INVARIANTS OF CHROMATIC GRAPHS

by
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1) On leave from Warsaw University
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ABSTRACT

In the paper we construct abstract algebras which yield invariants of graphs (including graphs with colored edges - chromatic graphs). We analyse properties of those algebras. We show that various polynomials of graphs are yielded by models of the algebras (including Tutte and matching polynomials). In particular we consider a generalization of Tutte's polynomial to a polynomial of chromatic graphs. We analyse relation of graph polynomials with recently discovered link polynomials.

It is known that computing of the Tutte polynomial is NP-hard. We show that a part of Tutte polynomial (and its generalization) can be computed faster than in exponential time.

1. Introduction

By a graph $G = (V(G), E(G))$ we understand a finite multigraph (i.e. we allow loops and multiedges). By m we denote the number of edges and by n the number of vertices of G. The number of connected components of G is denoted by p_0 and its cyclomatic number by p_1. By $G - e$ we denote the graph obtained from G by removing edge e, by G/e the graph obtained from $G - e$ by identifying the endpoints of e, and by $G//e$ the graph obtained from G by removing endpoints of edge e.

Let $\mathcal{G}$ be the family of all finite graphs and A be some set. By graph invariant we understand a function $W: \mathcal{G} \to A$ s.t. if G_1 is isomorphic to G_2 then $W(G_1) = W(G_2)$.

By chromatic graph we understand a graph with a function on edges $c: E(G) \to Z \times \{d, \ell\}$. The first element of the pair $c(e)$ will be called the color of e and the second the attribute (dark or light) of c. The family of finite chromatic graphs will be denoted by $\mathcal{G}'$.

By a plane graph we understand a planar graph together with its embedding on the plane. By a dual to a plane connected chromatic graph G we understand a graph G^* s.t. $V(G^*)$ and $E(G^*)$ are defined as for nonchromatic graphs and if $e^* \in E(G^*)$ is the dual edge to $e \in E(G)$ then we assign to e^* the same color as to e but the opposite attribute. The dual to a non-connected graph is the disjoint sum of duals to its connected components.

By an isthmus we understand an edge whose removal disconnects graph and by a loop we understand such an edge (v,w) that $v = w$. A (connected) graph whose edges are all either isthmus or loops is called a (tree) forest with loops.

The disjoint sum of the graphs G_1 and G_2 will be denoted by $G_1 \sqcup G_2$. A graph of i isolated vertices is denoted by N_i.

The paper is organised as follows:

In the next section we introduce abstract algebras for nonchromatic graphs and we show how these algebras yield known polynomials of graphs.

In the third section we generalize the algebraic approach to chromatic graphs and other graphs with some additional structure.

In the fourth section we introduce matched diagrams of links which (following the idea of Jaeger [J87]) establish a new relationship between links and graphs.

In the last section we address some computational aspects of graph polynomials.

The paper is an extended version of the manuscript [PP87]. In the meantime there appear papers dealing with similar problems, [M87], [K87], [Tr88], which affect the current version of the paper.

2. Algebraic approach to graphs invariants

One can observe that a very important class of graphs invariants, namely polynomials of graphs, including chromatic polynomial, Tutte polynomial [T84], Jaeger polynomial [A86], can be computed in a very similar manner. If we denote by W_G polynomial of graph G then W_G is a linear function of W_{G-e} and $W_{G/e}$. This gives a recursive definition of a polynomial for graph G: decompose G to G-e and G/e by removing and contracting some edge e and compute polynomials for obtained graphs. One of the differences between above polynomials is that computation rules for some of them forbid subtracting or deleting some kinds of edges (e.g. isthmus, loops). As a result differeent classes of graphs are considered as non-decomposable (i.e. polynomials for graphs in this class have to be computed explicitly).

In general we consider family of graph invariants which can be computed in a recoursive way (i.e. to compute polynomial $W(G)$ we compute polynomials $W(G_1),\dots,W(G_s)$ for simpler graphs $G_1,\dots,G_s$, and then combine the results). This computation process can be visualized with the help of a tree which has $W(G)$ in its root, $W(G_1),\dots,W(G_s)$ are children of $W(G)$ and so on. We will refer to such a tree as to a computation tree for $W(G)$. Note that $W(G)$ may have number of different computation trees (i.e. there may be many different ways of chosing $G_1,\dots,G_s$) but all of them must lead to the same result.

In this chapter we introduce abstract algebras whose models yield graphs invariants. In particular many known graph polynomials (including matching polynomial) are models of one of the algebras. The construction of the algebras is based on the Conway algebra introduced for links in [PT87].

Consider an abstract algebra of the following type:

$$\mathcal{A}1 = \langle U, |, \{a_i\}_{i\in I}\rangle$$

where U is a universum, $|$ is a two argument operation ($|: U \times U \to U$) and $\{a_i\}_{i\in I}$ are zero-argument operations (constants from U). Furthermore $|$ satisfies the following axiom:

$$\text{Ax1: } (a|b)(c|d) = (a|c)|(b|d)$$

We will use, for convenience, different index sets for the index set I, but all of them will be isomorphic to the set of natural numbers N.

As a model for the above abstract algebra we can consider the algebra which has polynomials of variables $A, B, \{z_i\}_{i\in N}$ as the universum. The operation $|$ is interpreted as follows:

$$a|b = Ba + Ab$$

and constants are defined as $a_i = z_i$. We will call this model the basic model.

Note that in this model we never multiply variables z_i. In fact it would be more natural to consider as a universum $Z[A,B]$-modul with free basis $\{z_i\}_{i\epsilon N}$. So we could define $U = Z[A,B,z]$, $a_i = z^i$ and obtain a model which is $Z[A,B]$ isomorphic to the above model.

An important property of the algebra $\mathcal{A}1$ is that the word problem is decidable (i.e. we can decide in a finite time whether two words define the same element of the algebra). It follows from the fact that left and right sides of the relation in Ax1 have the same length.

We will show how the above abstract algebra can be used to define graphs invariant.

Let $G/_Te$ denote the graph $G-e \sqcup T$, where T is a tree without loops, if e is a loop and G/e otherwise.

<u>Theorem 2.1</u> Let $\mathfrak{a} = \langle U,1,\{a_i\}_{i\epsilon N}\rangle$ be a model of algebra $\mathcal{A}1$ then the following function $W: \mathcal{G} \to U$ is well defined and therefore is a graph invariant

(i) $W(N_i) = a_i$ where N_i denotes a graph of n isolated vertices

(ii) $\forall G \epsilon \mathcal{G}$, $\forall e \epsilon E(G)$

$$W(G) = W(G-e)|W(G/_Te)$$

<u>Proof</u> The proof follows by induction on pairs (p_1,m) where p_1 is the cyclomatic number and m is the number of edges. We order the pairs lexicographically. For $p_1 = 0, m = 0$ the function W is well defined by point (i). Assume that W is well defined for all graphs with $(p_1,m) < (i,j)$. We have to show that the value $W(G)$ does not depend on the order of deleting and contracting edges. If $m = 1$ then we have no choice and the theorem follows immediately by the inductive step. Assume that $m \geq 2$ and let $e,f \epsilon E(G)$. Let

$$L = W(G-e)|W(G/_T e)$$

$$R = W(G-f)|W(G/_T f)$$

Then by the inductivehypothesis W(G-e),W(G/e),W(G-f),W(G/f) are well defined so we can assume

$$L = (W(g-e-f)|W(G-e/_T f))|(W(G/_T e-f)|W(G/_T e/_T f))$$

$$R = (W(g-f-e)|W(g-f/_T e))|(W(G/_T f-e)|W(G/_T f/_T e))$$

but

$$W(G-e-f) = W(G-f-e) \ ; \ W(G-e/_T f) = W(G/_T f-e)$$

$$W(G/_T e-f) = W(G-f/_T e) \ ; \ W(G/_T e/_T f) = W(G/_T f/_T e)$$

So by Ax1 L = R.

Note that if T is the empty tree $G/_T e = G/e$.

Choosing $T = N_1$ gives us a nice formulation of duality theorem for planar graphs:

Theorem 2.2. Let $\mathfrak{A} = \langle U, |, \{a_i\}_{i \in N}\rangle$ be the universal algebra of terms of algebra A1 and W: $\mathfrak{G} \to U$ be a function defined as in Theorem 2.1 for $T = N_1$ then for any planar graph G holds

$$W(G) = \overline{W(G^*)}$$

where $\overline{W}$ is the reverse of the word W and G* is a graph dual to G.

Proof. Denote by $G_1 \cdot G_2$ a graph obtained from G_1 and G_2 by identifying a vertex from G_1 with a vertex from G_2. Note that $G_1 \cdot G_2$ depends on the choice of vertices which are identified.

We start the proof of the theorem with the following lemma:

Lemma 2.3 Let $G = G_1 \cdot G_2$ and $G' = G_1 \sqcup G_2$ then $W(G') = W(G) + 1$ where $W + 1$ denotes the word obtained from a word W by increasing every index in W by 1.

So in particular $W(G_1 \cdot G_2)$ does not depend on the choice of points which are identified.

Proof. Consider a computation tree of $W(G)$. Construct a computation tree for $W(G')$ by contracting and removing edges in the same order as in $W(G)$. Then in each leaf of the computation tree of $W(G')$ occurs a graph with one more isolated vertex than in the corresponding leaf of the computation tree for $W(G)$. $\square$

We can continue now the proof of the theorem. The proof follows by

induction on the number, m, of edges. If $m = 0$ then G is a set of isolated vertices and $G = G^*$ so the theorem follows.

Assume that the theorem holds for any graph G with $|E(G)| < m$. Let e be an edge of G and e^* be the edge of G^* dual to e (i.e. e^* joins vertices of G^* corresponding to the faces of G separated by e).

If e is neither isthmus nor loop then

$$W(G) = W(G-e)|W(G/e)$$

$$W(G^*) = W(G^*-e^*)W(G^*/e)$$

and

$$W(G/e) \text{ is a dual to } W(G^*-e^*)$$

$$W(G-e) \text{ is a dual to } W(G^*/e)$$

and the theorem follows by induction.

Assume now that e is an isthmus (the case when e is a loop is symmetric). Then e^* is a loop. Then for $G-e = G_1 \sqcup G_2$

$$W(G) = W(G_1 \sqcup G_2)|W(G/e)$$

$$W(G^*) = W(G^*-e^*)|((W(G^*-e^*) \sqcup N_1) = W(G^*-e^*)|(W(G^*-e^*) + 1)$$

but G/e is dual to G^*-e^* and G^*-e^* is dual to $G_1 \cdot G_2$ so by Lemma 2.3

$$W(G_1 \sqcup G_2) = \overline{W(G^*-e^*) + 1}\ .$$ ∎

Note that the basic model yields (via Theorem 2.1) a graph polynomial (let us denote it B(G)). Also the following graph polynomials are yielded by algebra $\mathcal{A}1$ and Theorem 2.1.

2.1 Assume $U = Z[\mu, A, B]$, $a_i = \mu^{i-1}$, $a|b = Ba + Ab$, $T = N_1$ then this algebra and Theorem 2.1 yields the Kauffman bracket defined in [K87].

2.1 Assume $U = Z[t,x,y]$, $a_i = t^i$, $a|b = ya + xb$ and T be an arbitrary (fixed) tree. Then this algebra and Theorem 2.1 yields a family of polynomials $f_T(G)$. In particular $f_T(G)$ for $T = \emptyset$ was defined in [N87].

All these polynomials can be obtained from Tutte polynomial (see Remark 2.7).

Theorem 2.1 gives a scheme for recursive construction of graph invariants assuming that non-decomposable graphs (i.e. graphs for which invariant is computed explicitly) are graphs composed of isolated vertices. In general it does not have to be the case.

Consider now as the index set I the set $P = Z_+[x,y]$ (i.e. the set of two-invariable polynomials of non-negative integer coefficients). Consider also the following relation, R, on forests with loops.

R(G,H) <=> there exist such orderings of connected components of G and H

$$G = G_1 \cup \ldots \cup G_k$$

$$H = H_1 \cup \ldots \cup H_k.$$

s.t. G_i and H_i have the same number of edges and loops.

The equivalence classes of R are in one-to-one correspondence with elements of $Z_+[x,y]$: If $p \in Z_+[x,y], p = \sum_{ij} k_{ij}x^iy^j$ then the equivalence class corresponding to p (denoted by [p]) contains all forests which for all i,j contain k_{ij} connected components with i isthmus and j loops. The following theorem corresponds to Theorem 1.1:

Theorem 2.4 Let $\mathcal{a} = \langle U, |, \{a_p\}_{p\in P}\rangle$ be a model of algebra $\mathcal{A}1$ then the following function W: $\mathcal{G} \to U$ is well defined and therefore is a graph invariant:

(i) $\forall G \in [p], W(G) = a_p$

(ii) $\forall G \in \mathcal{G}$, $\forall e \in E(G)$ s.t. e is neither isthmus nor loop

$W(G) = W(G-e)|W(G/e)$

Proof. We start the proof of the theorem with the following lemma:

Lemma 2.5 Let $H \in [p]$, $Q \in [q]$. Let G be a graph obtained from H and Q by identifying a vertex from a connected component corresponding to a monomial $k_1x^{i_1}y^{j_1}$ of the polynomial p with a vertex from a connected component corresponding to a monomial $k_2x^{i_2}y^{i_2}$ of the polynomial q then $G \in [r]$ where

$$r = p + q - x^{i_1}y^{j_1}x^{i_2}y^{j_2} + x^{i_1+i_2}y^{j_1+j_2}$$

Proof. If is enough to notice that the equivalence class to which G belongs does not depend on choice of connected components in H and Q corresponding to the given monomials as well as on the choice of the vertices used for joining those components. ▯

Now we can continue the proof of Theorem 2.4. We have to show that W(G) does not depend on the order of contractions and deletions. Let k(G) be the number of edges of G which are neither loops nor isthmus. The proof will follow by induction on k(G).

For $k(G) = 0$ the theorem follows from point (i).

Assume that the theorem is true for all graphs G with $k(G) < i$. If $k(G) = 1$ then we have no choice and W(G) is well defined by the inductive hypothesis. Let G be a graph s.t. $k(G) = i \geq 2$ and $e, f \in E(G)$ s.t. e,f are not loops or isthmus. Consider the following cases.

1. Assume that e is not an isthmus of G-f and not a loop of G/f. Then f is also not an isthmus of G-e and not a loop of G/e. So let

$$L = W(G-e) | W(G/e)$$

$$R = W(G-f) | W(G/f)$$

By the induction hypothesis

$$L = (W(G-e-f) | W(G-e/f)) | (W(G/e-f) | W(G/e/f))$$

$$R = (W(G-f-e) | W(G-f/e)) | (W(G/f-e) | W(G/f/e))$$

and $W(G-e-f) = W(G-f-e)$, $W(G-e/f) = W(G/f-e)$, $W(G/e-f) = W(G-f/e)$, $W(G/e/f) = W(G/f/e)$. So by Ax1 $L = R$.

2. If f is a loop in G/e, then f and e are in the same multiedge so $W(G-e) = W(G-f)$ and $W(G/e) = W(G/f)$ and the theorem holds.

3. Assume that f is an isthmus of G-e. Then f and e must appear in G as in Figure 2.1.

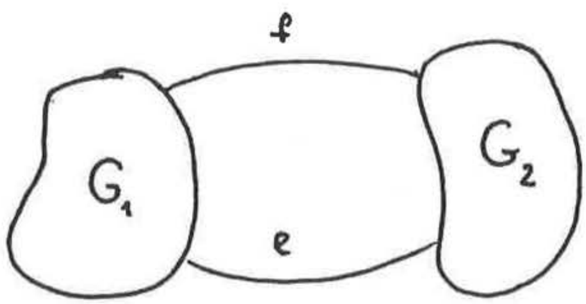


Figure 2.1

Let

$$L = W(G-e)|W(G/e) = W(G-e)|(W(G/e-f)|W(G/e/f))$$

$$R = W(G-f)|W(G/f) = W(G-f)|(W(G/f-e)|W(G/f/e))$$

then by Lemma 2.5

$$W(G/f-e) = W(G/e-f)$$

and (by double application of Lemma 2.5)

$$W(G-e) = W(G-f)$$

(see Figure 2.2)

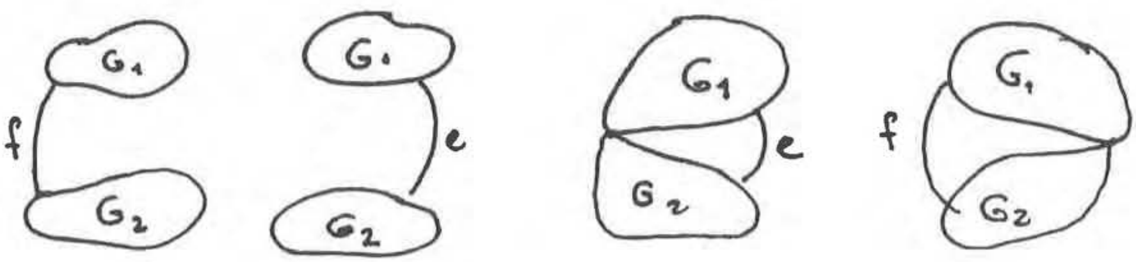


Figure 2.2

so L = R. ∎

In this case we also have a duality theorem similar to Theorem 2.2.

Theorem 2.6

Let $\mathfrak{A} = \langle U, 1, \{a_i\}_{i \in Z_+[x,y]} \rangle$ be the universal algebra of terms of algebra $\mathcal{A}1$ and let $W\colon \mathcal{G} \to U$ be a function defined as in Theorem 2.4. Then for any planar graph holds

$$W(G) = \overline{W(G^*)}^*$$

where w* is a word obtained from w by exchanging x for y (and the opposite) in every index of w.

Proof

The proof is similar to the proof of Theorem 2.2 and follows by induction on the k(G) (the number of edges of G which are neither isthmus or loops). Since a loop in G corresponds to an isthmus in G* and an isthmus in G corresponds to a loop in G* the theorem is true for k(G) = 0.

Assume that the theorem is true for any G with k(G) < i and let G be a graph with k(G) = i. Let e be an edge of G which is neither isthmus nor loop and let e* be the edge of G* dual to e. Then

$$W(G) = W(G-e)|W(G/e)$$

$$W(G^*) = W(G^*-e^*)|W(G^*/e^*)$$

but G-e is a dual to G*/e* and G/e is a dual to G*-e* so the theorem follows by induction. $\square$

From axiom Ax1 and Theorems 2.1, 2.4 it follows that both families of invariants cannot distinguish two 2-isomorphic graphs with the same number of components nor a pair of graphs s.t. one is a rotant of the other. [W33],[T80]

Consider now models of algebra $\mathcal{A}1$ which yield graphs invariants via Theorem 2.4. Note that the basic model is an example of such model. We have also the following polynomial models which lead to known graph polynomials:

2.4.1 $U = Z[x,y]$; $a|b = a+b$ for $p = \sum_{ij} k_{ij}x^iy^i$ $a_p = \prod_{ij}(x^iy^j)^{k_{ij}}$

This algebra yields Tutte polynomial (or dichromat) $\chi(G)$ [T.84].

2.4.2 $U = Z[t,z]$; $a|b = a+b$; for $p = \sum_{ij} k_{ij}x^iy^i$; $a_p = \prod_{ij} (t(1+t)^i(1+z)^j)^{k_{ij}}$

This algebra yields dichromatic polynomial Q(G)[T84].

<u>Remark 2.7</u>

Many graphs polynomials are equivalent (for connected graphs) to Tutte polynomial. We will show here how Tutte polynomial implies some other polynomials.

1. For the polynomial introduced by basic model and Theorem 1.4 we have

$$\text{if } \chi(G) = \sum_{ij} c_{ij} x^j y^j \quad \text{then} \quad B(G) = \sum_{ij} c_{ij} A^{m-p_1-i} B^{p_1-j} Z_{x_i y_i}$$

2. For $f_T(G) \in Z_+[t,\bar{x},\bar{y}]$ we have

$$f_T(G) = t^{p_0} \bar{y}^{p_1} \bar{x}^{m-p_1} \chi(G, 1 + \frac{\bar{y}}{\bar{x}} t,\ 1 + f_T(T) \cdot \frac{\bar{x}}{\bar{y}})$$

where $f_T(T) = t^{p_0} (\bar{x}+\bar{y}t)^{|E(T)|}$.

3. For Kauffman bracket we have

$$\langle G \rangle = \mu^{p_0-1} B^{p_1} A^{m-p_1} \chi(G,\ 1 + \frac{B}{A}\mu,\ 1 + \frac{A}{B}\mu)$$

To see that Kauffman bracket implies Tutte polynomial note that:

$$\frac{\langle G \rangle}{B^m} = \mu^{p_0-1} (\frac{A}{B})^{m-p_1} \chi(G,\ 1 + \frac{B}{A}\mu,\ 1 + \frac{A}{B}\mu)$$

let $\frac{A}{B} = t$ then

$$\frac{\langle G \rangle}{B^m} = \langle G \rangle' = \mu^{p_0-1} t^{m-p_1} \chi(G,\ 1 + \frac{\mu}{t},\ 1 + t \cdot \mu)$$

so $$\mu = \sqrt{(x-1)(y-1)} \qquad t = \sqrt{\frac{y-1}{x-1}}$$

$$\chi(G,x,y) = \frac{\langle G'\rangle}{(x-1)(y-1)^{\frac{p_0-1}{2}}\left(\frac{y-1}{x-1}\right)^{\frac{m-p_1}{2}}}.$$

It is interesting to note that as long as we apply a linear homogeneous function as an interpretation of | and compute polynomial according to the recursive formula in Theorem 1.1 or 1.4 we cannot obtain anything essentially more than Tutte polynomial.

Let us again change the index set I. Let I now be equal to $Z_+[y]$ (i.e. set of one-variable polynomials of non-negative integer coefficients).

Consider the following relation, R', on graphs whose all edges are loops:

R'(G,H) <=> there exist such orderings of connected components of G and H, say $G = G_1 \cup \ldots \cup G_k$, $H = H_1 \cup \ldots \cup H_k$, s.t. G_i and H_i have the same number of edges (loops).

The equivalence classes of R' are in one-to-one correspondence with elements of $Z_+[y]$. If $p \in Z_+[y]$ and $p = \sum_i k_i y^i$ then the equivalence class corresponding to p (denoted by [p]) contains all graphs whose all edges are loops and which for each i have k_i connected components with i loops. Then the following theorem holds.

<u>Theorem 2.8</u>

Let $\mathcal{Q} = \langle U, |, \{a_p\}_{p \in Z_+[y]}\rangle$ be a model of algebra $\mathcal{A}1$ then the following function $W: \mathcal{G} \to U$ is well defined and therefore is a graph invariant

(i) $\forall G \in [p]\ W(G) = a_p$

(ii) $\forall G \in$, $\forall \in E(G)$ s.t. e is not a loop

$W(G) = w(G-e)|W(G/e)$.

<u>Proof</u>

The proof follows by induction on the number of edges which are not loops. The proof is very similar to the proof of Theorem 2.4 but only the first two cases have to be considered.

Note that for the invariant defined by the universal algebra of terms and Theorem 2.8 we don't have a duality theorem similar to Theorem 2.2 or 2.6. In particular two different dual graph to the same graph do not have to have the same invariant. Consider as an example the graphs drawn on Figure 2.3.

Figure 2.3

Graphs G_1, G_2 from Figure 2.3 have the same duals but have different invariants:

$$W(G_1) = a_{1+y^2} | a_{y^2}$$

$$W(G_2) = a_{2y} | a_{y^2}$$

Let us consider the following model of algebra $\mathcal{A}1$:

$$U = Z[z_0, z_1, \ldots]$$

for $$p = \sum_i k_i y^i \quad a_p = \prod_i \Big(\sum_{k=0}^{i} \binom{i}{k} z_k \Big)^{k_i}$$

and $$a|b = a + b$$

then this algebra yields Jaeger polynomial $V(G)$ [A86]. It is known that Jaeger polynomial implies Tutte polynomial (substitute tz^j for z_j to obtain dichromatic polynomial which implies Tutte polynomial). The opposite is not true. For example G_1 and G_2 from the figure above have the same Tutte polynomial but different Jaeger polynomial.

Also the pair of graphs on the figure below have the same Tutte polynomial and different Jaeger polynomial.

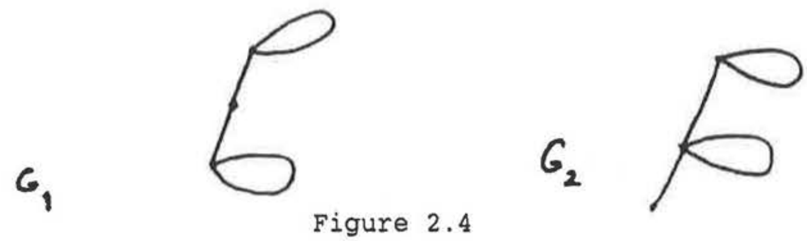


Figure 2.4

$$\chi(G_1) = \chi(G_2) = x^2y^2$$

$$V(G_1) = 3z_1z_0^2 + z_0^2 + 2z_1^2 + 4z_1z_0 + 3z_0 + 2z_1 + z_2$$

$$V(G_2) = z_0^3 + 2z_0^2z_1 + z_1^2 z_0 + 2z_0^2 + z_1^2 + 4z_1z_0 + z_2z_0 + z_0 + 2z_1 + z_2$$

Note that the above examples show also that invariants yielded by Theorem 2.8 and models of algebra $\mathcal{A}1$ may distinguish two 2-isomorphic graphs. In particular they may distinguish graphs with the same deposition onto two-connected components.

Consider now an abstract algebra obtained from $\mathcal{A}1$ by adding one more axiom. Let $\mathcal{A}2$ be the algebra:

$$\mathcal{A}2 = \langle U, |, \{a_i\}_{i\in N}\rangle$$

$$|: U \times U \to U$$

$$a_i \in U$$

with axioms:

$$\text{Ax1} \quad (a|b)|(c|d) = (a|c)|(b|d)$$

$$\text{Ax2} \quad (a|b)|c = (a|c)|b$$

Note that for the algebra $\mathcal{A}2$ the word problem is also decidable.

This algebra can be used to define graph invariants as follows.

Theorem 2.9

Let $\mathfrak{A} = \langle U, |, \{a_i\}_{i\in N}$ be a model of $\mathcal{A}2$ then the following function $W: \mathcal{G} \to U$ is well defined and therefore is a graph invariant:

(i) $W(N_i) = a_i$

(ii) for any edge $e \in E(G)$

$$W(G) = W(G-e)|W(G/\!/e)$$

when $G/\!/e$ denotes G without endpoints of e.

Proof

Proof follows by induction on the number of edges in $E(G)$. If $|E(G)| = 0$ then the theorem is implied by point (i). Assume that theorem holds for any graph G.s.t. $|E(G)| < m$. For $|E(G)| = 1$ function W is well defined by (i) and (ii). Let G be a graph with $|E(G)| = m \geq 2$ and let $e, f \in E(G)$. We will consider two cases.

1) Edges e and f are not adjacent then

$$L = w(G-e)|w(G/\!/e) = (w(G-e-f)|w(G-e/\!/f))|(w(G/\!/e-f)|w(G/\!/e/\!/f))$$

$$R = w(G-f)|w(G/\!/f) = (w(G-f-e)|(w(G-f/\!/e)|(w(G/\!/f-e)|(w(G/\!/e/\!/f))$$

and by Ax1 $L = R$.

2) Edges e and f are adjacent then

$$L = w(G-e)|w(G/\!/e) = (w(G-e-f)|w(G-e/\!/f))|(w(G/\!/e)$$

$R = w(G-f)|w(G/\!/f) = (w(G-f-e)|(w(G-f/\!/e)|(w(G/\!/f)$ but in this case

$$w(G-e//f) = w(G//f)$$

and

$$w(G-f//e) = w(G//e)$$

so by Ax2

$$L = R. \qquad \blacksquare$$

Consider the following model of the algebra $\mathcal{A}2$:

$$U = Z[w_1,w_2]$$

$$a_i = w_1{}^i$$

$$a|b = a + w_2 b$$

This model yields the polynomial of graphs called matching polynomial (see [F76]).

3. Abstract algebras for invariants of chromatic graphs

In this section we consider abstract algebras leading to invariants of graphs with some additional structure. In particular we consider graphs with colored edges. We associate with each edge color (number from Z) and the attribute dark (d) or light (ℓ). By $d^i(\ell^i)$ we will denote an arbitrary dark (light) edge of color i.

We will consider the following abstract algebra:

$$\mathcal{A}3 = \langle U,\ \{|_i\}_{i\in Z}\ ,\ \{a_i\}_{i\in I}\rangle$$

with the infinite set of axioms:

$$Ax_{ij}\colon\ (a|_i b)|_j(c|_i d) = (a|_j c)|_i(b|_j d) \text{ for all } i,j\in Z.$$

As in the previous section we will use for convenience as the index set I different sets isomorphic to N.

The following model of $\mathcal{A}3$ will be considered as the basic model:

$$= Z[\{A_i, B_i\}_{i\in Z}, \{a_i\}_{i\in I}]$$

$$a|_i b = B_i a + A_i b$$

We can generalize Theorem 2.1 for algebra $\mathcal{A}3$ as follows:

Theorem 3.1

Let $\mathfrak{A} = \langle U, \{|_i\}_{i\in Z}, \{a_i\}_{i\in N}\rangle$ be a model of algebra $\mathcal{A}3$ then the following function W: $\mathcal{G}' \to U$ is well defined and therefore is an invariant of chromatic graphs.

(i) $\forall_i$ $W(N_i) = A_i$ where N_i is a graph of n isolated vertices.

(ii) $\forall_i$ $d^i \in E(G)$

$$W(G) = W(G-d^i)\,|_i\, W(G/_T d^i)$$

(iii) $\forall_i$ $\ell^i \in E(G)$

$$W(G) = W(G/_T \ell^i)\,|_i\, W(G-\ell^i)$$

where T is a fixed forest without loops.

Proof

The proof follows by induction on pairs (p_1, m) assuming their lexicographic order and is similar to the proof of Theorem 2.1. For $p_1 = 0$, $m = 0$ the function W is well defined by point (i). Assume that the theorem holds for all graph with $(p_1, m) < (i,j)$ set G be a graph with $p_1 = i$ and $m = j$. If $m = 1$ then we have no choice and the theorem follows by the inductive hipothesis. Assume that $m \geq 2$

and let e and f be two edges of G. Assume that color of e is equal to i and color of f is equal to j. Consider the following cases:

1. Assume that attribute of e is equal to attribute of f. Assume that this attribute is dark (for the attribute light the argument is symmetric).

   Let

$$L = W(G-e)\,|_i\,W(G/_T e)$$

$$R = W(G-f)\,|_j\,W(G/_T e)$$

   by the inductive hypothesis

$$L = (W(G-e-f)\,|_j\,W(G-e/_T f))\,|_i\,(W(G/_T e-f)\,|_j\,W(G/_T e/_T f))$$

$$R = (W(G-f-e)\,|_i\,W(G-f/_T e))\,|_j\,(W(G/_T f-e)\,|_i\,W(G/_T f/_T e))$$

   so by axiom Ax_{ij} L = R.

2. Assume that the attribute of e is different than the attribute of f. Assume that the attribute at e is dark (the case when the attribute of e is light is symmetric).

   Let

$$L = W(G-e)\,|_i\,W(G/_T e)$$

$$R = W(G/_T f)\,|_j\,W(G-f)$$

   then

$$L = (W(G-e/_T f)\,|_j\,W(G-e-f))\,|_i\,(W(G/_T e/_T f)\,|_j\,W(G/_T e-f)$$

$$R = (W(G/_T f-e)\,|_i\,W(G/_T e/_T f))\,|_j\,(W(G-f-e)\,|_i\,W(G-f/_T e)$$

   so by Ax_{ij} L = R. ∎

   We define as a dual to a chromatic graph G graph G* defined as follows:

- vertices and edges are defined as in a nonchromatic dual
- if $e \epsilon E(G)$ and $e^* \epsilon E(G^*)$ is the dual edge to e then color of e^* is equal to the color of e and the attribute of e^* is opposite to the attribute of e.

For such defined dual graph we have the following duality theorem corresponding to Theorem 2.2 in the previous section.

Theorem 3.2

Let $\mathcal{A} = \langle U, \{|_i\}_{i \epsilon Z}, \{a_i\}_{i \epsilon N} \rangle$ be the universal algebra of terms of algebra $\mathcal{A}3$ and W be defined as in Theorem 3.1 for $T = N_1$ then for any plan a graph G

$$W(G) = W(G^*).$$

Proof

Note that Lemma 2.3 holds also for chromatic graphs. As in the proof of Theorem 1.3 we will proceed by induction on the number, m, of edges.

For m = 0 G = G* and the theorem is true. Assume it is true of all graphs with E(G) < m. Let G be a graph with m edges. Let $e \epsilon E(G)$ and let $e^* \epsilon E(G^*)$ be the dual edge to e. Assume that e is neither a loop nor an isthmus. Let $e = d^i$ for some i (the case when $e = \ell^i$ is similar), then

$$W(G) = W(G-e)\,|_i W(G/e)$$

$$W(G^*) = W(G^*/e^*)\,|_i W(G^*-e^*)$$

but W(G/e) is dual to $W(G^*-e^*)$ and W(G-e) is a dual to $W(G^*/e^*)$.

If e is an isthmus then for $G-e = G_1 \cup G_2$

$$W(G) = W(G_1 \quad G_2)\,|_i W(G/e)$$

$$W(G^*) = W(G^*-e^* \quad N_1)\,|_i W(G^*-e^*) =$$

$$W(G^*-e^*) + 1|_i W(G^*-e)$$

and by the modified version of Lemma 2.3 the theorem holds.

∎

Consider now polynomials of chromatic graphs yielded by algebra $\mathcal{A}3$ and Theorem 3.1.

3.1.1 Let $T = N_1$, $U = Z[\{A_i, B_i\}_{i\in Z}, \mu]$, $a_i = \mu^{i-1}$ and $a|_i b = B_i a + A_i b$.

This model gives Kauffman bracket $\langle G\rangle$ for chromatic graphs.

3.1.2. Let T be any fixed tree and $U = Z[\{w_i\}_{i\in Z}, t]$, $a_i = t^i$

$$a|_i b = a + w_i b$$

Then Theorem 3.1 gives a graph polynomial which we denote by Q_T.

This should be compared with dichromatic polynomial Q for weighted graphs [Tr88] defined as follows:

Let w(e) be the weight associated with edge e (an element from some communtative ring with unity) then

(i) if $G = N_i$ then $Q(G;t,z) = t^i$

(ii) if e is not a loop then

$$Q(G,t,z) = Q(G-e;t,z) + w(e)Q(G/e,t,z)$$

(iii) if e is a loop then

$$Q(G;t,z) = (1+w(e)z)\ Q(G-e;t,z).$$

<u>Lemma 3.3</u>

The dichromatic polynomial, Q, for a weighted graph can be computed by computing a finite number of polynomials Q_T for different T.

<u>Proof</u>

To obtain the dichromatic polynomial for weighted graphs assume that the set of different weights of G is taken as the index set for colors (we remove edges with weight equal to zero from the graph). Now compute $Q_T(G)$ for $T = N_0, N_1, \ldots, N_{p_1(G)}$. This gives us the following sequence of polynomials

$$Q(G;t,1), Q(G;t,t) \ \ldots, \ Q(G,t,t^{p_1(G)}).$$

Since the highest degree of z in Q is equal to $p_1(G)$ this determines the polynomial $Q(G;t,z)$.

Assume now that the index set I is equal to $P' = Z[\{x_i, y_i\}_{i\in Z}]$. Let $p\in P'$. Denote by [p] the class of chromatic forests with loops s.t.

$T\in[p]$ iff

T has k connected components with n_i isthmuses of color i and m_i loops of color i iff the monomial $k\prod_i x_i^{n_i} y_i^{m_i}$ occurs in p.

Consider the following quotient algebra of algebra $\mathcal{A}3$:

$$\mathcal{A}4 = \langle U, \{|_i\}_{i\in Z}, \{a_p\}_{p\in P'}\rangle$$

with the following three groups of axioms:

(A) $\forall i,j\in Z, (a|_i b)|_j (c|_i d) = (a|_j c)|_i (b|_j d)$

(B) $\forall i,j\in Z, p_0, p_1, q \in P'$

$$a_{p_0+p_1x_j}|_i (a_{p_0+p_1}|_j a_q) = a_{p_0+p_1x_i}|_j (a_{p_0+p_1}|_i a_q)$$

(C) $\forall i,j \in Z$, $p_1, p_2 \in P'$; let a be any word (term) of algebra $\mathcal{A}4$ and let $a\cdot y_j$ denote the word obtained from a by multiplying every index in a by y_j then

$$(a_{p_1}|_j a_{p_2})|_i a\cdot y_j = (a_{p_1}|_i a_{p_2})|_j a\cdot y_i.$$

This algebra introduces a class of invariants of chromatic graphs via the following theorem:

Theorem 3.4

Let $\mathcal{A} = \langle U, \{|\}_{i\in I}, \{a_p\}_{p\in P}\rangle$ be a model of algebra $\mathcal{A}4$ then the following function $W: \mathcal{G}' \to U$ is well defined and therefore is a graph invariant:

(i) $\forall G\in [p]\quad w(G) = a_p$

(ii) $\forall G\in \mathcal{G}'$, $\forall e\in E(G)$ s.t. e is neither isthmus nor loop and col(e) = i.

$$W(G) = W(G-e)|_i W(G/e).$$

Proof

Proof of the theorem is similar to the proof of Theorem 2.9 and follows by induction on number, K(G), of edges which are neither isthmuses nor loops.

If K(G) = 0 it follows from (i).

If K(G) = 1 then we have no choice and the theorem follows immediately.

Assume that the theorem is true for any graph G with K(G) < n. Let G be a graph with $K(G) = n\geq 2$. Let $e,f\in E(G)$ and let e has color i and f has color j. Consider the following cases:

1. e is not an isthmus of G-f or a loop of G/f. Then the theorem follows immediately from (A).

2. e is a loop of G/f

$$L = W(G-e)|_i W(G/e) = (W(G-e-f)|_j W(G-e/f))|_i W(G/e)$$

$$R = W(G-f)|_j W(G/f) = (W(G-e-f)|_i W(G-f/e))|_j W(G/f)$$

$$\text{but } W(G-e/f) = W(G-f/e)$$

and G/e is equal to G/f with loop of color j replaced by loop of color i. So by (C) L = R.

3. Assume e is an isthmus of G-f but not a loop of G/f. Then

$$L = W(G-e)|_i W(G/e) = W(G-e)|_i (W(G/e-f)|_j W(G/e/f))$$

$$R = W(G-f) |_j W(G/f) = W(G-f) |_j (W/G/f-e) |_i W(G/e/f))$$

but by the argument similar as in Lemma 2.5

$$W(G/e-f) = W(G/f-e).$$

To see that $W(G-f) = W(G-e)$ consider computation trees for $W(G-f)$ and $W(G-e)$ obtained by deleting and contracting the same edges and in the same order. Then corresponding leaves of those trees look the same except one isthmus which corresponds to f in the tree of $W(G-e)$, and to e in the tree of $W(G-f)$. So by (B) the theorem follows.

∎

In the above algebra we have ignored the fact that edges also have attributes dark or light. In order to define an algebra which leads to invariants of chromatic graphs with attributes on edges light or dark we need some additional duality axioms. Define index set I to be $P'' = Z[\{x_i, y_i, \bar{x}_i, \bar{y}_i\}_{i \in Z}]$. For $p \in P''$ define $[p]$ as class of forest with self loops s.t. $T \in [p]$ iff

T has k connected components with

n_i isthmuses of color i and attribute light

$\bar{n}_i$ isthmuses of color i and attribute dark

m_i self loops of color i and attribute light

$\bar{m}_i$ self loops of color i and attribute dark

iff

$$k \prod_{i \in Z} x_i^{n_i} \bar{x}_i^{\bar{n}_i} y_i^{m_i} \bar{y}_i^{\bar{m}_i}$$

occurs in p.

The additional duality axioms can be formulated as follows.

(B^1) $\forall_{ij} \epsilon Z$, $\forall p_0, p_1, q$

$$a_{p_0+p_1\bar{x}_j} |_i (a_{p_0+p_1} |_j a_q) = (a_{p_0+p_1} |_i a_q) |_j a_{p_0+p_1 x_i},$$

$$(a_{p_0+p_1} |_j a_q) |_i \; a_{p_0+p_1\bar{x}_j} = (a_{p_0+p_1} |_i a_q) |_j \; a_{p_0+p_1\bar{x}_i}$$

(C^1) $\forall, i, j \epsilon Z$, $\forall p_1, p_2 \epsilon P''$. Let a be a word then

$$(a_{p_1} |_j a_{p_2}) |_i \; a \cdot \bar{y}_j = a \cdot y_i |_j \; (a_{p_1} |_i \; a_{p_2})$$

$$a \cdot \bar{y}_j |_i (a_{p_1} |_j \; a_{p_2}) = a \cdot \bar{y}_i |_j (a_{p_1} |_i a_{p_2})$$

Let us call this new algebra $\mathcal{A}5$. For this algebra we can reformulate Theorem 3.3 as follows.

Theorem 3.5

Let $\mathfrak{A} = \langle U_1 \{|_i\}_{i\epsilon Z}, \{a_p\}_{p\epsilon P''}\rangle$ be a model of algebra $\mathcal{A}5$ then the following function $W: \mathcal{G}' \to U$ is well defined and therefore is a graph invariant.

(i) $\forall G \epsilon [p]$ $W(G) = a_p$

(ii) $\forall_i$ if d^i, ℓ^i are not isthmus or loop then

$$W(G) = W(G-d^i) |_i W(G|d^i)$$

$$W(G) = W(G|\ell^i) |_i W(G-\ell^i)$$

Proof

The proof is similar, up to technical details, to the proof of Theorem 3.4 and is left to the reader. ∎

For planar graphs we have also the following duality theorem.

Theorem 3.6

Let $\mathfrak{A} = \langle U\{|\}_{i\epsilon Z}, \{a_p\}_{p\epsilon P''}\rangle$ be the universal algebra of terms of the algebra $\mathfrak{A}5$ and let W be defined as in Theorem 3.4 then for any planar chromatic graph G and its dual G*

$$W(G) = W(G^*)^*$$

where w* is equal to w with all x changed to y and the opposite.

Proof

Proof of this theorem is very similar to the proof of Theorem 3.4 and is left to the reader. ∎

Algebra $\mathfrak{A}5$ together with Theorem 3.6 leads for example, to the following graph polynomials.

Let $U = Z[t,z,\{x_i\}_{i\epsilon I}]$ and let $G\epsilon[p]$ (for $p\epsilon P''$) then

$$a_p = t^{p_0(G)} \prod_i ((1+x_i z)^{k_i}(x_i+t)^{r_i})$$

where

r_i - number of isthmuses of color i,

k_i - number of loops of color i.

Let also

$$a|_i b = a + x_i b$$

Then if G is a weighted graph and I is the set of different weights, this model yields the dichromatic polynomial for weighted graphs [Tr88].

If in the above we assume number of colors equal to two and additionally $x_1 = x_2^{-1}$, then we obtain an invariant for signed graphs $P_\Gamma(x,y,z)$ defined in

[M87] as follows:

$$P_{\Gamma}(x,y,z) = y^{-1}Q(G;y,z,x,x^{-1})$$

We may also consider invariants of graphs with a homomorphism h: $\Pi_1(G) \to H$ where H is a group and $\Pi_1(G)$ is the fundamental group of G. This situation occurs, for example, for embedded graphs. Then we may consider as H the fundamental group of the surface. Then we can associate with graphs G-e and G/e homomorphisms induced by H. To obtain an invariant of embedded graphs we have to associate with each pair (graph, homomorphism) a properly chosen element from some universum U.

4. Dichromatic graphs and link diagrams

We will consider two methods of assigning to a plane graph a link diagram. The first method is based on the idea of Jaeger [J87] and the second is the classical one (see [BZ85] or [K87a]).

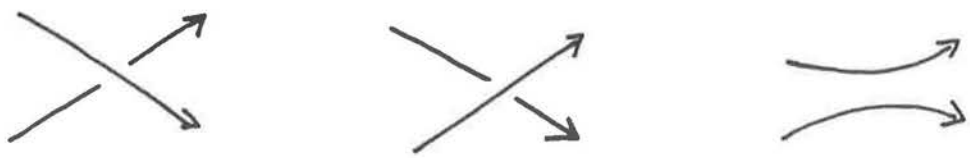
By a (oriented) link we understand several (oriented) circles embedded in S^3. We say that two links L_1 and L_2 are isotopic iff there exists an isotopy $F: S^3 \times I \to S^3 \times I$ such that $F_0 = Id$ and $F_1(L_1) = L_2$. If the links are oriented the isotopy must preserve the orientation. Informally two links are isotopic if one can be continuously transformed to the other.

By links invariant we understand link isotopy classes invariant.

A diagram D of a link L is a regular projection of L in the plane together with an overcrossing-undercrossing structure denoted as in Figure 4.1

Figure 4.1

By L_+, L_-, L_0 we will denote diagrams of links which are identical, except near one crossing point where they look like in the figure below



L_+ L_- L_0

Figure 4.2

We associate a sign (+ or -) to each crossing according to the above convention. Important isotopy invariants are link polynomials. We will define here three such polynomials.

1. The skein (named also Flypmoth, Homfly, generalized Jones, 2-variable Jones, Jones-Conway, twisted Alexander) polynomial, $P_L \in Z[a^{\mp 1}, z^{\mp 1}]$, of oriented links is defined recursively in the following way [FYHLMO85], [PT87].

   (i) $P_{\bigcirc} = 1$

   (ii) $aP_{L_+} + a^{-1}P_{L_-} = z\, P_{L_0}$

   where $P_{\bigcirc}$ corresponds to a trivial link of one component (i.e. to a single circle).

2. Jones polynomial of oriented links (the precursor of the skein polynomial) [Jo85], [Jo87], $V(t) \in Z[t^{\mp 1/2}]$, is defined recursively by

   (i) $V_{\bigcirc}(t) = 1$

(ii) $\frac{1}{t} V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})\, V_{L_0}(t)$.

In particular $V_L(t) = P_L(it^{-1}, i(\sqrt{t} - \frac{1}{\sqrt{t}}))$.

3. The classical Alexander polynomial of oriented links [A28] (as normalized by J. Conway [Co69]), $\nabla_L(z) \in Z[z]$, is defined recursively by

(i) $\nabla_{\bigcirc}(z) = 1$

(ii) $\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z)$

In particular $\nabla_L(z) = P_L(ia, iz)$.

We may also assign a polynomial to a nonoriented link diagram. Such a polynomial has been defined in [K87a] and is called Kauffman bracket. It is defined recursively by

(i) $\langle\ \rangle = B\langle\ \rangle + A\langle)(\rangle$

(ii) $\langle\ \rangle = A\langle\ \rangle + B\langle)(\rangle$

(iii) $\langle 0 \ldots 0 \rangle = \mu^{i-1}$

where , , and $)($ denote four diagrams of links which are identical exept near one crossing, as shown on the diagram. If we assign $B = A^{-1}$, $\mu = -(A^2+A^{-2})$ then Kauffman bracket gives a variant of the Jones polynomial (for oriented links). Namely for $A = t^{-\frac{1}{4}}$

$$V_D(t) = (-A^3)^{-w(D)} \langle D \rangle$$

where $w(D)$ is the planar writhe (or twist) of D defined by taking algebraic sum of the crossings, counting and as +1 and -1 respectively.

Consider now a chromatic graph with number of colors equal to two. Such a graph will be called dichromatic graph. The two colors of the dichromatic graph will be denoted + and −.

By d^+ (resp. d^-) we will denote a dark edge of color + (resp.−) and by ℓ^+ (resp.ℓ^-) we will denote a light edge of color + (resp. −). Let G be a dichromatic plane graph. We can associate with G an oriented diagram of a link (D(G)) (together with chess-like coloring of the plane) according to the following rules:

(4.1)

1) $v \overset{d^+}{\text{———}} w$ (two positive crossings)

2) $v \overset{d^-}{\text{———}} w$ (two negative crossings)

3) $v \overset{\ell^+}{\text{———}} w$ (two positive crossings)

4) $v \overset{\ell^-}{\text{———}} w$ (two negative crossings)

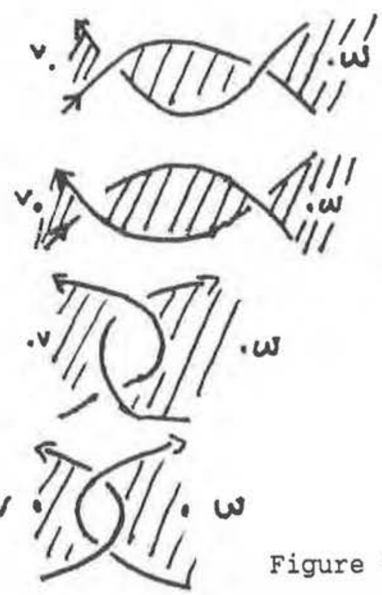


Figure 4.3

i.e. the edge (v,w) is replaced by one of the above diagrams so that vertices v and w remain in the black regions as in the Figure 4.3. In particular

Figure 4.4

We assume also $D(\cdot) =$. Diagrams of links obtained in the above manner are called matched diagrams [APR87].

Remark 4.2

If G^* is the dual to the chromatic plane graph G then $D(G) = -D(G^*)$ with black and white regions exchanged and $-D(G)$ denotes the link diagram with reversed orientation of each link component.

Define $W(G)$ to be $W(G) = P_{D(G)}(a,z)$ then the following holds:

(4.3) $$W(G) = a^{-1}zW(G-d^+) - a^{-2}W(\cdot G/d^+)$$

$$W(G) = -a^{-2}W(G-\ell^+) + a^{-1}zW(G/\ell^+)$$

$$W(G) = azW(G-d^-) - a^2W(G/d^-)$$

$$W(G) = -a^2W(G-\ell^-) + azW(G/\ell^-)$$

$$W(N_i) = \left(\frac{a+a^{-1}}{z}\right)^{i-1}$$

Note that function W satisfies Theorem 3.1 for dichromatic graphs assuming that + and - denote colors. So it defines invariant for all (not necessarily plane) graphs. This is in fact an equivalent definition of Kauffman bracket of dichromatic graphs (Example 3.1.1 of Theorem 3.1), for $\mu = \frac{a+a^{-1}}{z}$, $A_1 = -a^{-2}$, $B_1 = a^{-1}z$, $A_2 = -a^2$, $B_2 = az$.

Now let us associate with dichromatic graph G an unoriented link diagram $D_N(G)$ (together with chess-like coloring of the plane) according to the following rules (4.4):

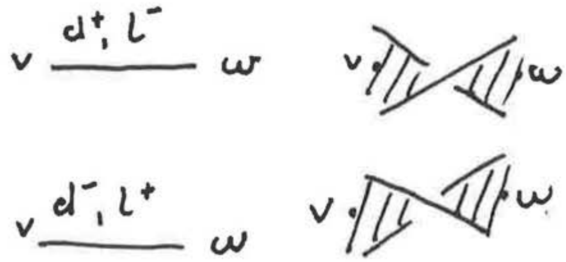


Figure 4.5

Lemma 4.5

Let G be a dichromatic planar graph, $D_N(G)$- unoriented link diagram obtained from G by rules 4.4 and D(G)-oriented link diagram obtained by rules 4.1. Let in_+(resp. in_-) be the number of positive (resp. negative) edges of G. For an oriented link diagram D define w(D) (writhe or twist of D) to be equal to $\sum_p \operatorname{sgn} p$ where the sum is taken over all crossings of D. Then for $B^{-1} = A = a^{1/2}$, $\mu = -A^2-A^{-2}$ holds:

$$\langle D_N(G)\rangle = (-a^{\frac{3}{2}})^{in_+-in_-} P_{D(G)}(a,-1)$$

$$= (-a^{\frac{3}{2}})^{\frac{1}{2}w(D(G))} P_{D(G)}(a,-1)$$

Proof

Note that for $z = -1$ we can rewrite 4.3 as follows:

$$-a^{\frac{3}{2}} P_{D(G)}(a,-1) = a^{\frac{1}{2}} P_{D(G-d^+)}(a,-1) + a^{-\frac{1}{2}} P_{D(G/d^+)}(a,-1)$$

$$-a^{\frac{3}{2}} P_{D(G)}(a,-1) = a^{-\frac{1}{2}} P_{D(G-\ell^+)}(a,-1) + a^{\frac{1}{2}} P_{D(G/\ell^+)}(a,-1)$$

$$-a^{-\frac{3}{2}} P_{D(G)}(a,-1) = a^{-\frac{1}{2}} P_{D(G-d^-)}(a,-1) + a^{\frac{1}{2}} P_{D(G/d^-)}(a,-1)$$

$$-a^{-\frac{3}{2}} P_{D(G)}(a,-1) = a^{\frac{1}{2}} P_{D(G-\ell^-)}(a,-1) + a^{-\frac{1}{2}} P_{D(G/\ell^-)}(a,-1)$$

$$P_{D(N_1)}(a_1-1) = [-(a+a^{-1})]^{i-1}$$

Note also that operations $G-d^+, G-d^-, G/\ell^+$ and G/ℓ^- change to , and operations $G/d^+, G/d^-, G-\ell^+$ and $G-\ell^-$ change to .

Combining this with the definition of the Kauffman bracket for $B = A^{-1}$, $\mu = (-A^2-A^{-2})$ Lemma 4.5 follows. ∎

In particular if we can orient $D_N(G)$ (to get $D_N^{or}(G)$) in such a way that its orientation agrees with that of $D(G)$ (i.e. positive crossings of D_N^{or} correspond to positive crossings of $D(G)$) then the following holds:

<u>Corollary 4.6</u>

$$V_{D_B^{or}(G)}(t) = P_{D(G)}(a,-1), \text{ for } a = t^{-\frac{1}{2}}$$

In [Ko87] there are given necessary and sufficient conditions for existence of $D_N^{or}(G)$.

It is an open question whether any link has a matched diagram. It is very unlikely, however, on the other hand, any 2-bridge link (BZ85] possesses a matched diagram. One can hope to use Lemma 4.5 and some properties of the skein polynomial and the Kauffman bracket to find a link without a matched diagram.

One more observation should be mentioned. If we change ℓ^- edge in a planar graph G to the edge d^+ (denote the new graph G') then $D_N(G') = D_N(G)$ and therefore $\langle D_N(G')\rangle = \langle D_N(G)\rangle$. Now by Lemma 4.5 $P_{D(G')}(a,-1) = a^{-3}\, P_{D(G)}(a,-1)$. The last equality can be put in more general context as follows:

Consider the following move on the oriented link diagrams (called in [P86]

$-t_3$ move).

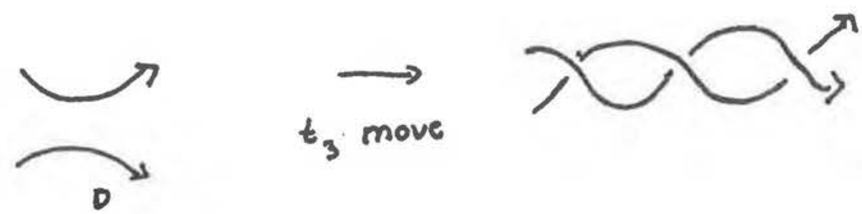


Figure 4.6

It can be easily checked (see [P86]) that $P_{t_3(D)}(a,-1) = a^{-3}P_D(a,-1)$. On the other hand one can go from D(G) to D(G') by a t_3 move and isotopy as it is illustrated in Figure 4.7.

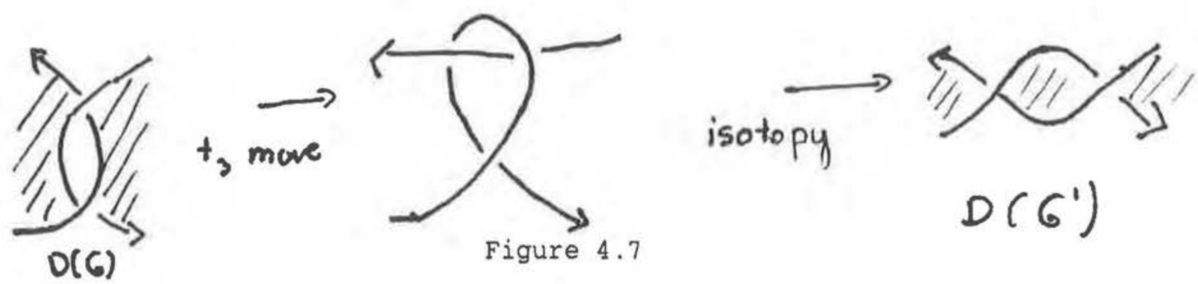


Figure 4.7

5. Complexity aspects of computing the polynomial for chromatic graph

Let $g,f\colon N \to R^+$. We will say that f is $O(g)$ iff there exist constants $c > 0$, $n_0 \in N$ st. for every $n > n_0, f(n) \le cg(n)$.

It is well known that computing Tutte polynomial for graphs is NP-hard (even if we restrict ourselves to planar graphs). It is also easy to see that if $\chi(G;x,y)$ is the Tutte polynomial of graph G and $\chi(G;x,y) = \sum_{i=0}^{p_1(G)} q_i(x)y^i$ then polynomial $q_{p_1(G)-k}$ can be computed in $O(m^{k+1})$ steps, each of them being a summation of two polynomials. This follows from the fact that if we consider a computation tree which has in its leaves forests with loops then a polynomial of degree $p_1(G)-k$ may occur in at most $\binom{m}{k}$ leaves. So the number of internal nodes involved in the computation of $q_{p_1(G)-k}$ is $O(m^{k+1})$.

Similarly for a graph G the k^{th} coefficient (starting from the highest one) of the chromatic polynomial P where

$$P(G;\lambda) = (-1)^{|V(G)|}(-\lambda)\chi(G;1-\lambda,0)$$

can be computed in a polynomial time on the number of edges (assuming that k is a constant).

In this section we address the question: How difficult it is to compute coefficients of the polynomial of a chromatic graph? We also show a substitution which reduces ⟨G⟩ to a one-variable polynomial which can be computed in a polynomial time.

Some obvious properties of ⟨G⟩ are given by the following lemma:

Lemma 5.1

(i) $$\langle G\rangle = \sum_{S\in 2^{E(G)}} \mu^{||S||-1} \left(\prod_{i=1}^{n} A_i^{\alpha_i+\alpha_i'} B_i^{\beta_i+\beta_i'}\right)$$

where

$||S||$ - the number of points after removing edges in S and contracting the rest of the edges (in this section we use G/e to denote $G/_{N_1}e$).

α_i - the number of contracted dark i^{th} colored edges

α_i' - the number of deleted light i^{th} colored edges

β_i - the number of deleted dark i^{th} colored edges

$\beta_i{}'$ - the number of contracted light i^{th} colored edges.

(ii) The highest power of A_i in $\langle G\rangle$ is equal to the number of i^{th} colored edges.

Proof

(i) is a natural state model for $\langle G\rangle$ and (ii) follows immediately from (i).∎

Let $B_i{}' = B_i\mu$. Then we can rewrite the recursive definition of $\langle G\rangle$ as follows:

$$\langle\langle G\rangle\rangle = B'_i\mu^{-1}\langle\langle G-d^i\rangle\rangle + A_i\langle\langle G/d^i\rangle\rangle$$

(5.2) $$\langle\langle G\rangle\rangle = A_i\ \langle\langle G-\ell^i\rangle\rangle + B'_i\mu^{-1}\langle\langle G/\ell^i\rangle\rangle$$

$$\langle\langle N_i\rangle\rangle = \mu^{i-1}$$

Denote by G_d (respectively G_ℓ) the graph obtained from G by removing all light edges (respectively all dark) and by $\deg_\mu P$ maximal degree of μ in the polynomial P.

$$\text{Let } \langle\langle G\rangle\rangle = \sum_{j=0}^{\deg \mu \langle\langle G\rangle\rangle} q_j(A_1, B'_1, A_2, B'_2, \ldots)\mu^j.$$

Lemma 5.3

Proof

The lemma is immediately true if $G = G_d$. If G has a light edge ℓ^i then an inductive arugument show that max $(\deg_\mu(A_i\langle\langle G-\ell^i\rangle\rangle, \deg_\mu B'_i\mu^{-1}\langle\langle G/\ell^i\rangle\rangle =$ $P_0(G_d) + P_1(G_d) -1$, so the lemma follows.

Immediately from this lemma and from 5.2 follows:

Lemma 5.4

Let ℓ be a light edge and d be a dark edge, then

$$\deg_\mu\langle\langle G-\ell\rangle\rangle = \deg_\mu\langle\langle G\rangle\rangle$$

$$\deg_\mu\langle\langle G/\ell\rangle\rangle = \begin{cases} \deg_\mu\langle\langle G\rangle\rangle + 1 & \text{if both ends of } \ell \text{ belong to the same connected component of } G_d \\ \deg_\mu\langle\langle G\rangle\rangle - 1 & \text{otherwise} \end{cases}$$

$$\deg_\mu\langle\langle G-d\rangle\rangle = \begin{cases} \deg_\mu\langle\langle G\rangle\rangle + 1 & \text{if } d \text{ is an isthmus in } G_d \\ \deg_\mu\langle\langle G\rangle\rangle - 1 & \text{otherwise} \end{cases}$$

$$\deg_\mu\langle\langle G/d\rangle\rangle = \deg_\mu\langle\langle G\rangle\rangle$$

Proof

Immediately from Lemma 5.3. ∎

Immediately from the above lemma we have the following corollary:

Corollary 5.5

All powers of μ with non-zero coefficients have the same parity.

Let $b(G) = p_0(G) + p_1(G) - 1$.

Note that in Lemmas 5.2 and 5.3 only the attribute of an edge was important, not its color. In particular if $G = G_d$ (or $G = G_\ell$) (all the edges are black) then we can argue, in a way similar as in the case of Tutte polynomial, that cost of computing $q_{b(G)-k}$ is polynomial.

We will show that if G is a planar graph the cost of computing $q_{b(G)-k}$ can be bounded by a function which grows slower than exponentially.

Definition 5.6

A rooted tree T of high m is called (s,k,α)-unbalanced (where $0 < \alpha < 1$ and s,k are natural numbers) iff it satisfies the following:

If $m \leq s$ then T is a regular binary tree. Otherwise each internal node has two children or is a root of at most k (s,k,α)-unbalanced trees of high at most $\lceil \alpha m \rceil$ (where $\lceil x \rceil$ denotes the smallest integer greater or equal to x). Furthermore, every path in T which starts in the root of T and goes s times right (and arbitrary number of time left) contains a vertex being a root of (s,k,α)-unbalanced tree of high $\lceil \alpha m \rceil$.

Informally this definition says that (s,k,α) unbalanced tree of high m is a tree in which after going at most s times right we always reach a root of (s,k,α) unbalanced tree of high $\lceil \alpha m \rceil$. The lemma below says that such a tree has number of nodes much smaller than full binary tree of high m.

Lemma 5.7

A (s,k,α)-unbalanced tree of high m has $O(m^{c \ln m + s-1})$ nodes where $c = \dfrac{1}{\ln \frac{1}{\alpha}}$.

Proof

The proof follows by induction on s. The first step of the induction bases on the observation that the function which counts the number of leaves in a $(1,k,\alpha)$-unbalanced tree of high m, is bounded by function f(m) satisfying $f(m) - f(m-1) = k \cdot f(\alpha m)$. So it grows slower than a function g(x) such that $g'(x) = k \cdot g(\alpha x)$ and that in some interval $(\lceil \alpha n \rceil, n)$, $g(x) \geq f(x)$. But such a function g(x) grows slower than $x^{c \ln x}$

for $c \geq \frac{1}{\ell n \frac{1}{\alpha}}$. The inductive step follows by the similar argument. The details of the proof are left to the reader.

∎

We will use this lemma to prove the following theorem:

Theorem 5.8

Let G be a planar connected chromatic graph with m_d dark edges and m_ℓ light edges then the polynomial q_{b-2j} can be computed in $O(m_d^{c_1 \max(\ln m_d, 1)+4j+3} + c_2 \cdot m_\ell^{j+1})$ where c_1, c_2 are same constants.

Proof

First note that the only reason to introduce max $(\ln m_d, 1)$ was to obtain a correct formula for $m_d = 0$. In this case the result $O(c_2 m_\ell^{j+1})$ follows by the same argument as for Tutte polynomial. If $m_\ell = 0$ then we can use the duality theorem to reduce this to the case when $m_d = 0$. So assume $m_d \neq 0$ and $m_\ell \neq 0$.

We will start with the proof for $j = 0$. By Lemma 5.4, it follows that computing the polynomial of a chromatic graph G can be reduced in a polynomial time to computing the polynomial for a graph G' s.t. $|E(G_d')| \leq |E(G_d)|$, $|E(G_\ell')| \leq |E(G_\ell')|$, $|V(G')| \leq |V(G)|$ and G_d' forms a spanning tree for G'. This follows from the fact that if an edge, say e, is not a dark isthmus of G_d or is a light edge connecting two components of G_d, then the result of the computation of one of either G-e or G/e does not influence the value of q_b.

In the rest of the proof we will show that if G is such a graph that G_d is a spanning tree for G then we can remove and contract edges of G in such an order that corresponding computation tree is isomorphic to a $(4,8,\frac{5}{6})$-unbalanced tree.

Furthermore, the next edge to be deleted or contracted can always be determined in a polynomial time.

The basic fact used by our algorithm is that the polynomial of a disconnected graph is equal to the product of polynomials for its connected components times μ^{p_0-1} and that the polynomial of connected graph is equal to the product of polynomials of its biconnected components. We will choose for the consecutive steps of the computation a sequence of edges which leads quickly to a decomposition of the original graph into connected or biconnected components of size at most $\frac{5}{6} m_d$ where by the size of G we understand the number of dark edges.

Assume that G is embedded on the plane. A construction of a $(4,8,\frac{5}{6})$-unbalanced computation tree of high m_d is given by the following algorithm:

1. Find an edge $e \in E(G_d)$ whose removal disconnects G_d into two subtrees G_d' and G_d'', each of them of size in the range $\langle \frac{1}{3} m_d, \frac{2}{3} m_d \rangle$ or a vertex v s.t. one can draw a simple closed curve going through v which divides G_d into two parts, G_d' and G_d'', of sizes in the same range.

   This can be done by the following simple algorithm:

   1.1 Choose a vertex, say u, to be the root of G_d.

   1.2 Let $v_1,\ldots,v_k$ be the children of u listed in an order defined by the embedding. Compute recursively for each v_i the number $m(v_i)$ equal to the number of edges in the subtree rooted at v_i plus one (for the edge (u,v_i)).

   1.3 If for some $i, \frac{1}{3} m_d \leq m(v_i) \leq \frac{2}{3} m_d$ then the algorithm returns edge (u,v_i).

1.4 If for some $i, m(v_i) > \frac{2}{3} m_d$ then go to step 1.2 with $u = v_i$.

1.5 If neither 1.3 nor 1.4 holds then we can split G_d in the vertex u into G_d' and G_d'' of required sizes. The algorithm returns vertex u.

2. If the algorithm from step 1 returned an edge, say e, then the computation of the polynomial for G-e can be reduced (by Lemma 5.3) in a polynomial time to computation of polynomials of few connected components spanned by G_d' and G_d'', each of them of size at most $\frac{2}{3} m_d$. So by induction computation tree for polynomials of G_d', G_d'' is $(4,8,\frac{5}{6})$-unbalanced.
3. Assume that in step 1 we have found a split vertex v. Perform the algorithm defined in step 1 for the bigger of G_d' and G_d''.
4. If the algorithm in step 3 returned an edge, say e, or a vertex, say w, different than v then removing e (or any edge e' on the path from v to w) disconnects G_d into two parts, each of them at size in the range

$$\langle \frac{1}{6} m_d, \frac{5}{6} m_d \rangle.$$

Now similarly as in step 2, we can reduce computation of polynomial for G-e (or G-e') to computation of polynomials of two connected components, each of them of size at most $\frac{5}{6} m_d$.
5. Assume (opposite to the previous step) that algorithms from steps 1 and 3 returned the same vertex, say v. Then we can draw on the plane two (closed) Jordan curves ℓ_1, ℓ_2 which divides G_d according to the splits found in steps 1 and 3 such that:
   (i) neither of them cut a light edge more than once,
   (ii) both of them are going through vertex v and some vertex, w, on the external face.

Curves ℓ_1, ℓ_2 divide the plane into four parts, I, II, III and III', containing respectively dark subtrees T_1, T_2, T_3, T_3' (see Figure 5.1).

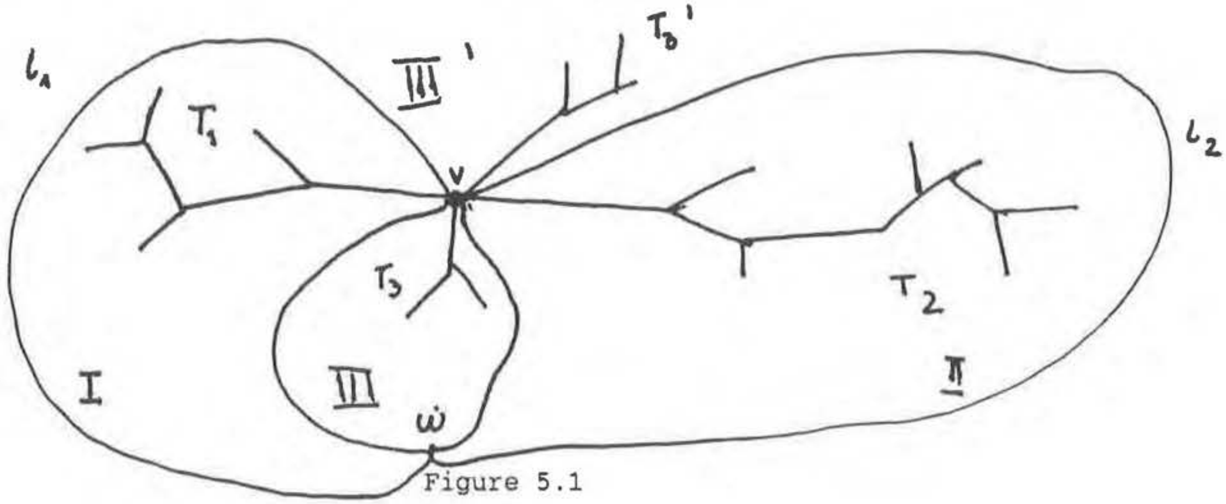
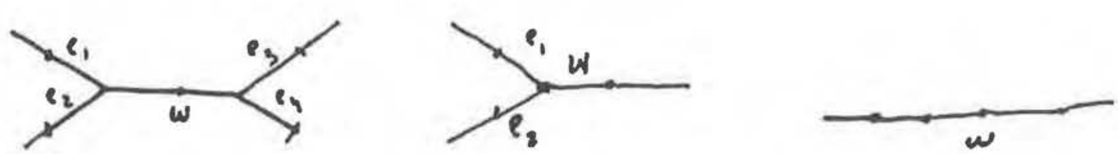


Figure 5.1

Consider now graph G* dual to G. Note that dark edges of G* (corresponding to light edges of G) form a spanning tree for G*. In particular light edges which are cut by ℓ_1 and ℓ_2 correspond to a dark subtree in G* denoted by L_{12}. This subtree may look like in Figure 5.2. On this figure w denotes the vertex corresponding to the external face.



a) b) c)

Figure 5.2

By removing four edges in the case of a) and two edges in the case of b) this subtree breaks down to linear pieces. These four (respectively two) edges will be called branching edges. Removing an edge in G dual to a branching edge will be called a critical step.

Consider first case c. Since there is no branching edges, curves ℓ_1 and ℓ_2 meet exactly the same light edges. This means that there are no light edges between nodes in subtree T_3 or subtree T_3' and subtree T_1 or subtree T_2. So vertices in parts I and II are in different biconnected components than vertices in parts III and III'. So v splits G into two (non-necessary connected) parts each of them of size at most $\frac{5}{6}\, m_d$. So computation of the polynomial for G can be reduced to computation of polynomials of those parts.

Consider now case a (case b is similar). If we remove branching edges then $(G^*)_d$ breaks down into five connected components. By Lemma 5.3 we can reduce the computation of the polynomial for G* without branching edges to the computation of five connected components. By the duality the computation of the polynomial for G without edges dual to branching edges can be reduced to computation of polynomial of five connected components. Note that with the exception of the connected component containing vertex v the vertices of all connected components may lay on at most two parts of the plane. These two parts correspond to left and right side of the parts of ℓ_1, ℓ_2 corresponding to one of the paths of L_{12} obtained by removing branching edges. So all connected components except one are of size at most $\frac{5}{6}\, m_d$. On the other hand, vertex v is a cut vertex for the remaining connected component. So computation of the polynomial for this connected component can be reduced to the computation of polynomials for four graphs, each of them of size at most $\frac{5}{6}\, m_d$. This finishes the case a.

We have shown that after at most four consecutive critical steps the computation on the branch of the computation tree on which those steps occur can be reduced to the computation of polynomials of at most eight graphs of size at most $\frac{5}{6}\, m_d$.

We can think about critical steps as about steps going right in the computation tree. To finish the proof we should show that if between those critical steps will be some number of steps "going left", then after this mixed sequence of steps the computation can also be reduced to computing polynomials of at most eight graphs of size at most $\frac{5}{6}\, m_d$. But if going right on computation tree means to contract a light edge, then going left means to delete this edge. Since deleting a light edge does not increase sizes of connected components we can use the graph L_{12} (with one edge contracted) to determine the next branching edge (if any). So even if critical steps (i.e. steps right) are alternated with removing light edges crossed by ℓ_1 or ℓ_2, (i.e. by steps left) four critical steps suffices to reduce the computation to computation of polynomials of at most 8 graphs of size at most 5/6 m, so the computation tree obtained in this way is $(4,8,\frac{5}{6})$-unbalanced.

This finishes the proof for j = 0. For $j \neq \emptyset$ we have to extend the computation tree used for j = 0. We can do it in such a way that the "extended" tree will be $(4j+3,8,\frac{5}{6})$-unbalanced.

∎

One should notice that from the fact that Theorem 4.7 is dealing with planar graph it follows (by duality theorem) that we can replace m_d by $\min(m_\ell,m_d)$ and m_ℓ by $\max(m_\ell,m_d)$.

Theorem 4.7 has its application to knot theory.

It follows from the fact that every planar graph defines (by the rules 4.1) a link (more exactly a matched diagram of oriented link). So we can translate Theorem 4.7 to polynomials of links in the following way:

<u>Corollary 5.7</u>

Consider a matched diagram of an oriented link (4.1) and its skein polynomial $P_L(a,z) = \sum_{i=m}^{M} b_i(a) z^i$ where $b_m(a), b_M(a) \neq 0$. Then

(i) $m = 1 - S(L)$ where $S(L)$ is the number of components of the link L,

(ii) $b_{m+j}(a)$ can be computed in $0(n(D)^{c \ln n(D)+3+4j}$ time for some constance c. $n(D)$ denotes the number of crossings of the matched diagram D.

Proof

(i) follows from Lemma 4.2, namely if G is a planar dichromatic graph and D(G) its matched diagram then $p_0(G_b) + p_1(G_b)$ is equal to the number of components of the link with matched diagram D(G). Point (i) is also shown in [LM87].

(ii) follows from Theorem 5.6 and the relation between bracket $\langle\langle G\rangle\rangle$ of a planar dichromatic graph G and skein polynomial $P_{D(G)}(a,z)$ of the corresponding matched diagram D(G). The variable z in $P_{D(G)}(a,z)$ correspond to the variable μ^{-1} in $\langle\langle G\rangle\rangle$. ∎

It is an open question whether corollary 5.7(ii) holds for any oriented link diagram. The relation between matched diagrams and planar graphs introduced by 4.1 allows us also to translate results concerning knots to graph theory.

If we substitute $\mu = 0$, $A_i = 1$ and $B_i = (-1)^{i+1}B$ in the Kauffman bracket then for the simplified bracket $\langle G\rangle_B \in Z[B]$ we have the following:

Proposition 5.7 For planar graphs $\langle G\rangle_B$ can be computed in polynomial time.

Proof. Operations $|_i$ from the definition of the Kauffman bracket, Example 3, are reduced to

$$\langle G\rangle_B - \langle G/_d i\rangle_B = (-1)^{i+1}B\ \langle G-d^i\rangle_B$$

4.7 $$\langle G\rangle_B - \langle G-\ell^i\rangle_B = (-1)^{i+1}B\ \langle G/_\ell i\rangle$$

$$\langle\cdot\rangle = 1.$$

For a planar graph G and associated oriented link diagram D(G), 4.7 can be translated as

$$\langle \; \rangle_B - \langle \; \rangle_B = B\langle \; \rangle_B$$

$$\langle 0 \rangle = 1.$$

This is exactly the Alexander polynomial for $B = z$, Example 3 of Section 4. It is well known that the Alexander polynomial can be computed in a polynomial time (first computing the Seifert matrix V of the link L and then $\mathrm{Det}(\sqrt{t}^{-1} V^T - \sqrt{t}\, V)$. This is exactly the Alexander polynomial of L for $z = \sqrt{t} - \frac{1}{\sqrt{t}}$; see [BZ85].

■

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Appendix

to the manuscript: "Invariants of chromatic graphs" by T.Przytycka and J.H.Przytycki

Algebraic structures underlying Tutte polynomial of graphs

In the paper we considered abstract algebras which yielded, via corresponding theorems, invariants of graphs. This concept can be generalized to invariants of matroids in a similar way as graphs invariants can be generalized to invariants of matroids [BO88]. We present a further generalization which allows to define invariants for other families of subsets of a given set.

A.1 Notation and definitions

By a graph G = (V(G),E(G)) we understand a finite multigraph. G-e denotes the graph obtained from graph G by removing edge e, and G/e denotes the graph obtained from G by contracting edge e (i.e removing e and identifing its endpoints).

Definition A.1: A *setoid* is a pair $(E,\mathfrak{F})$ where E is a set and $\mathfrak{F} \subset 2^E$.

For example, a graph matroid, where E is a set of edges of a graph G and $\mathfrak{F}$ is a set of spanning forests, is a setoid. The notions introduced below are easily seen to be analogous to the corrresponding notions for graphs. Let $(E,\mathfrak{F})$ be a setoid.

Definition A.2 : An element $e \in E$ such that for every $T \in \mathfrak{F}$, $e \in T$ is called *an isthmus*. An element $e \in E$ such that for every $T \in \mathfrak{F}$, $e \notin T$ is called *a loop*.

Definition A.3 : *The dual* of a setoid $G=(E,\mathfrak{T})$ is the setoid $G^* = (E,\mathfrak{T}^*)$ where $\mathfrak{T}^*=\{T \subset E \mid (E-T)\in\mathfrak{T}\}$.

Remark A.4 : An element $e\in T$ is an isthmus of G if and only if it is a loop of G^*.

Definition A.5 : Let $G=(E,\mathfrak{T})$be a setoid and $e\in E$ be neither an isthmus nor a loop. We define setoids G-e and G/e in the following way:

$G\text{-}e = (E\text{-}\{e\}, \{T \subset E\text{-}\{e\} \mid T\in\mathfrak{T}\})$,

$G/e = (E\text{-}\{e\}, \{T \subset E\text{-}\{e\} \mid T\cup\{e\}\in\mathfrak{T}\})$.

We say that G-e (resp. G/e) is obtained from G by *removing* (resp. *contracting*) element e. Setoids which can be obtained from a given setoid by a sequence of *removing* and *contracting* elements are called *minors* of the given setoid. A property *P* of a setoid is said to be *closed under minors* if *P*(G) implies *P*(G') where G' is a minor of G.

Remark A.6 : $G^*\text{-}e = (G/e)^*$ and $G^*/e = (G\text{-}e)^*$.

Assume that E is a countable set. We introduce an order I on its elements: $a_1,a_2,a_3,\ldots$.

Definition A.7 : *A computation tree $C_I(G)$ of the setoid G with respect to the order I* is a rooted binary tree (possibly infinite) whose leaves are setoids and which satisfy the following conditions:

(i) if $\forall e\in E$, e is either an isthmus or a loop then $C_I(G)$ is a leaf equal to G,

(ii) otherwise let e be the smallest element of E which is neither an isthmus nor a loop; the left subtree of the root of $C_I(G)$ is the tree $C_I(G\text{-}e)$ and the right subtree of the root of $C_I(G)$ is the tree $C_I(G/e)$.

Note that every leaf of $C_I(G)$ corresponds to a set in $\mathfrak{F}$. An element e chosen in point (ii) is called *a branching element* of $C_I(G)$.

Definition A.8 : Let $T \in \mathfrak{F}$. An element $e \in T$ (resp. $e \notin T$) is called *internally* (resp. *externally*) *stable* for set T with respect to an order I if e does not occur as a branching element of $C_I(G)$ on the path from the root to the leaf corresponding to T.

Remark A.9 : An element $e \in T$ (resp. $e \notin T$) is internally (resp. externally) stable for a set T with respect to an order I if and only if e is externally (resp. internally) stable for the set T* in the dual setoid with respect to the order I.

Note that there is an obvious correspondence between the above notions and the notions used to define Tutte polynomials for graphs, namely an internally (resp. externally) stable element corresponds to an internally (resp. externally) active edge.

From the definitions of an isthmus and a loop it follows immediately :

Lemma A.10:

(i) f is an isthmus of G/e $\Leftrightarrow$ e is a loop of G-f,

(ii) f is an isthmus of G-e $\Rightarrow$ e is an isthmus of G-f,

(iii) f is a loop of G/e $\Rightarrow$ e is a loop of G/f.

Note that in graph theory contracting an edge never introduces an isthmus and that removing an edge never introduces a loop.

Consider a class of setoids that satisfy the additional properties *P1* and *P2* defined as follows:

P1: If f is not an isthmus (resp. a loop) of G then, for any $e \in E$ that is neither an isthmus nor a loop, f is not an isthmus of G/e (resp. a loop of G-e).

P2: If e is neither an isthmus nor a loop of a setoid G and f is an isthmus (resp. a loop) of the setoid G-e (resp. G/e) then interchanging element e and element f in each set of $\mathfrak{F}$ gives a setoid identical to G.

Definition A.11: A setoid for which properties P1 and P2 hold and are closed under minors is called *a P1/2-restricted setoid.*

Definition A.12: Let $\mathcal{G}$ be a family of setoids. A function W: $\mathcal{G} \to X$ is called an *invariant* of setoids in this family if and only if for any two isomorphic setoids $G_1, G_2 \in \mathcal{G}$, $W(G_1) = W(G_2)$.

A.2. Invariants of P1/2-restricted setoids

Assume that the set E is finite. Consider an abstract algebra $\mathcal{A}$ defined as follows:

$$\mathcal{A} = (U, \mid, \{a_i\}_{i \in Q})$$

where $\mid : U \times U \to U$ satisfies the axiom:

$$Ax1 : \quad (a|b)|(c|d) = (a|c)|(b|d).$$

and Q is the set of two variable monomials with coefficient one.

For $p = x^i y^j$ define [p] to be the class of setoids $(E, \mathfrak{F})$ such that every element $e \in E$ is either a loop or an isthmus and such that the number of isthmuses is equal to i and the number of loops is equal to j. The following theorem shows that models of algebra $\mathcal{A}$ yield invariants of P1/2-restricted setoids.

Theorem A.13 : Let $\mathfrak{A} = \langle U, \mid, \{a_i\}_{i \in Q} \rangle$ be a model of algebra $\mathcal{A}$ and let $\mathcal{G}$ be the class of all P1/2-restricted setoids, then the function W: $\mathcal{G} \to U$, given by conditions (i) and (ii) below, is well defined and therefore is an invariant of setoids in $\mathcal{G}$:

(i) if $G \in [p]$ then $W(G) = a_p$,

(ii) otherwise let $e \in E$ be an element which is neither an isthmus nor a loop then

$$W(G)=W(G\text{-}e) \mid W(G/e).$$

Proof: To prove the theorem we have to show that W(G) does not depend on the method of computation (i.e on the choice of an element in step (ii)). The proof follows by induction on the number, say k, of elements that are neither isthmuses nor loops. If $k \leq 1$ then the theorem follows immediately from (i) and (ii). Assume that the theorem holds for all numbers less than k. Consider a setoid which has k ($k \geq 2$) elements which are neither isthmuses nor loops and let e and f be two such elements. We will show that W(G) does not depend on which of e,f is used in point (ii).

If f is neither an isthmus of G-e nor a loop of G/e then formula (ii) applied to e gives:

$$L = W(G\text{-}e) \mid W(G/e)$$

Simmilarly formula (ii) applied to f gives:

$$R=W(G\text{-}f) \mid W(G/f)$$

By the inductive hypothesis invarants for G-e, G/e, G-f, G/f are well defined so we can use formula (ii) to get

$$L = W(G\text{-}e\text{-}f) \mid W(G\text{-}e/f)) \mid (W(G/e\text{-}f) \mid W(G/e/f) \text{ and}$$

$$R = W(G\text{-}f\text{-}e) \mid W(G\text{-}f/e) \mid (W(G/f\text{-}e) \mid (W(G/f/e).$$

But G-e-f = G-f-e, G/e/f = G/f/e, G-e/f = G/f-e, G-f/e = G/e-f. So by Ax1 we have:

$$(W(G\text{-}e\text{-}f) \mid W(G\text{-}e/f)) \mid (W(G/e\text{-}f) \mid W(G/e/f)) = (W(G\text{-}f\text{-}e) \mid W(G\text{-}f/e) \mid (W(G/f\text{-}e) \mid (W(G/f/e))$$

and L=R in this case.

If f is an isthmus of G-e (the case when f is a loop of G/e is similar) then again formula (ii) applied to e gives:

$$L = W(G\text{-}e) \mid W(G/e)$$

Similarly formula (ii) applied to f gives:

$$R=W(G\text{-}f) \mid W(G/f).$$

By the inductive hypothesis, invarants for G-e, G/e, G-f, G/f are well defined. By *P2* W(G-e) = W(G-f) and W(G/e) = W(G/f) so L=R.

By property *P1* these are all the cases we have to consider. []

Example A.14: Chose a model $\mathfrak{A}1 = \langle U, \mid, \{a_i\}_{i\in Q}\rangle$ as follows:

$$U=Z[x,y];\ a|b = a+b;\ \text{for } p= x^iy^j \text{ define } a_p= x^iy^j$$

then W(G) is a generalization of the Tutte polynomial for graphs to P1/2-restricted setoids. Denote this polynomial by $\chi(G)$. It is easy to check that if $\mathfrak{F}$ is a basis of a matroid $(E,\mathfrak{F}')$ then $(E,\mathfrak{F})$ defines a P1/2-restricted setoid. Note that *P1* does need to hold when $\mathfrak{F}$ is a basis of a greedoid (the set of maximal elements), where a greedoid is a setoid $(E,\mathfrak{K})$ satisfying the following conditions ([KL82]) :

(i) $\varnothing\in\mathfrak{K}$,

(ii) if $\varnothing\neq X\in\mathfrak{K}$ then $\exists a\in X$, $X-\{a\}\in\mathfrak{K}$,

(iii) $X,Y\in\mathfrak{K}$, $|X|>|Y|$ then $\exists a\in X-Y$, $Y\cup\{a\}\in\mathfrak{K}$.

An example of such a greedoid is given by $G1=(E,\mathfrak{K})$ where $E=\{a,b,c,d\}$, $\mathfrak{K}=\{\varnothing,\{b\},\{c\},\{d\},\{a,b\},\{b,c\},\{d,b\},\{c,d\}\}$ and $\mathfrak{F}=\{\{a,b\},\{b,c\},\{d,b\},\{c,d\}\}$. Note that polynomial χ is undefined for this greedoid (orders a,c,b,d and c,a,b,d lead respectively to polynomials xy^2+y+6 and $y^2+2y+xy+4$).

The class of P1/2-restricted setoids is, however, larger then matroids. An example is given by the following setoid: $G2=(E,\mathfrak{F})$ where $E=\{a,b,c,d\}$, $\mathfrak{F}=\{\{a\},\{b\},\{c\},\{d\},\{a,b,c\},\{a,c,d\},\{a,b,d\},\{b,c,d\}\}$. It is easy to check that this is not a matroid but it is a P1/2 restricted setoid and $\chi(G2)=4x+4y$.

Note that by Definition A.9 for any P1/2-restricted setoid we have:

$$\chi(G)=\sum_{T\in\mathfrak{F}} x^{i_T}y^{j_T}$$

where i_T is the number of internally stable elements in T and j_T is the number of externally stable elements of T with respect to a given order I of elements of E.This formula generalized Tutte's state model for χ to the class of P1/2-restricted setoids([T84]).

Example A.15: Consider the following model $\mathfrak{A}2 = \langle U, |, \{a_i\}_{i\in Q}\rangle$ of algebra $\mathcal{A}$:

$$U = Z[x^{\pm 1}, y^{\pm 1}]; \quad a|b = y^{-1}a + x^{-1}b; \quad a_p = 1.$$

Denote by S(G) (Laurent) polynomial invariant of P1/2-restricted setoids yelded by this model. Observe that if all elements of $\mathfrak{F}$ have the same cardinality,say h, then $\chi(G) = x^h y^{|E|-h} S(G)$. The invariant S(G) can be extended to countable setoids (values of S(G) will be in (Laurent) infinite series of variables x and y). The state formula for S(G) can be written:

$$S(G) = \sum_{T\in\mathfrak{F}} x^{-i_T} y^{-j_T}$$

where i_T (resp. j_T) is the number of externally (resp. internally) unstable elements of T with respect to a given order I of elements of E. We assume that $x^{-\infty} = y^{-\infty} = 0$. For example if E is the set of natural numbers, $\mathfrak{F} = \{\ \{i,j\} \mid i \neq j\ \}$ then $S(G) = x^{-2}y^2 \frac{1}{(1-y)^2}$.

References to the appendix

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