Spatial and Spectral Descriptions of Stationary Gaussian Fractals

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Abstract

A general treatment of stationary Gaussian fractals is presented. Relations are established between the fractal properties of an n-dimensional random field and the form of its correlation function and power spectrum. These relations are used to show that the second-order parameter H commonly used to describe fractal texture (e.g., in [4][5]) is insufficient to characterize all fractal aspects of the field. A larger set of measures — based on the power spectrum — is shown to provide a more complete description of fractal texture.

Several interesting types of "non-fractal" self-similarity are also developed. These include a generalization of the fractional Gaussian noises of Mandelbrot and van Ness [6], as well as a form of "locally" self-similar behaviour. It is shown that these have close relations to the Gaussian fractals, and consequently, that textures containing these types of self-similarity can be described by the same set of measures as used for fractal texture.

I. Introduction

The piecewise-differentiable curves and surfaces commonly used to describe form do not adequately capture all aspects of natural structure. For many obects, interesting structure exists at all scales of measurement. Over the last few years, increasing attention has given to the use of fractals [1] for describing such structure.

Fractals are usually defined as sets having a non-integral Hausdorff-Besicovitch dimension D [1][2]. For surfaces in three-dimensional Euclidean space, the value of D ranges between 2 and 3 — when $D \rightarrow 2$, the surface is smooth and almost planar; when $D \rightarrow 3$, it appears extremely rough and jagged. Since other types of dimension are also possible, fractals are usually described by the more general ***similarity parameter*** H, which must be in the range $0 < H < 1$ [2]. For fractal surfaces in three-dimensional space, it is conjectured that $D = 3 - H$ [1].

Image texture has often been modelled — both in computer graphics and image analysis — by ***self-similar*** fractals (e.g., [1][3][4][5]). These fractals have properties that match themselves under rescaling by some ***scaling ratio*** $h > 0$. More precisely, a self-similar stochastic fractal $\mathrm{a}(\vec{x})$ is characterized by the equation

$$\mathrm{a}(\vec{x}_1 + h\vec{x}) - \mathrm{a}(\vec{x}_1) = h^H[\mathrm{a}(\vec{x}_1 + \vec{x}) - \mathrm{a}(\vec{x}_1)],$$

for arbitrary position $\vec{x}_1$ and displacement $\vec{x}$ [2][6]. Image texture models have typically been based on Gaussian (Brownian) fractals, for which the random fields are composed of Gaussian random variables.

This paper provides a general treatment of stationary self-similar Gaussian fractals, based on the use of the self-similar random fields described in section II. Sufficient conditions for the existence of stationary Gaussian fractals are established in section III, where their properties are linked to the form of their power spectra and correlation functions. It is shown that the second-order parameter H is not sufficient to describe a stochastic fractal — other parameters, such as the scaling ratio h are also required. This approach is carried out for the general n-dimensional case, enabling a natural description of two-dimensional fractal textures to be given in terms of two-dimensional power spectra. As such, this generalizes some of the work described in [4].

Section IV contains an extension of these ideas to random fields that are non-fractal, but still contain interesting forms of self-similar behaviour. These fields are shown to be a generalization of the stationary "fractional Gaussian noises" of Mandelbrot and van Ness [6]. Some classes of texture with H outside the usual range for fractals (as found in e.g., [4]) are therefore well-defined. The more general question of how to interpret values of H falling outside the bounds required for self-similar random fields is addressed in section V, where the idea of "local" self-similarity is briefly examined.

II. Self-Similar Random Fields

To cast the descriptions of self-similar stochastic fractals into more conventional form, it is useful to consider them as special cases of ***self-similar random fields.*** These are defined here to be stationary real-valued n-dimensional Gaussian random fields with power spectra $S(\vec{k})$ such that for some $H \in \Re$, $h \in (0, \infty)$

$$S(h\vec{k}) = h^{-n-2H} S(\vec{k}) \qquad \vec{k} \neq 0,$$

where $\vec{k}$ is the n-dimensional spatial frequency. Anticipating the relation of these fields to stochastic fractals, the quantity h will be called the ***scaling ratio*** of the field, and the quantity H its ***similarity parameter.***

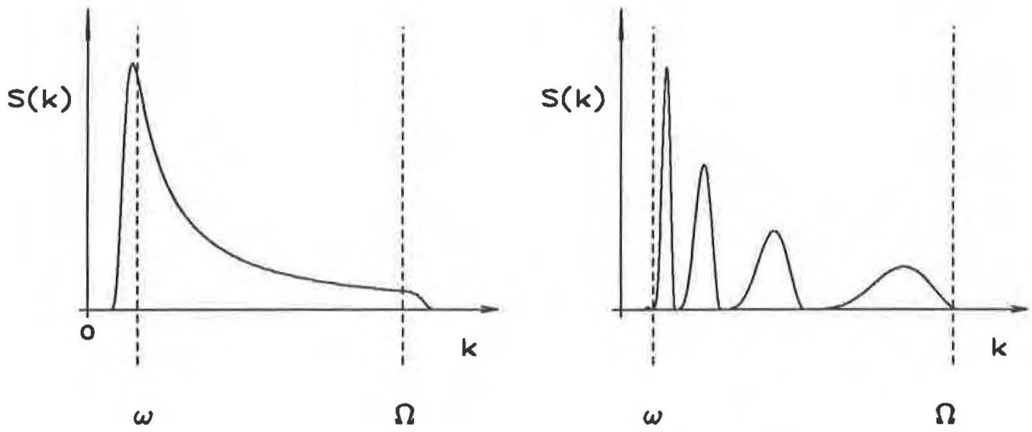
The scaling ratio describes the "scaling periodicity" of the spectrum, i.e., the amount of rescaling needed for $S(\vec{k})$ to be similar to its rescaled form. Owing to the reciprocal roles played by h and $1/h$, removing the interval $(0, 1)$ from the domain of h would have no effect on the class of spectra captured by the definition. To allow a unique value to be given to the scaling ratio of a field, then, the value of h will be taken to be the minimum possible value greater than one.

As is evident from the definition, self-similar spectra are completely specified by the parameters h and H, together with a pattern function $P(\vec{k})$ defined over the region $k_{base} \leq |\vec{k}| < hk_{base}$, where $k_{base} \neq 0$ is some arbitrary spatial frequency. Examples of spectra for the one-dimensional case are shown in figure 1, the spectra approaching a self-similar form as the bounds ω and Ω approach 0 and ∞ respectively. In figure 1(a), $S(k) \propto k^{-1-2H}$ in the interval (ω, Ω), corresponding to the case $h \to 1$. More generally, when $h \to 1$, the pattern $P(\vec{k})$ is a function of direction only, viz., $P(\vec{k}) = P(\vec{k}/k)$. In contrast, the pattern function shown in figure 1(b) may take on a variety of forms in the interval $k_{base} \leq k < hk_{base}$, since $h - 1$ is a finite positive quantity.

Sufficient conditions for the existence of these fields must now be established. As the following two sections show, not only the existence but also the nature of these fields is determined by the value of the similarity parameter H.

III. Self-Similar Stochastic Fractals

In general, stochastic fractals are described in terms of various topological and metric properties [1][2]. For a stationary Gaussian fractal, however, the description can be recast into more a conventional form based on power spectra and correlation functions.



(a) $H = 0.20,\ h \to 1$ (b) $H = 0.00,\ h = 2$

Figure 1: examples of self-similar power spectra

Theorem 1: A stationary Gaussian random field $\mathbf{f}(\vec{x})$ has a correlation function $R(\vec{x})$ such that within some range $\lambda < |\vec{x}| < \Lambda/h$

$$R(h\vec{x}) - R(\vec{0}) = h^{2H}[R(\vec{x}) - R(\vec{0})];$$

iff within that range the field behaves as a stationary stochastic fractal, with scaling ratio h, and similarity parameter $H \in (0,1)$.

Proof: If the random field $\mathbf{f}(\vec{x})$ is stationary and Gaussian, the behaviour of its increments can be described by

$$\mathbf{f}(\vec{x}_1 + h\vec{x}) - \mathbf{f}(\vec{x}_1) = w(h, H, \vec{x})[\mathbf{f}(\vec{x}_1 + \vec{x}) - \mathbf{f}(\vec{x}_1)]$$

where $\vec{x}_1$ is an arbitrary point, $\vec{x}$ an arbitrary displacement, and $w(h, H, \vec{x})$ is some scalar function as yet undetermined. Taking the expected value of the square of both sides and using the symmetry of the correlation function yields

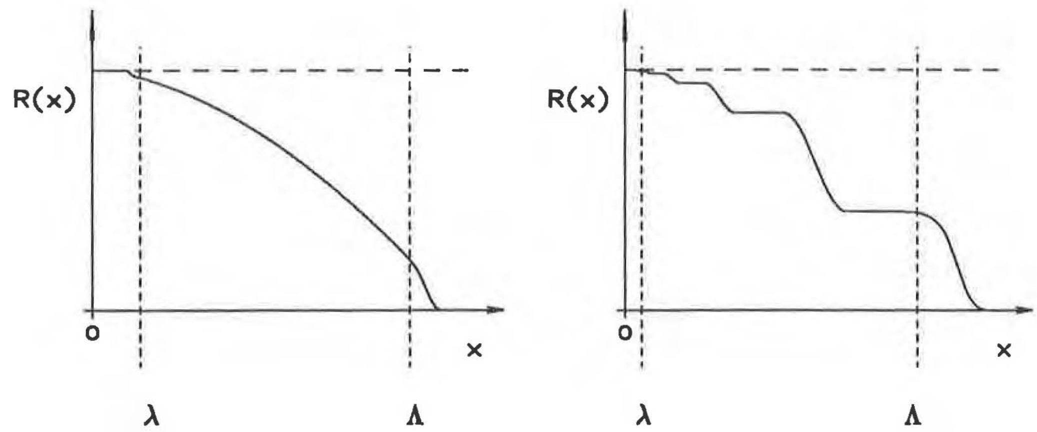
$$[R(h\vec{x}) - R(\vec{0})] = w(h, H, \vec{x})^2[R(\vec{x}) - R(\vec{0})].$$

When $\lambda < |\vec{x}| < \Lambda/h$, $w(h, H, \vec{x})$ can be identified as h^H; the random field therefore exhibits fractal behaviour in this range.

Conversely, if the field is a Gaussian fractal, its increments are such that

$$\mathbf{f}(\vec{x}_1 + h\vec{x}) - \mathbf{f}(\vec{x}_1) = h^H[\mathbf{f}(\vec{x}_1 + \vec{x}) - \mathbf{f}(\vec{x}_1)]; \quad \lambda < |\vec{x}| < \Lambda.$$

Taking the expected value of the square of both sides leads to a correlation function of the appropriate form. ■



(a) $H = 0.85$, $h \to 1$ (b) $H = 0.85$, $h = 2$

Figure 2: examples of self-similar correlation functions

Examples of one-dimensional self-similar correlation functions are shown in figure 2. Note that only *differences* of $R(\vec{x})$ are involved. Consequently, the mean μ of the random field has no effect on its self-similar behaviour. Also, the random field need not be isotropic — the only requirement on $R(\vec{x})$ is that it be symmetric about the origin in order for the field to exist.

More generally, the conditions sufficient for the existence of a self-similar Gaussian fractal are given in the following theorem:

Theorem 2: Let $\mathbf{f}(\vec{x})$ be a stationary n-dimensional real-valued random field with a power spectrum $S(\vec{k})$ integrable in some region $b \leq |\vec{k}| < bh$, $b \in \Re^+$. The power spectrum then approaches a form such that

$$S(h\vec{k}) = h^{-n-2H} S(\vec{k}); \qquad \vec{k} \neq 0$$

iff the behaviour of $\mathbf{f}(\vec{x})$ approaches that of a stochastic fractal with scaling ratio h and similarity parameter $0 < H < 1$.

Proof: Consider the function

$$\tilde{S}_a(\vec{x}) = \int_a^\infty S(\vec{k}) \exp\{i2\pi(\vec{x} \cdot \vec{k})\} d\vec{k}.$$

The central area $|\vec{k}| < a$ of $S(k)$ has been deleted, and its Fourier transform obtained. This transform can be rewritten as a function of radial distance $k = |\vec{k}|$

and $n-1$ angular parameters. Since $S(\vec{k})$ is integrable over $b \leq |\vec{k}| < bh$, it follows from self-similarity that the integral of $S(\vec{k})$ over $\vec{k}$ may be bounded from above by the integral of the function $A|\vec{k}|^{-n-2H}, \vec{k} \neq 0$, where A is some positive real-valued constant. This leads to the observation that the integral of $S(\vec{k})$ over the $n-1$ angular parameters is bounded from above by $A'k^{-2H-1}$, where A' is some positive real constant. Given that $a > 0$ and $H > 0$, $\tilde{S}_a(0)$ must be finite, and consequently $\tilde{S}_a(\vec{x})$ must exist for all $\vec{x}$.

Subtracting the term $\tilde{S}_a(0)$ and rescaling yields

$$\tilde{S}_a(h\vec{x}) - \tilde{S}_a(0) = h^{-n} \int_{ah}^{\infty} S(\vec{k}/h)[\exp\{i2\pi(\vec{x}\cdot\vec{k})\} - 1]d\vec{k}.$$

Due to the term $[\exp\{i2\pi(\vec{x}\cdot\vec{k})\} - 1]$, there is no contribution of $S(0)$ to the integral. Using the relation $h^{n+2H}S(\vec{k}) = S(\vec{k}/h)$ in the above equation leads to

$$\begin{aligned}\Delta_a(\vec{x}) &= h^{2H}[\tilde{S}_a(\vec{x}) - \tilde{S}_a(0)] - [\tilde{S}_a(h\vec{x}) - \tilde{S}_a(0)] \\ &= h^{2H} \textstyle\int_a^{ah} S(\vec{k})[\exp\{i2\pi(\vec{x}\cdot\vec{k})\} - 1]d\vec{k},\end{aligned}$$

where the error term $\Delta_a(\vec{x})$ describes the deviation from true self-similarity.

Since the field is real-valued, its power spectrum is necessarily symmetric about the origin. Owing to this symmetry, the exponent of $\tilde{S}_a(\vec{x})$can be replaced by a cosine. Thus, for $a|\vec{x}| << 1/h$, the magnitude of $\Delta_a(\vec{x})$ obeys the inequality

$$\begin{aligned}|\Delta_a(\vec{x})| &< |h^{2H} \textstyle\int_a^{ah} S(\vec{k})(2\pi\vec{x}\cdot\vec{k})^2 d\vec{k}| \\ &\leq 4\pi^2 A h^{2H}|\vec{x}|^2 \textstyle\int_a^{ah} |(|\vec{k}|^{-n-2H+2}d\vec{k}|\end{aligned}$$

Given that the volume of an n-dimensional sphere of radius ρ is [1]

$$\frac{2\pi^{n/2}}{n\Gamma(n/2)} \cdot \rho^n,$$

where $\Gamma(x)$ is the gamma function, the integral over the angular parameters is limited by $2\pi^{n/2}/\Gamma(n/2)$. It therefore follows that

$$|\Delta_a(\vec{x})| \leq \frac{8\pi^{2+n/2}A|h^2 - h^{2H}|a^{2-2H}}{\Gamma(n/2)(2-2H)}|\vec{x}|^2.$$

The deviation from true self-similarity thus has an upper bound proportional to the square of the distance from the origin. For any given amount of error, a spatial range $|\vec{x}| < \Lambda$ can be found within which $\tilde{S}_a(\vec{x})$ has sufficiently good self-similar behaviour.

Since $0 < H < 1$, the size of the error term decreases with decreasing a. Correspondingly, the range of self-similar behaviour exhibited by $\tilde{S}_a(\vec{x})$ increases. Since

$$R(\vec{x}) = \lim_{a\to 0} \tilde{S}_a(\vec{x}),$$

the behaviour of $R(\vec{x})$ approaches

$$[R(h\vec{x}) - R(0)] = h^{2H}[R(\vec{x}) - R(0)].$$

Theorem 1 may then be invoked to show that the field exhibits self-similar fractal behaviour within a range that increases without bound as $a \to 0$.

For the converse case, consider a random field with correlation function $R(\vec{x})$ such that

$$R(h\vec{x}) - R(0) = h^{2H}[R(\vec{x}) - R(0)]$$

for $|\vec{x}| \leq b/h$, and zero outside $|\vec{x}| > b$. From Theorem 1, it follows that such a field exhibits fractal behaviour in the range $|\vec{x}| \leq b$.

Consider now the integral

$$\tilde{R}_b(\vec{k}) = \int_0^b [R(\vec{x}) - R(0)] \exp\{-i2\pi(\vec{x}\cdot\vec{k})\}d\vec{x}.$$

Rescaling, the difference

$$\begin{aligned} \Delta_b(\vec{k}) &= h^{-n-2H}\tilde{R}_b(\vec{k}/h) - \tilde{R}_b(\vec{k}) \\ &= h^{-n-2H}\textstyle\int_0^b [R(\vec{x}) - R(0)] \exp\{-i2\pi(\vec{x}\cdot\vec{k}/h)\}d\vec{x} \\ &\quad - \textstyle\int_0^b [R(\vec{x}) - R(0)] \exp\{-i2\pi(\vec{x}\cdot\vec{k})\}d\vec{x}, \quad \vec{k} \neq 0 \end{aligned}$$

becomes

$$\Delta_b(\vec{k}) = \int_{b/h}^b [R(0) - R(\vec{x})] \exp\{-i2\pi(\vec{x}\cdot\vec{k})\}d\vec{x}, \quad \vec{k} \neq 0.$$

Replacing the limit b by bh^j ($j \geq 1$), this difference becomes

$$\Delta_{bh^j}(\vec{k}) = \int_{bh^{j-1}}^{bh^j} [R(0) - R(\vec{x})] \exp\{-i2\pi(\vec{x}\cdot\vec{k})\}d\vec{x}, \quad \vec{k} \neq 0.$$

Note that the constraint $|R(\vec{x})| \leq R(0)$ [7:10-7] may require that $R(\vec{x})$ be displaced upwards as $j \to \infty$. However, this affects neither differences of the correlation function nor the value of $S(\vec{k})$ at $\vec{k} \neq 0$.

Introducing the variable $\vec{x}' = h^{-j}\vec{x}$, the error term becomes

$$\begin{aligned} \Delta_{bh^j}(\vec{k}) &= h^{jn}\textstyle\int_{b/h}^b [R(0) - R(\vec{x})]\exp\{i2\pi(\vec{x}'h^j\cdot\vec{k})\}d\vec{x}' \\ &= h^{(n-2H)j}\textstyle\int_{b/h}^b [R(0) - R(\vec{x}')]\exp\{i2\pi(\vec{x}'h^j\cdot\vec{k})\}d\vec{x}'. \end{aligned}$$

This integral can be written as

$$\begin{aligned} &\textstyle\int\ldots\int [R(0) - R(\vec{x}')] \cos(2\pi x_1 k_1 h^j)\ldots\cos(2\pi x_n k_n h^j)dx_1'\ldots dx_n' \\ &\leq\ h^{-nj} I_R, \end{aligned}$$

where the term I_R is finite, owing to the boundedness of $R(0) - R(\vec{x}')$ over the fixed region of integration. Since $H > 0$, it follows that $\Delta_{bhj}(\vec{k}) \to 0$ as $j \to \infty$. The relation

$$S(\vec{k}) = \lim_{j \to \infty} \tilde{R}_{bhj}(\vec{k}); \vec{k} \neq 0$$

then leads to the required result. ■

IV. Self-similar Noises

The Gaussian fractals described above do not exhaust the types of self-similar random field possible. When $-n/2 < H < 0$, a different type of self-similar random field results, for which correlation functions with the same form of self-similarity as the power spectra. Restricted cases of such random fields were first introduced by Mandelbrot and van Ness[6], under the name of "fractional Gaussian noises". The fields developed here are a generalization of these; they will be referred to as *self-similar noises.*

Theorem 3: Let $\mathbf{f}(\vec{x})$ be a stationary n-dimensional random field with a power spectrum $S(\vec{k})$ integrable over some region $c \leq |\vec{k}| < ch$, for $c \in (0, \infty)$. For a similarity parameter $H \in (-n/2, 0)$, the power spectrum approaches a form such that

$$S(h\vec{k}) = h^{-n-2H} S(\vec{k}); \quad \vec{k} \neq 0$$

iff $\mathbf{f}(\vec{x})$ has a correlation function $R(\vec{x})$ such that

$$[R(h\vec{x}_1) - R(h\vec{x}_2)] = h^{2H}[R(\vec{x}_1) - R(\vec{x}_2)]; \quad \vec{x}_1, \vec{x}_2 \neq 0,$$

Proof: Consider the function

$$\tilde{S}_c(\vec{x}) = \int_0^c S(\vec{k}) \exp\{i2\pi(\vec{x} \cdot \vec{k})\} d\vec{k}.$$

Since $S(\vec{k})$ is integrable in the region $c \leq |\vec{k}| < ch$, it follows from self-similarity that the integral of $S(\vec{k})$ is bounded from above by the integral of $B|\vec{k}|^{-n-2H}, \vec{k} \neq 0$, where B is some real-valued positive constant. Integrating over the $n-1$ angular parameters, this leads to the upper bound $S(k) \leq B' k^{-2H-1}$ for some positive value $B' \in \Re$. Since $H < 0$ and $c > 0$, it follows that $\tilde{S}_c(\vec{k})$ must be finite.

Via a rescaling procedure similar to that used for theorem 2, the difference

$$\Delta_c(\vec{x}_1, \vec{x}_2) \;=\; h^{-2H}[\tilde{S}_c(h\vec{x}_1) - \tilde{S}_c(h\vec{x}_2)] - [\tilde{S}_c(\vec{x}_1) - \tilde{S}_c(\vec{x}_2)]$$

can be written as

$$\Delta_c(\vec{x}_1, \vec{x}_2) = \int_c^{ch} S(\vec{k})[\exp\{i2\pi(\vec{x}_1 \cdot \vec{k})\} - \exp\{i2\pi(\vec{x}_2 \cdot \vec{k})\}]d\vec{k}.$$

Replacing the limit c by ch^j, $j \geq 1$, this difference becomes

$$\Delta_{ch^j}(\vec{x}_1, \vec{x}_2) = \int_{ch^j}^{ch^{j+1}} S(\vec{k})[\exp\{i2\pi(\vec{x}_1 \cdot \vec{k})\} - \exp\{i2\pi(\vec{x}_2 \cdot \vec{k})\}]d\vec{k}.$$

Letting $\vec{k'} = h^{-j}\vec{k}$, this can be written

$$\begin{aligned} \Delta_{ch^j}(\vec{x}_1, \vec{x}_2) &\leq h^{jn} \textstyle\int_c^{ch} S(\vec{k'}h^j)[\exp\{i2\pi(\vec{x}_1 \cdot \vec{k'}h^j)\} - \exp\{i2\pi(\vec{x}_2 \cdot \vec{k'}h^j)\}]d\vec{k'} \\ &= h^{-2jH} \textstyle\int_c^{ch} S(\vec{k'})[\exp\{i2\pi(\vec{x}_1 \cdot \vec{k'}h^j)\} - \exp\{i2\pi(\vec{x}_2 \cdot \vec{k'}h^j)\}]d\vec{k'} \end{aligned}$$

Consider now the first term of this difference. The integral of the first term can be written

$$\begin{aligned} &\textstyle\int \ldots \int S(k) \cos(2\pi x_1 k_1 h^j) \ldots \cos(2\pi x_n k_n h^j) dk_1 \ldots dk_n; \quad c < |\vec{k}| \leq ch \\ &\leq h^{-nj} I_S, \end{aligned}$$

where the term I_S is finite, owing to the boundedness of $S(\vec{k})$ over the finite region of integration. The second term of the difference may be treated in a similarly. Thus,

$$\Delta_{ch^j}(\vec{x}_1, \vec{x}_2) \leq h^{(-2H-n)j}[I_{S_1} + I_{S_2}].$$

Since $-n/2 < H < 0$, it follows that $\Delta_{ch^j}(\vec{x}_1, \vec{x}_2) \to 0$ as $j \to 0$. The correlation function

$$R(\vec{x}) = \lim_{j \to \infty} \tilde{S}_{ch^j}(\vec{x}),$$

consequently approaches a self-similar form.

As is evident from this procedure, replacing the differences $R(h\vec{x}_1) - R(h\vec{x}_2)$ and $R(\vec{x}_1) - R(\vec{x}_2)$ by the corresponding quantities $S(h\vec{k})$ and $S(\vec{k})$ does not affect the general pattern of the proof. The converse case can therefore be established in a manner similar to that used for the forward direction. ■

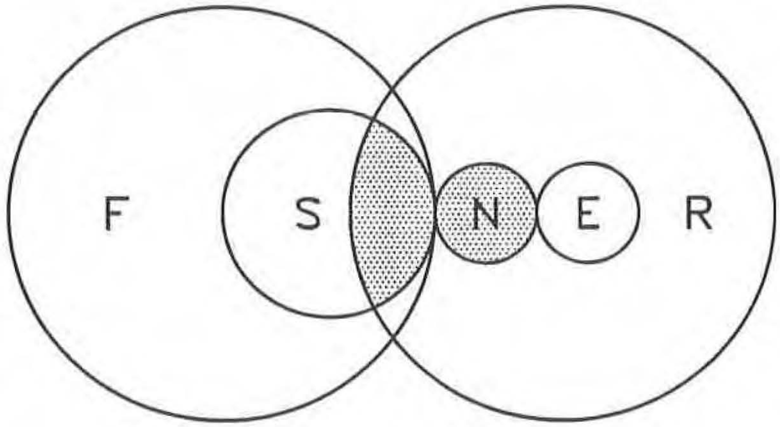


Figure 3: relation between fractals and random fields

V. Effectively Self-Similar Random Fields

Random fields with a power spectrum such that

$$S(h\vec{k}) = h^{-n-2H} S(\vec{k}); \quad \vec{k} \neq 0.$$

exist when $-n/2 < H < 0$ and $0 < H < 1$. These fields have a "global" self-similarity in the spectral and spatial realms, i.e., a rescaled function is similar to itself everywhere in the domain. However, it is unlikely that global self-similarity is attainable in most physical processes. Rather, what is required is a description of the degree of "local" self-similarity present. Such structure is captured by *effectively self-similar random fields*, which are self-similar only between some finite limits ω and Ω. These fields can exist provided $S(k)$ approaches zero quickly enough as $k \to 0$ and $k \to \infty$.

An interesting observation is that for effectively self-similar fields, the similarity parameter H may take on any real value, positive or negative. Thus, given that a random field has a power spectrum self-similar over the relevant range of frequencies, a much larger range of self-similar behaviour is possible locally than could exist globally. One instance of this has been found in computer graphics, where effectively self-similar fields have been discovered to give rise to textures having much the same qualitative structure as fractals and self-similar noises [8].

A schematic description of the general relation between self-similar stochastic fractals, self-similar noises, and effectively self-similar random fields is given in figure 3.

VI. Summary

This paper has provided a general description of Gaussian fractals, based on the use of self-similar random fields. A few simple conditions have been shown sufficient for the existence of a wide variety of Gaussian fractals, and treatment has been generalized to the n-dimensional case. Descriptions have been cast into the conventional language of power spectra and correlation functions. A natural consequence of this reformulation is the description of Gaussian fractals in terms of their similarity parameter H, scaling ratio h, and pattern function $P(\vec{k})$. A general set of measures for describing image textures in terms of their fractal properties follows naturally from this approach.

Self-similar random fields have also been shown to contain a class of self-similar noises. For these random fields, the self-similar behaviour of the power spectrum and correlation function are the same. The most important parameter determining the existence and nature of the random field is the similarity parameter H:

$0 < H < 1$: stochastic self-similar fractal

$-n/2 < H < 0$: self-similar noise

The properties of these self-similar noises have been related to the form of their power spectra and correlation functions. It has been shown that they may be described with the same set of measures as used for the Gaussian fractals. These descriptions also apply to "local" self-similarity as well, with some of the constraints on "global" self-similarity being relaxed.

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