γ is also contained in the same half-plane. Now rotate L and Γ so that L becomes parallel to the y-axis. L is now described by the equation $x = c$. Since L does not intersect Γ, it follows that $x(w_0) \geq c$ for every point w_0 on Γ. Let Γ_σ be an arc length evolved version of Γ. Every point of Γ_σ is a weighted average of all the points of Γ. Therefore $X(W_0,\sigma) \geq c$ for every point W_0 on Γ_σ and Γ_σ is also contained in the same half-plane. This result holds for *every* line tangent to G therefore Γ_σ is contained inside the intersection of all the left (or right) half-planes created by the tangent lines of G. It follows that Γ_σ is also inside G. □

Theorem 1. Let $\Gamma = (x(w), y(w))$ be a planar curve in C_1 and let $x(w)$ and $y(w)$ be polynomial functions representing the arc length parametrization of Γ. A single point on one curvature zero-crossing contour in the resampled curvature scale space image of Γ determines Γ uniquely up to uniform scaling, rotation and translation (except on a set of measure zero).

Proof: The proof of this theorem is similar to the proof of theorem **1** in [Mokhtarian 1988b]. Only the differences will be explained here. Recall that derivatives at one point (at any scale) on any curvature zero-crossing contour in the curvature scale space of Γ were computed and it was shown that the resulting equations can be solved for the coefficients of expansion of the curvature function of Γ in functions related to the Hermite polynomials.

As before, we choose a point on a zero-crossing contour at any scale of the resampled curvature scale space image of Γ and compute the necessary derivatives. The value of σ in the resulting equations is then set to zero. Consequently, the arc length evolved curve Γ_σ, where σ corresponds to the scale at which the derivatives were computed, is reconstructed modulus uniform scaling, rotation and translation.

The next step is to recover the original curve Γ. This is done by applying *reverse* arc length evolution to Γ_σ. Let the arc length evolved curve Γ_σ be defined by:

$$\Gamma_\sigma = \{(X(W,\sigma), Y(W,\sigma)) \,|\, W \in [0,1]\}$$

A reverse arc length evolved curve Γ is defined by:

$$\Gamma = \{(x(w), y(w)) \,|\, w \in [0,1]\}$$

where

$$x(w) = X(w,\sigma) \circledast D_N(w,\sigma)$$

$$y(w) = Y(w,\sigma) \circledast D_N(w,\sigma)$$

where D_N is a deblurring operator defined in [Hummel *et al.* 1987] and

$$w(W,t) = \int_0^t \int_0^w \kappa^2(w,t)\,dw\,dt$$

where $t = \sigma^2/2$. As a result, Γ is recovered modulus uniform scaling, rotation and translation. □

Theorem 2. Let Γ be a planar curve in C_2. If all arc length evolved curves Γ_σ are in C_2, then all extrema of contours in the resampled curvature scale space image of Γ are maxima.

Proof: Since by assumption all arc length evolved curves Γ_σ are in C_2, the conditions of the implicit function theorem are satisfied on contours $\kappa(W,t) = 0$ in the resampled curvature scale space image of Γ and the proof is similar to the proof of theorem **1** in [Mackworth & Mokhtarian 1988]. □

Theorem 3. Let $\Gamma = (x(w), y(w))$ be a planar curve in C_1 and let $x(w)$ and $y(w)$ be polynomial functions of w. Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma))$ be an arc length evolved version of Γ with a cusp point at W_0. There is a $\delta>0$ such that $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of point W_0.

Proof: Theorem **2** in [Mackworth & Mokhtarian 1988] showed theorem **3** to be true about *any* parametrization of the curve therefore it must also be true about arc length parametrization or close approximations. □

Theorem 4. Simple curves remain simple during arc length evolution.

Proof: Assume by contradiction that Γ is a simple curve which intersects itself during arc length evolution. Γ must touch itself at point P before self-intersection. Let Γ_σ be the first arc length evolved version of Γ which touches itself such that $\Gamma_{\sigma+\delta}$ is self-intersecting. There are two distinct, non-overlapping neighborhoods of Γ_σ which contain point P. Let these neighborhoods be S_1 and S_2. Let u=0 at point P. It follows that S_1 and S_2 can be approximated using the lowest non-zero terms in the polynomial representation of their coordinate functions:

$$S_1 = (u^m, u^n)$$

$$S_2 = (u^p, u^q)$$

Assume w.l.o.g. that P is at the origin. It follows that m, n, p and q are at least equal to one. Assume further w.l.o.g. that $n>m$ and $q>p$. We will now find approximations to arc length parametrizations of S_1 and S_2. On segment S_1:

$$X(u,\sigma) = u^m$$

$$Y(u,\sigma) = u^n$$

Therefore

$$X_u(u,\sigma) = mu^{m-1}$$

$$Y_u(u,\sigma) = nu^{n-1}$$

and

$$s = \int_0^u \sqrt{X_u^2 + Y_u^2}\, du = \int_0^u \sqrt{m^2u^{2(m-1)} + n^2u^{2(n-1)}}\, du = \int_0^u mu^{m-1}\left(1 + \frac{n^2}{m^2}u^{2(n-m)}\right)^{1/2} du$$

It follows from Taylor's theorem that about u=0:

$$\left(1 + \frac{n^2}{m^2}u^{2(n-m)}\right)^{1/2} \simeq 1 + \frac{n^2}{2m^2}u^{2(n-m)} .$$

Hence

$$s = \int_0^u (mu^{m-1} + \frac{n^2}{2m}u^{2n-m-1})du = u^m + \frac{n^2}{4nm-2m^2}u^{2n-m}.$$

It follows from the assumption $n>m$ that $2n-m>m$. Therefore s can be approximated as:

$$s \simeq u^m$$

so

$$u \simeq (s)^{1/m}$$

and

$$u^n \simeq (s)^{n/m}$$

and an approximation to the arc length parametrization of S_1 in a neighborhood of P is given by:

$$X(s,\sigma) = s$$

$$Y(s,\sigma) = (s)^{n/m}$$

Now let $r = s-1$. Then

$$X(r,\sigma) = r+1$$

$$Y(r,\sigma) = (r+1)^{n/m}$$

It follows from Taylor's theorem that about $r=0$:

$$Y(r,\sigma) \simeq 1 + \frac{n}{m}r + \frac{n}{2m}(\frac{n}{m}-1)r^2.$$

Therefore the new arc length parametrization of S_1 is given by:

$$X(r,\sigma) = 1 + r$$

$$Y(r,\sigma) = 1 + \frac{n}{m}r + \frac{n}{2m}(\frac{n}{m}-1)r^2.$$

We now deblur this arc length parametrization of S_1 by an infinitesimal amount t. This is done by convolving each of X_1 and Y_1 with the function $\frac{2}{\sqrt{\pi}}e^{-v^2}(1-v^2)$, an approximation to the deblurring operator derived in [Hummel *et al.* 1987]. This approximation is good for small values of t, the scale factor controlling the amount of deblurring. Note that $t = \sigma^2/2$. On the deblurred segment:

$$X_1(r,\sigma) = 1 + r$$

$$Y_1(r,\sigma) = 1 + \frac{n}{m}r + \frac{n}{2m}(\frac{n}{m}-1)(r^2 - 2t).$$

Similarly, an arc length parametrization for the deblurred segment S_2 is given by:

$$X_2(r,\sigma) = 1 + r$$

$$Y_2(r,\sigma) = 1 + \frac{q}{p}r + \frac{q}{2p}(\frac{q}{p}-1)(r^2 - 2t).$$

Let $\frac{n}{m}$ be larger than $\frac{q}{p}$. It follows that at $r=0$, Y_1 is less than Y_2. However, as r

grows, Y_1 becomes larger than Y_2. Therefore the curve intersects itself just before touching itself. This is a contradiction of the assumption that the curve was simple before touching itself. It follows that a simple curve remains simple during arc length evolution. $\square$

Theorem 5: Let $\Gamma = (x(w), y(w))$ be a planar curve in C_1 and let $x(w)$ and $y(w)$ be polynomial functions of w. Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma))$ be an arc length evolved version of Γ with a cusp point at W_0. There is a $\delta>0$ such that $\Gamma_{\sigma+\delta}$ has two new curvature zero-crossings in a neighborhood of W_0.

Proof: It will be shown that this theorem holds for an arbitrary parametrization of Γ_σ. Therefore it must also be true of arc length parametrization or close approximations.

Let $(x(w), y(w))$ be an arbitrary parametrization of Γ_σ with a cusp point at u_0. Using a case analysis similar to the one in the proof of theorem **2** in [Mackworth & Mokhtarian 1988] to characterize all possible kinds of singularities of Γ_σ at u_0, we can again conclude that only the singular points in cases **1** and **4** are cusp points. In case **1**, the curve is approximated by (u^m, u^n) in a neighborhood of u_0 where m and n are both even. As shown in the proof of theorem **2** in [Mackworth & Mokhtarian 1988], this type of cusp point can not arise on Γ_σ if Γ is in C_1. We now turn to the cusp points of case **4**. Recall that in case **4**, the curve Γ_σ is approximated, in a neighborhood of u_0, by (u^m, u^n) where m is even and n is odd. Observe that

$$\dot{x}(u) = m\,u^{m-1} \qquad \ddot{x}(u) = m(m-1)\,u^{m-2}$$

$$\dot{y}(u) = n\,u^{n-1} \qquad \ddot{y}(u) = n(n-1)\,u^{n-2}$$

and

$$\kappa(u) = \frac{\dot{x}(u)\,\ddot{y}(u) - \dot{y}(u)\,\ddot{x}(u)}{(\dot{x}(u)^2 + \dot{y}(u)^2)^{3/2}} = \frac{mn(n-1)\,u^{m+n-3} - m(m-1)\,n\,u^{m+n-3}}{(m^2u^{2m-2} + n^2u^{2n-2})^{3/2}}$$

Since $n>m$, $\kappa(u)$ is always positive on either side of the cusp point in a neighborhood of u_0. Therefore no curvature zero-crossings exist in that neighborhood on Γ_σ.

We now derive analytical expressions for $\Gamma_{\sigma+\delta}$ so that it can be analyzed in a neighborhood of u_0. To blur function $f(u) = u^k$, we convolve a rescaled version of that function with the function $\frac{1}{\sqrt{\pi}}e^{-x^2}$, the deblurring operator, as follows:

$$F(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} f(u+2x\sqrt{t})\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}(u+2x\sqrt{t})^k\,dx$$

where t is the scale factor and controls the amount of blurring. Solving the integral above yields

$$F(u) = \sum_{\substack{p=0 \\ (p\ even)}}^{k} 1.3.5 \cdots (p\text{-}1) \frac{(2t)^{p/2} k(k-1) \cdots (k-p+1)}{p!} u^{k-p}.$$

The following are four functions of the form $f(u) = u^k$ and their blurred versions:

a. $f(u) = u^2$ $\quad F(u) = u^2 + 2t$
b. $f(u) = u^3$ $\quad F(u) = u^3 + 6tu$
c. $f(u) = u^4$ $\quad F(u) = u^4 + 12tu^2 + 12t^2$
d. $f(u) = u^5$ $\quad F(u) = u^5 + 20tu^3 + 60t^2u$

An expression for $\Gamma_{\sigma+\delta}$ in a neighborhood of the cusp point can be obtained by by blurring each of its coordinate functions:

$$X(u) = u^m + c_1 t u^{m-2} + c_2 t^2 u^{m-4} + \cdots + c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^2 + c_{\frac{m}{2}} t^{\frac{m}{2}}$$

$$Y(u) = u^n + c_1' t u^{n-2} + c_2' t^2 u^{n-4} + \cdots + c'_{\frac{n-1}{2}} t^{\frac{n-1}{2}} u.$$

Note that all constants are positive, all powers of u in $X(u)$ are even and all powers of u in $Y(u)$ are odd. It follows that all powers of u in

$$\dot{X}(u) = m u^{m-1} + (m-2) c_1 t u^{m-3} + \cdots + 2 c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u$$

are odd, all powers of u in

$$\ddot{X}(u) = m(m-1) u^{m-2} + (m-2)(m-3) c_1 t u^{m-4} + \cdots + 2 c_{\frac{m-2}{2}} t^{\frac{m-2}{2}}$$

are even, all powers of u in

$$\dot{Y}(u) = n u^{n-1} + (n-2) c_1' t u^{n-3} + \cdots + c'_{\frac{n-1}{2}} t^{\frac{n-1}{2}}$$

are even and all powers of u in

$$\ddot{Y}(u) = n(n-1) u^{n-2} + (n-2)(n-3) c_1' t u^{n-4} + \cdots + c'_{\frac{n-3}{2}} t^{\frac{n-3}{2}}$$

are odd.

The curvature of $\Gamma_{\sigma+\delta}$ in a neighborhood of u_0 is given by

$$\kappa(u) = \frac{\dot{X}(u)\ddot{Y}(u) - \dot{Y}(u)\ddot{X}(u)}{(\dot{X}(u)^2 + \dot{Y}(u)^2)^{3/2}}. \tag{3}$$

Since the denominator of (3) never goes to zero in a neighborhood of u_0, the zero-crossings of $\kappa(u)$ are the same as those of

$$\kappa'(u) = \dot{X}(u)\,\ddot{Y}(u) - \dot{Y}(u)\ddot{X}(u).$$

Observe that the term with the highest power of u in $\dot{X}(u)\,\ddot{Y}(u)$ is $mn(n-1)u^{m+n-3}$ and the term with the highest power of u in $\dot{Y}(u)\ddot{X}(u)$ is $m(m-1)nu^{m+n-3}$ and that in both $\dot{X}(u)\,\ddot{Y}(u)$ and $\dot{Y}(u)\ddot{X}(u)$, all powers of u are even and all constants are positive. Furthermore, note that at u=0, $\dot{X}(u)\,\ddot{Y}(u)$ is zero and $\dot{Y}(u)\ddot{X}(u) > 0$. Therefore at u=0, $\kappa < 0$. As u grows larger in absolute value, the terms in $\dot{X}(u)\,\ddot{Y}(u)$ and $\dot{Y}(u)\ddot{X}(u)$ with highest powers of u become dominant (all other terms have positive powers of t=δ in them). Since the dominant terms have equal powers of u, the one with the larger coefficient becomes the larger term. Since $n > m$, the largest term in $\dot{X}(u)\,\ddot{Y}(u)$ becomes larger than the largest term in $\dot{Y}(u)\ddot{X}(u)$. Therefore as u grows in absolute value, κ becomes positive. It follows that there are two curvature zero-crossings in a neighborhood of u_0 on $\Gamma_{\sigma+\delta}$. These zero-crossings are new since it was shown that no zero-crossings exist in a neighborhood of u_0 on Γ_σ.

This completes the proof of theorem **5**. □

B.IV. Discussion

Lemma **1** showed that arc length evolution of a planar curve is invariant under rotation, uniform scaling and translation of the curve. This shows that the resampled curvature scale space of a planar curve has the *invariance* property [Mokhtarian & Mackworth 1986]. The invariance property is essential since it makes it possible to match a planar curve to another of similar shape which has undergone a transformation consisting of arbitrary amounts of rotation, uniform scaling and translation.

Lemmas **2** and **3** showed that connectedness and closedness of a planar curve are preserved during arc length evolution. These lemmas show that arc length evolution of a planar curve is a physically plausible operation. Consider a closed, connected planar curve that represents the boundary of a two-dimensional object. If such a curve is not closed or connected after arc length evolution, then it can no longer admit a physically plausible interpretation.

Lemma **4** showed that the center of mass of a planar curve does not move as the curve evolves and lemma **5** showed that a planar curve remains inside its convex hull during arc length evolution. Together, lemmas **4** and **5** impose constraints on the physical location of a planar curve as it evolves. These constraints become useful whenever the physical location of curves in an image or their locations with respect to each other is important. A possible application area is stereo matching in which it is advantageous to carry out matching at coarser levels of detail first and then match at fine detail levels to increase accuracy.

Theorem **1** showed that the resampled curvature scale space of a planar curve determines that curve uniquely modulus uniform scaling, rotation and translation. This shows that the resampled curvature scale space satisfies the *uniqueness* property [Mokhtarian & Mackworth 1986]. This property ensures that curves of different shapes do not have the same representation.

Theorems **3** and **5** together locally characterize the behaviour of a planar curve just before and just after the formation of a cusp point during arc length evolution. This behaviour can be used to detect any cusp points that form during the arc length evolution of a planar curve. Such cusp points can then be used effectively to facilitate matching since they provide us with a set of distinctive and easily recognizable features.

Theorem **2** showed that if a planar curve remains smooth during arc length evolution, then no new curvature zero-crossings will be observed in its resampled curvature scale space image. Theorem **3** showed that every planar intersects itself during arc length evolution just before the formation of a cusp point and theorem **4** showed that simple curves remain simple during arc length evolution. Combining theorems **2, 3** and **4**, we conclude that no new curvature zero-crossing points are created during arc length evolution of simple curves. This is an important result since simple curves are a very important subclass of planar curves. Note that a subclass of self-crossing curves also shares this property.

The result stated by theorem **4** is also very important. Simple planar curves usually represent the boundaries of two-dimensional objects. Arc length evolved versions of those curves can only have physical plausibility if they are also simple. Theorem **4** shows that this is in fact the case. Figure **5** shows a simple curve and its evolved versions as defined in [Mokhtarian & Mackworth 1986]. It can be seen that the curve intersects itself during evolution. Figure **6** shows the same curve and its arc length evolved versions. As expected, the curve remains simple during arc length evolution.

C.I. Multi-Scale Representations of Space Curves

A multi-scale representation for space curves [Mokhtarian 1988c] can be obtained by generalizing the concepts described in section **B.I.** A space curve is represented by the continuous, vector-valued and locally one-to-one function

$$\mathbf{r}(u) = (x(u), y(u), z(u)).$$

An *evolved* version of a space curve

$$\Gamma = \{(x(w), y(w), z(w)) \,|\, w \in [0,1]\}$$

where w is the normalized arc length parameter, is defined by

$$\Gamma_\sigma = \{(X(u,\sigma), Y(u,\sigma), Z(u,\sigma)) \,|\, u \in [0,1]\}$$

where

$$X(u,\sigma) = x(u) \circledast g(u,\sigma)$$

$$Y(u,\sigma) = y(u) \circledast g(u,\sigma)$$

and

$$Z(u,\sigma) = z(u) \circledast g(u,\sigma).$$

It can be shown [Goetz 1970] that the curvature of each Γ_σ is given by:

$$\kappa(u,\sigma) = \frac{\sqrt{A^2 + B^2 + C^2}}{(X_u(u,\sigma)^2 + Y_u(u,\sigma)^2 + Z_u(u,\sigma)^2)^{3/2}}$$

where

$$A = \begin{vmatrix} Y_u(u,\sigma) & Z_u(u,\sigma) \\ Y_{uu}(u,\sigma) & Z_{uu}(u,\sigma) \end{vmatrix} \qquad B = \begin{vmatrix} Z_u(u,\sigma) & X_u(u,\sigma) \\ Z_{uu}(u,\sigma) & X_{uu}(u,\sigma) \end{vmatrix}$$

$$C = \begin{vmatrix} X_u(u,\sigma) & Y_u(u,\sigma) \\ X_{uu}(u,\sigma) & Y_{uu}(u,\sigma) \end{vmatrix}$$

and that the torsion of each Γ_σ is given by:

$$\tau(u,\sigma) = \frac{\begin{vmatrix} X_u(u,\sigma) & Y_u(u,\sigma) & Z_u(u,\sigma) \\ X_{uu}(u,\sigma) & Y_{uu}(u,\sigma) & Z_{uu}(u,\sigma) \\ X_{uuu}(u,\sigma) & Y_{uuu}(u,\sigma) & Z_{uuu}(u,\sigma) \end{vmatrix}}{A^2 + B^2 + C^2}.$$

The function defined implicitly by

$$\kappa(u,\sigma) = c$$

is the curvature scale space image of Γ and the function defined implicitly by

$$\tau(u,\sigma) = 0$$

is the *torsion scale space image* of Γ. The curvature and torsion scale space images of a space curve constitute a multi-scale representation of that curve.

Every evolved curve Γ_σ can be reparametrized by its normalized arc length parameter w using the function $\Phi_\sigma(u)$ defined by

$$w = \Phi_\sigma(u) = \frac{\int_0^u |\mathbf{R}_v(v,\sigma)|\,dv}{\int_0^1 |\mathbf{R}_v(v,\sigma)|\,dv}$$

where

$$\mathbf{R}(u,\sigma) = (X(u,\sigma), Y(u,\sigma), Z(u,\sigma)).$$

The function defined implicitly by

$$\kappa(w,\sigma) = c$$

is the renormalized curvature scale space image of Γ and the function defined implicitly by

$$\tau(w,\sigma) = 0$$

is the renormalized torsion scale space image of Γ.

C.II. The resampled curvature and torsion scale space of space curves

The resampled curvature and torsion scale space images of a space curve can be obtained by generalizing the concepts defined in section **B.II.**

An arc length evolved version of

$$\Gamma = \{(x(w), y(w), z(w)) | w \in [0,1]\}$$

is defined by

$$\Gamma_\sigma = \{(X(W,\sigma), Y(W,\sigma), Z(W,\sigma)) | W \in [0,1]\}$$

where

$$X(W,\sigma) = x(W) \circledast g(W,\sigma)$$

$$Y(W,\sigma) = y(W) \circledast g(W,\sigma)$$

$$Z(W,\sigma) = z(W) \circledast g(W,\sigma)$$

and $W(w,\sigma_0)$, where σ_0 is any value of σ, is a continuous and monotonic function of w. Furthermore, W always remains the arc length parameter as the curve evolves. Again, an explicit formula for W can be derived. Let

$$\mathbf{r}(u) = (x(u), y(u), z(u)).$$

The Frenet equations for a space curve are given by:

$$\frac{\partial \mathbf{t}}{\partial u} = |\frac{\partial \mathbf{r}}{\partial u}| \kappa \mathbf{n}$$

$$\frac{\partial \mathbf{n}}{\partial u} = -|\frac{\partial \mathbf{r}}{\partial u}| \kappa \mathbf{t} + |\frac{\partial \mathbf{r}}{\partial u}| \tau \mathbf{b}.$$

Let $t = \sigma^2/2$. Observe that

$$\frac{\partial}{\partial t}(|\frac{\partial \mathbf{r}}{\partial u}|^2) = \frac{\partial}{\partial t}(\frac{\partial \mathbf{r}}{\partial u}.\frac{\partial \mathbf{r}}{\partial u}) = 2(\frac{\partial \mathbf{r}}{\partial u}.\frac{\partial^2 \mathbf{r}}{\partial u \partial t}).$$

Note that

$$\frac{\partial \mathbf{r}}{\partial u} = |\frac{\partial \mathbf{r}}{\partial u}| \mathbf{t}$$

and

$$\frac{\partial \mathbf{r}}{\partial t} = \kappa \mathbf{n}$$

since the Gaussian function satisfies the heat equation. Therefore

$$\frac{\partial}{\partial t}(|\frac{\partial \mathbf{r}}{\partial u}|^2) = 2(|\frac{\partial \mathbf{r}}{\partial u}| \mathbf{t}.\frac{\partial}{\partial u}(\kappa \mathbf{n})) = 2(|\frac{\partial \mathbf{r}}{\partial u}| \mathbf{t}.(\frac{\partial \kappa}{\partial u}\mathbf{n} - |\frac{\partial \mathbf{r}}{\partial u}| \kappa^2 \mathbf{t} + |\frac{\partial \mathbf{r}}{\partial u}| \kappa \tau \mathbf{b})) = -2|\frac{\partial \mathbf{r}}{\partial u}|^2 \kappa^2$$

which is identical to what was derived in section **B.II.** Therefore it follows again that

$$W(w,t) = -\int_0^t \int_0^W \kappa^2(W,t)\, dW\, dt. \tag{2}$$

The function defined implicitly by

$$\kappa(W,\sigma) = c$$

is the resampled curvature scale space of Γ and the function defined implicitly by

$$\tau(W,\sigma) = 0$$

is the resampled torsion scale space of Γ.

In practice, however, the resampled curvature and torsion scale space images of a space curve are computed using a procedure similar to the one described in section **B.II**. The arc length evolution of the curve continues until the number of curvature level-crossings drops to zero and the number of torsion zero-crossings drops to two. The resampled curvature and torsion scale space images of a space curve together constitute a multi-scale representation of that curve.

C.III. Arc length evolution properties of space curves

This section contains a number of results on the arc length evolution of space curves. These results are generalizations of results obtained for an earlier formulation of evolution of space curves [Mokhtarian 1988c].

The first five lemmas express a number of fundamental properties of arc length evolution.

Lemma 6. Arc length evolution of a space curve is invariant under rotation, uniform scaling and translation of the curve.

Proof: Similar to proof of lemma **1** in section **B.III** and to proof of lemma **1** in [Mokhtarian 1988a]. □

Lemma 7. A closed space curve remains closed during arc length evolution.

Proof: Similar to proof of lemma **2** in section **B.III** and to proof of lemma **3** in [Mokhtarian 1988a]. □

Lemma 8. A connected space curve remains connected during arc length evolution.

Proof: Similar to proof of lemma **3** in section **B.III** and to proof of lemma **2** in [Mokhtarian 1988a]. □

Lemma 9. The center of mass of a space curve is invariant during arc length evolution.

Proof: Similar to proof of lemma **4** in section **B.III** and to proof of lemma **4** in

[Mokhtarian 1988a]. □

Lemma 10. Let Γ be a closed space curve and let G be its convex hull. Γ remains inside G during arc length evolution.

Proof: Same as proof of lemma **5** in [Mokhtarian 1988a]. □

Theorem 6. Let $\Gamma = (x(w), y(w), z(w))$ be a space curve in C_1 and let $x(w)$, $y(w)$ and $z(w)$ be polynomial functions representing the arc length parametrization of Γ. A single point on one torsion zero-crossing contour in the resampled torsion scale space image of Γ determines Γ modulus function $\beta(w) = \kappa^2(w)\tau(w)$ (except on a set of measure zero).

Proof: Similar to proof of theorem **2** in [Mokhtarian 1988b]. The only difference is that an arc length evolved curve Γ_σ (rather than Γ) is recovered, modulus function $\beta(W,\sigma)$, to which reverse arc length evolution is applied to recover Γ modulus $\beta(w)$. The procedure is similar to the one described in the proof of theorem **1** in section **B.III.** □

Theorem 7. Let $\Gamma = (x(w), y(w), z(w))$ be a space curve in C_1 and let $x(w)$, $y(w)$ and $z(w)$ be polynomial functions of w. Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma), Z(W,\sigma))$ be an arc length evolved version of Γ with a cusp point at W_0. There exists a $\delta>0$ such that either $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of point W_0, or two projections of $\Gamma_{\sigma-\delta}$ intersect themselves in a neighborhood of W_0.

Proof: Theorem **1** in [Mokhtarian 1988a] showed theorem **7** to be true of any parametrization of the curve therefore it must also be true of arc length parametrization or close approximations. □

Theorem 8. let $\Gamma = (x(w), y(w), z(w))$ be a space curve in C_1 and let $x(w)$, $y(w)$ and $z(w)$ be polynomial functions of w. Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma), Z(W,\sigma))$ be an arc length evolved version of Γ with a cusp point at W_0. There exists a $\delta>0$ such that either a torsion zero-crossing point exists at W_0 on curves $\Gamma_{\sigma-\delta}$ and $\Gamma_{\sigma+\delta}$, or $\Gamma_{\sigma+\delta}$ has two new torsion zero-crossings in a neighborhood of W_0.

Proof: Theorem **2** in [Mokhtarian 1988a] showed theorem **8** to be true of any parametrization of the curve therefore it must also be true of arc length parametrization or close approximations. □

C.IV. Discussion

The arguments made in this section are similar to some of the arguments made in section **B.IV**. Lemma **6** shows that the resampled torsion scale space of a space curve has the *invariance* property [Mokhtarian 1988c]. Lemmas **7** and **8** show that arc length evolution of a space curve is a physically plausible operation and lemmas **9** and **10** impose constraints on the physical location of a space curve during arc length evolution. Theorem 6 shows that the resampled torsion scale space of a space curve is usually sufficient to distinguish that curve from other space curves to which it is being compared. Theorems 7 and 8 together locally characterize the behaviour of a space curve just before and just after the formation of a cusp point during arc length evolution.

D. Conclusions

The concept of arc length evolution was defined in this paper and the resampled curvature and torsion scale space representations were proposed as new multi-scale representations for planar and space curves.

It was shown that the resampled curvature scale space representations are more suitable than the renormalized curvature scale space representations for matching a curve to another curve of similar shape with added non-uniform noise.

A number of arc length evolution properties of planar and space curves were also investigated in this paper. A new result obtained is that simple planar curves remain simple during arc length evolution. Combining this with other results, we conclude that no new curvature zero-crossing points can exist at higher scales in the resampled curvature scale space representation of simple curves.

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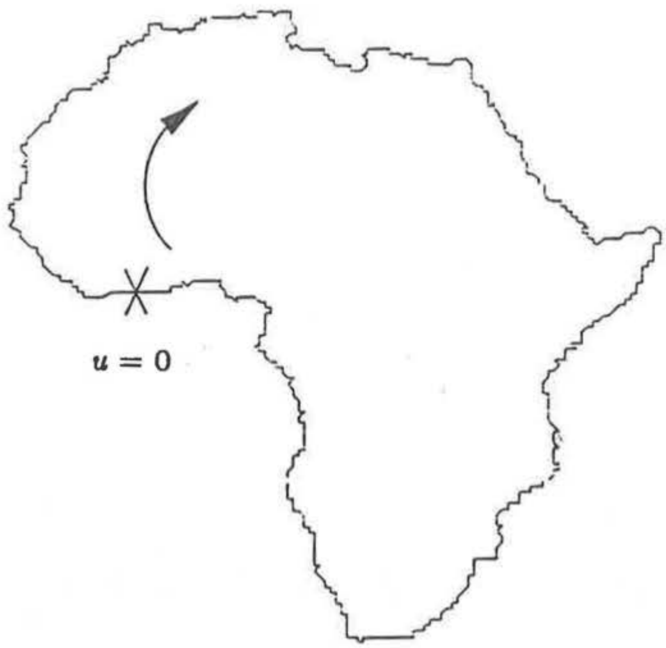
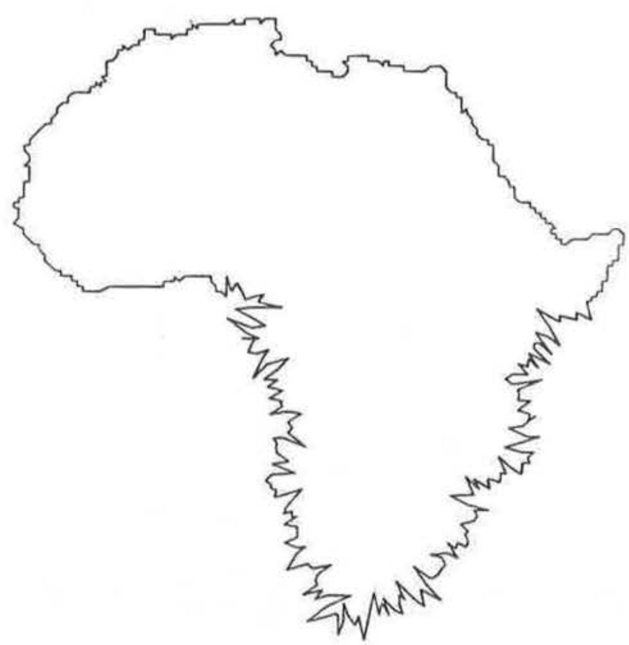
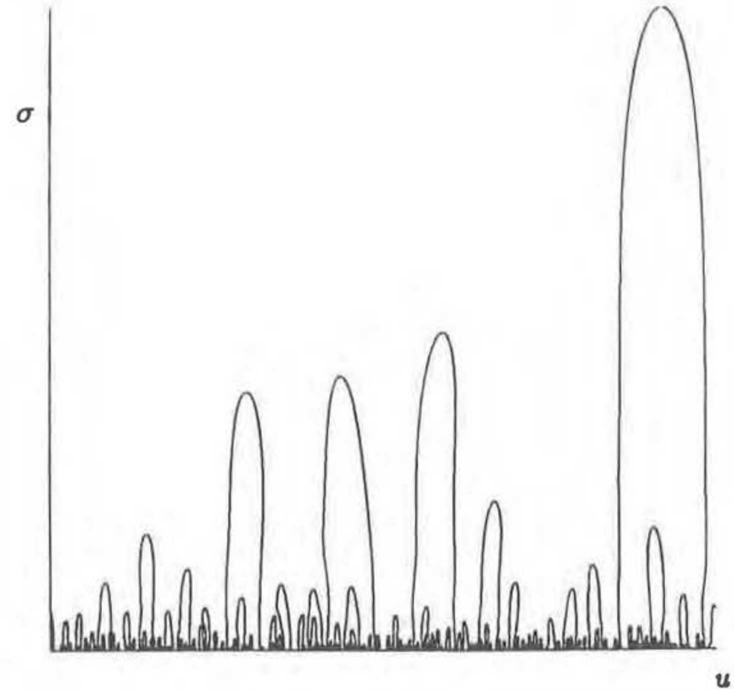


(a) Coastline of Africa.

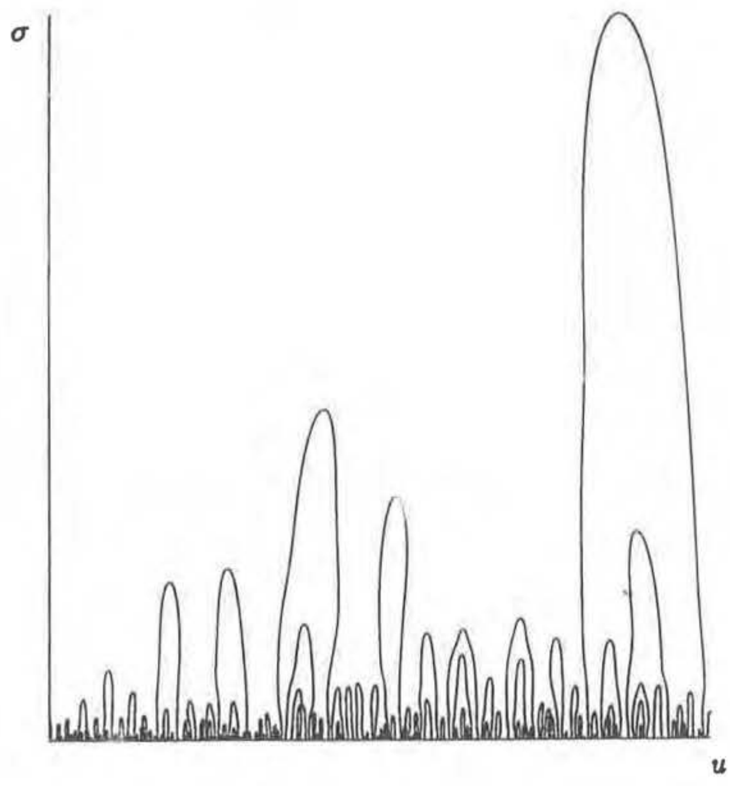


(b) Coastline of Africa with added noise.

Figure 1. Two planar curves used as test data.

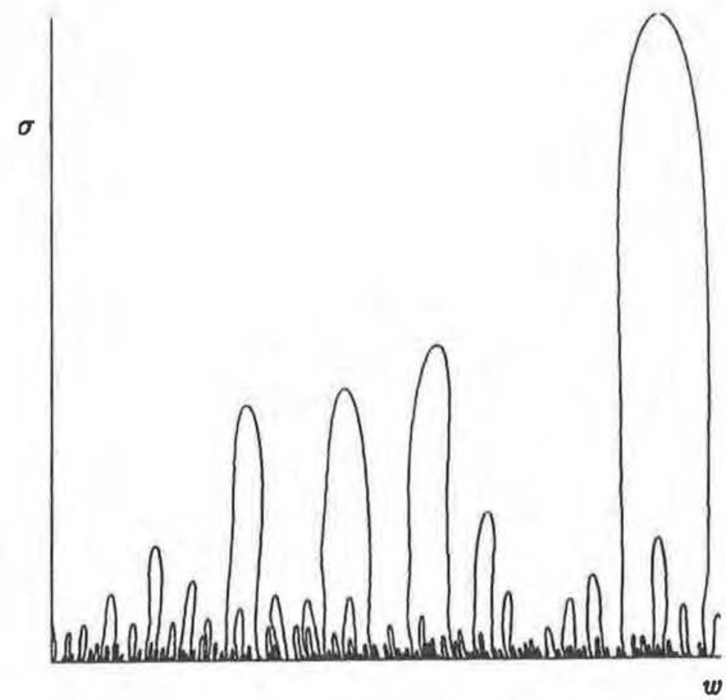


(a) The curvature scale space image of Africa.

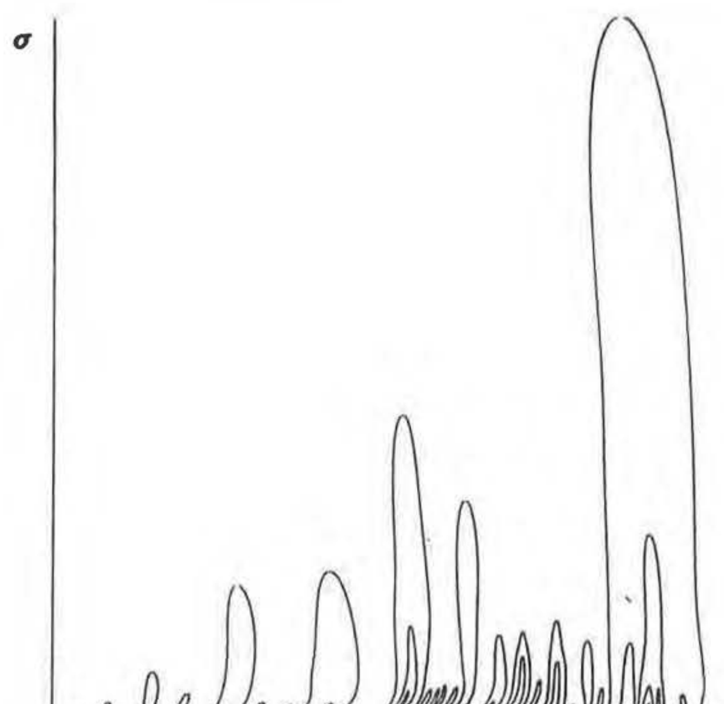


(b) The curvature scale space image of noisy Africa.

Figure 2. The curvature scale space images of Africa and noisy Africa.

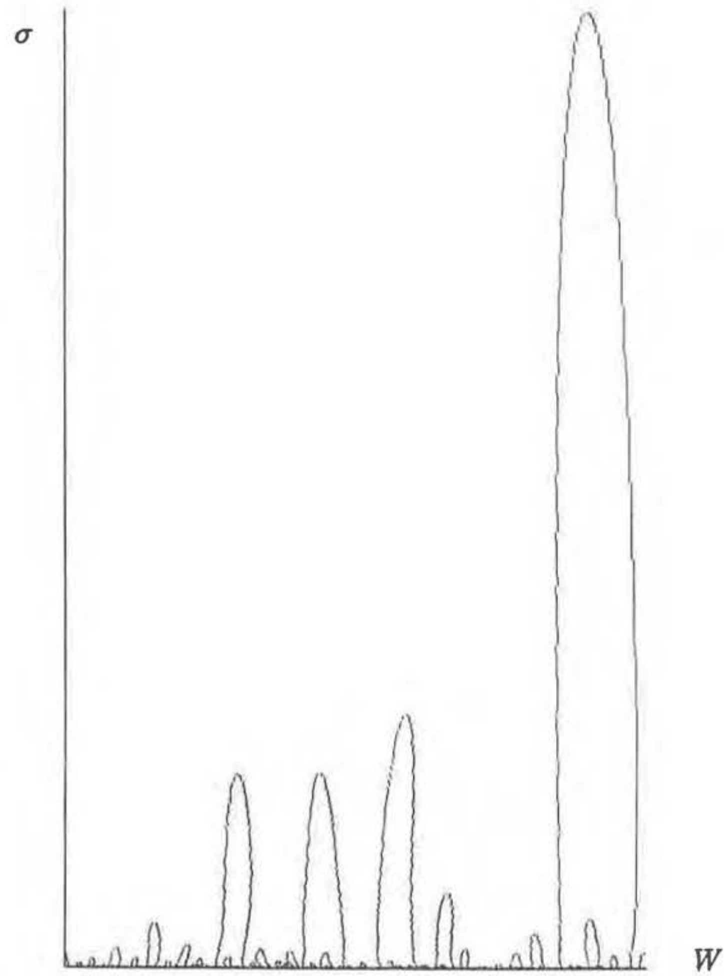


(a) The renormalized curvature scale space image of Africa.

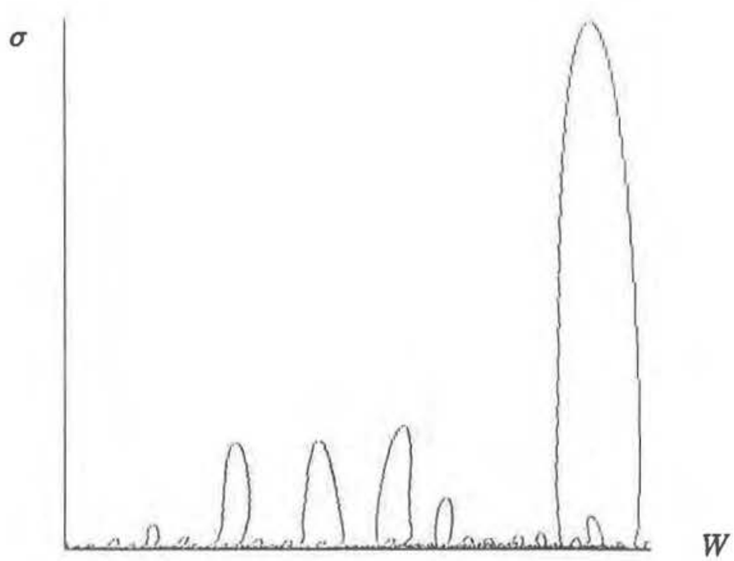


(b) The renormalized curvature scale space image of noisy Africa.

Figure 3. The renormalized curvature scale space images of Africa and noisy Africa.

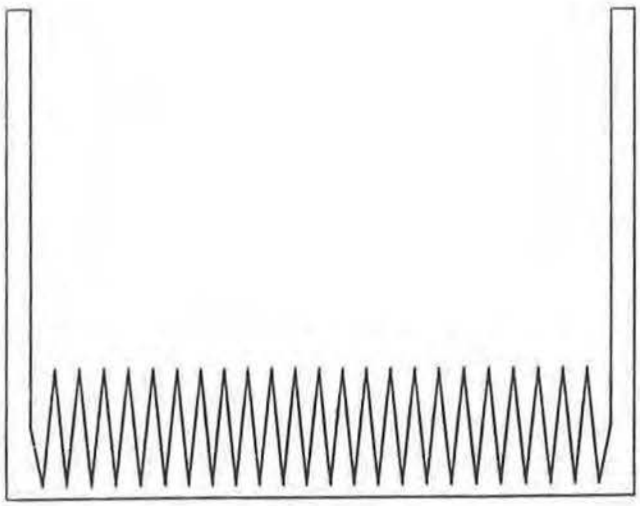


(a) The resampled curvature scale space image of Africa.

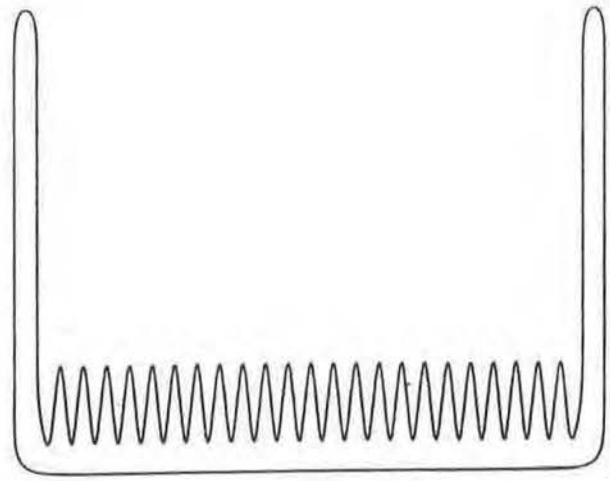


(b) The resampled curvature scale space image of noisy Africa.

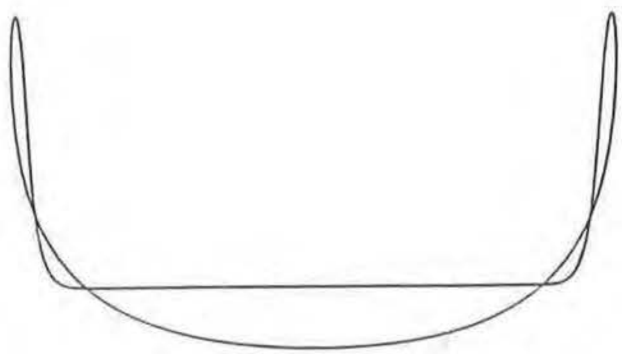
Figure 4. The resampled curvature scale space images of Africa and noisy Africa.



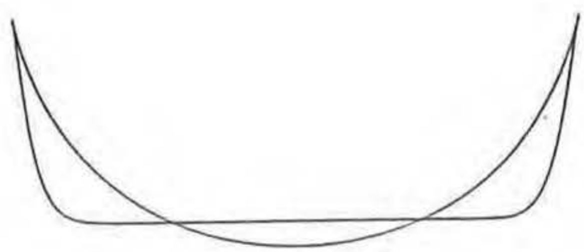
(a) A simple curve



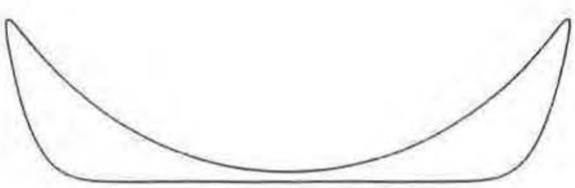
(b) Convolved with σ=4



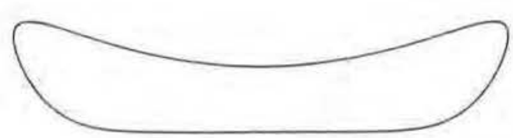
(c) Convolved with σ=16



(d) Convolved with σ=25

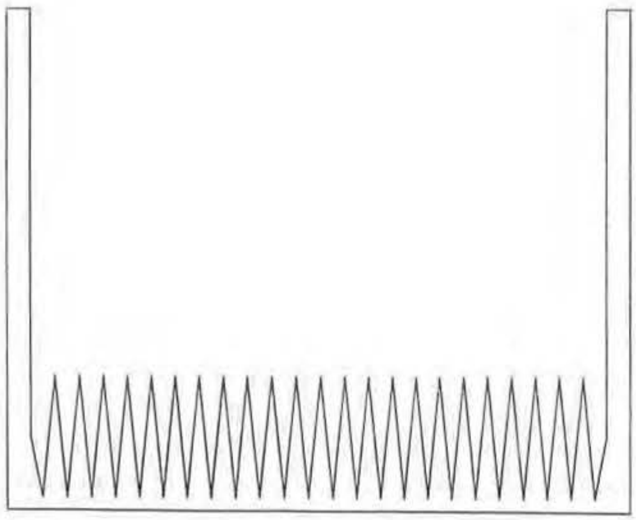


(e) Convolved with σ=32

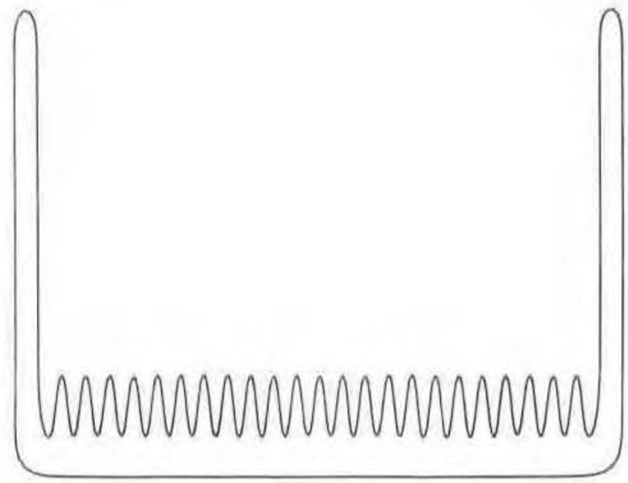


(f) Convolved with σ=48

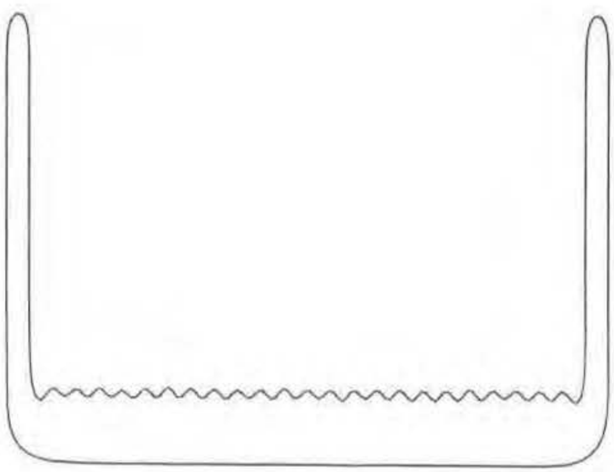
Figure 5. A simple curve during [regular] evolution.



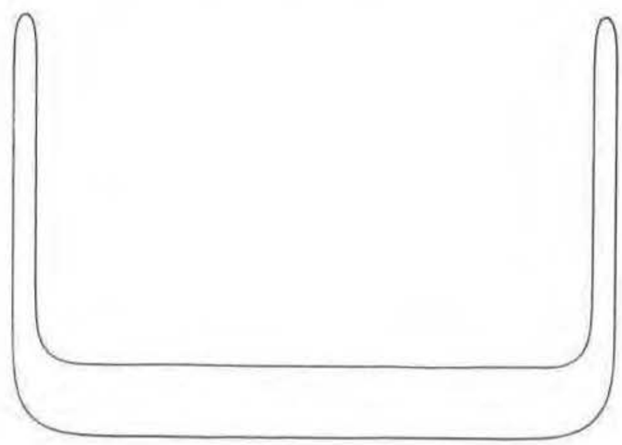
(a) A simple curve



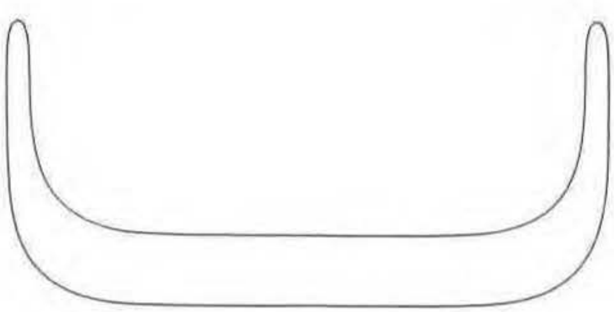
(b) After 3 iterations



(c) After 6 iterations



(d) After 10 iterations



(e) After 30 iterations

(f) After 50 iterations

Figure 6. A simple curve during arc length evolution.