Fingerprint Theorems for Curvature and Torsion Zero-Crossings

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Abstract

The *scale space image* of a signal $f(x)$ is constructed by extracting the zero-crossings of the second derivative of a Gaussian of variable size σ convolved with the signal, and recording them in the x–σ map.

Likewise, the *curvature scale space image* of a planar curve is computed by extracting the curvature zero-crossings of a parametric representation of the curve convolved with a Gaussian of variable size. The curvature level-crossings and torsion zero-crossings are used to compute the *curvature* and *torsion scale space images* of a space curve respectively.

It has been shown [Yuille and Poggio 1983] that the scale space image of a signal determines that signal uniquely up to constant scaling and a harmonic function. This paper presents a generalization of the proof given in [Yuille and Poggio 1983]. It is shown that the curvature scale space image of a planar curve determines the curve uniquely, up to constant scaling and a rigid motion. Furthermore, it is shown that the torsion scale space of a space curve determines the function $\tau(u)\kappa^2(u)$ modulus a scale factor where $\tau(u)$ and $\kappa(u)$ represent the torsion and curvature functions of the curve respectively. Our results show that a 1-D signal can be reconstructed using only one point from its scale space image. This is an improvement of the result obtained by Yuille and Poggio.

The proofs are constructive and assume that the parametrizations of the curves can be represented by polynomials of finite order. The scale maps of planar and space curves have been proposed as representations for those curves [Mokhtarian and Mackworth 1986, Mokhtarian 1988]. The result that such maps determine the curves they are computed from uniquely, shows that they satisfy an important criterion for any shape representation technique.

I. Introduction

Witkin [1983] and Stansfield [1980] introduced a *scale space* description of signals which shows the location of the zero-crossings of the second derivative of the signal convolved with a Gaussian filter of varying size. That is, the scale space description of a signal $f(x)$ consists of zero-crossing contours in the x–σ plane where σ is the width of the Gaussian filter.

Yuille and Poggio [1983, 1984] have shown that the zero-crossing contours in the scale space image of a signal of class P, at two distinct points at the same scale, almost always determine that signal up to constant scaling. This is an important result and shows that the scale space image of a signal represents that signal uniquely.

Mokhtarian and Mackworth [1986] generalized the scale space concept to planar curves. The generalized scale space, referred to as the *curvature scale space*, consists of curvature zero-crossing contours in the u–σ plane where u is the arc-length parameter along the curve and σ is again the width of the Gaussian filter. Mokhtarian [1988] further generalized the scale space concept to space curves. The scale space description of a space curve consists of the curvature scale space and the *torsion scale space* images of that curve which contain the curvature level-crossing and torsion zero-crossing contours of the curve respectively.

The scale space descriptions of planar and space curves have been proposed as representations for those curves [Mokhtarian and Mackworth 1986, Mokhtarian 1988]. One important property of such a representation is *completeness* [Nishihara 1981] or *uniqueness* [Mackworth 1987]. In the absence of this property, it would be impossible to differentiate between a curve and all other curves which are also represented by its representation. This paper shows that the scale space descriptions of planar curves represent those curves uniquely up to constant scaling and a rigid motion. However, the scale space description of a space curve only represents that curve up to a class which is larger than the class obtained by scaling and/or applying a rigid motion to the curve. Our results can be summarized in the following two theorems:

Theorem 1: Let $\Gamma = (x(u), y(u))$ be a planar curve in C_1 and let $x(u)$ and $y(u)$ be functions of class P representing the arc-length parametrization of Γ. A single point on one curvature zero-crossing contour in the curvature scale space image of Γ determines Γ uniquely up to constant scaling, rotation and translation (except on a set of measure zero).

Note that theorem **1** applies only to those curves which have at least one curvature zero-crossing contour in their curvature scale space images.

Theorem 2: Let $\Gamma = (x(u), y(u), z(u))$ be a space curve in C_1 and let $x(u)$, $y(u)$ and $z(u)$ be functions of class P representing the arc-length parametrization of Γ. Let $\tau(u)$ and $\kappa(u)$ represent the torsion and curvature functions of Γ respectively. A single point on one torsion zero-crossing contour in the torsion scale space image of Γ determines function $\beta(u) = \tau(u)\kappa^2(u)$ uniquely modulus a scale factor (except on a set of measure zero).

Again note that theorem **2** applies only to those curves which have at least one torsion zero-crossing contour in their torsion scale space images.

The proofs of theorems **1** and **2** are generalizations of the proof of the 1-D theorem in [Yuille and Poggio 1983]. The proofs start by taking derivatives along the curvature and torsion zero-crossing contours at a certain point. It is shown that the equations obtained are in terms of moments of functions related to the coordinate functions of the curve. Those moments are in turn shown to be related to the coefficients of expansion of the coordinate functions of the curve in functions related to the Hermite polynomials. The result is a system of homogeneous quadratic or cubic equations converted into a homogeneous linear system the solution to which determines the curve uniquely modulus constant scaling and a rigid motion in the 2-D case and function $\beta(u)$ modulus scaling in the 3-D case. In each case, a single point is sufficient for the reconstruction process.

II. Reconstructing a planar curve from its curvature scale space

This section contains a proof of theorem **1**. Section **II.A** shows that the derivatives at a point on a curvature zero-crossing contour provide homogeneous equations in the moments of the Fourier transforms of the coordinate functions of the curve. Section **II.B** shows that the moments are related to the coefficients of expansion of the coordinate functions of the curve in functions related to the Hermite polynomials. Section **II.C** shows that the moments at one curvature zero-crossing point can be related to the moments at other curvature zero-crossing points. Section **II.D** shows that the quadratic equations obtained in section **II.A** can be converted into a homogeneous linear system of equations which can be solved uniquely for the curvature function of the curve.

II.A. Constraints from the curvature zero-crossing contours

Let $\Gamma = (x(u), y(u))$ be the arc-length parametrization of the curve with Fourier transform $\tilde{\Gamma} = (\tilde{x}(\omega), \tilde{y}(\omega))$. The Fourier transform of the Gaussian filter $G(u,t) = \frac{1}{\sqrt{2t}} e^{-u^2/4t}$ is $\tilde{G}(\omega) = e^{-\omega^2 t}$.

Let $\Gamma_{t_0} = (x(u,t_0), y(u,t_0))$ be a curve obtained by convolving $x(u)$ and $y(u)$ with $G(u,t_0)$. Assume that Γ_{t_0} is in C_∞. Such a t_0 exists since Γ is in C_1. The curvature zero-crossings in a neighborhood of t_0 are given by solutions of $\alpha(u,t) = 0$ where

$$\alpha(u,t) = \dot{x}(u,t)\,\ddot{y}(u,t) - \dot{y}(u,t)\,\ddot{x}(u,t)$$

where . represents derivative with respect to u. Using the convolution theorem, the terms in $\alpha(u,t)$ can be expressed as following:

$$\dot{x}(u,t) = \int e^{-\omega^2 t} e^{i\omega u} (i\omega)\, \tilde{x}(\omega)\, d\omega$$

$$\ddot{y}(u,t) = \int e^{-\omega^2 t} e^{i\omega u} (-\omega^2)\, \tilde{y}(\omega)\, d\omega$$

$$\dot{y}(u,t) = \int e^{-\omega^2 t} e^{i\omega u} (i\omega)\, \tilde{y}(\omega)\, d\omega$$

$$\ddot{x}(u,t) = \int e^{-\omega^2 t} e^{i\omega u} (-\omega^2)\, \tilde{x}(\omega)\, d\omega$$

The Implicit Function Theorem guarantees that the contours $u(t)$ are C_∞ in a neighborhood of t_0. Let ξ be a parameter of the curvature zero-crossing contour. Then

$$\frac{d}{d\xi} = \frac{du}{d\xi}\frac{\partial}{\partial u} + \frac{dt}{d\xi}\frac{\partial}{\partial t}$$

On the curvature zero-crossing contour, $\alpha = 0$ and $\dfrac{d^k}{d\xi^k}\alpha = 0$ for all integers k. Furthermore, since the curvature zero-crossing contour is known, all the derivatives of u and t with respect to ξ are known as well.

We can now compute the derivatives of α with respect to ξ at (u_0, t_0). The first derivative is given by:

$$\begin{aligned}
\frac{d}{d\xi}\alpha(u_0,t_0) = \frac{du}{d\xi}\Big(&\int e^{-\omega^2 t}e^{i\omega u}(i\omega)^3\tilde{y}(\omega)\,d\omega \int e^{-\omega^2 t}e^{i\omega u}(i\omega)\tilde{x}(\omega)\,d\omega \\
&- \int e^{-\omega^2 t}e^{i\omega u}(i\omega)^3\tilde{x}(\omega)\,d\omega \int e^{-\omega^2 t}e^{i\omega u}(i\omega)\tilde{y}(\omega)\,d\omega\Big) \\
+ \frac{dt}{d\xi}\Big(&\int e^{-\omega^2 t}e^{i\omega u}(i\omega)^3\tilde{x}(\omega)\,d\omega \int e^{-\omega^2 t}e^{i\omega u}(-\omega^2)\,\tilde{y}(\omega)\,d\omega \\
&+ \int e^{-\omega^2 t}e^{i\omega u}(\omega^4)\,\tilde{y}(\omega)\,d\omega \int e^{-\omega^2 t}e^{i\omega u}(i\omega)\tilde{x}(\omega)\,d\omega \\
&- \int e^{-\omega^2 t}e^{i\omega u}(i\omega)^3\tilde{y}(\omega)\,d\omega \int e^{-\omega^2 t}e^{i\omega u}(-\omega^2)\,\tilde{x}(\omega)\,d\omega \\
&- \int e^{-\omega^2 t}e^{i\omega u}\omega^4\tilde{x}(\omega)\,d\omega \int e^{-\omega^2 t}e^{i\omega u}(i\omega)\tilde{y}(\omega)\,d\omega\Big)
\end{aligned} \tag{1}$$

Note that the moment of order k of function $f(\omega) = e^{-\omega^2 t}e^{i\omega u}(i\omega)\tilde{x}(\omega)$ is defined by:

$$M_k = \int_{-\infty}^{\infty} (i\omega)^k e^{-\omega^2 t}e^{i\omega u}(i\omega)\tilde{x}(\omega)\,d\omega$$

and the moment of order k of function $f'(\omega) = e^{-\omega^2 t}e^{i\omega u}(i\omega)\,\tilde{y}(\omega)$ is defined by:

$$M'_k = \int_{-\infty}^{\infty} (i\omega)^k e^{-\omega^2 t}e^{i\omega u}(i\omega)\,\tilde{y}(\omega)\,d\omega$$

As a result, equation (1) can be re-written as:

$$\frac{d}{d\xi}\alpha(u_0,t_0) = \frac{du}{d\xi}(M_2'M_0 - M_2M_0') \qquad (2)$$
$$+ \frac{dt}{d\xi}(M_2M_1' + M_3'M_0 - M_2'M_1 - M_3M_0')$$

Furthermore, the second derivative is given by:

$$\frac{d^2}{d\xi^2}\alpha(u_0,t_0) = \frac{d^2u}{d\xi^2}(M_2'M_0 - M_2M_0') \qquad (3)$$
$$+ \frac{d^2t}{d\xi^2}(M_2M_1' + M_3'M_0 - M_2'M_1 - M_3M_0')$$
$$+ \left(\frac{du}{d\xi}\right)^2(M_3'M_0 + M_1M_2' - M_3M_0' - M_1'M_2)$$
$$+ 2\frac{du}{d\xi}\frac{dt}{d\xi}(M_4'M_0 - M_4M_0')$$
$$+ \left(\frac{dt}{d\xi}\right)^2(M_4M_1' + 2M_3'M_2 + M_5'M_0 - M_4'M_1 - 2M_3M_2' - M_5M_0')$$

Since the parametric derivatives along the curvature zero-crossing contours are zero, equations (2) and (3) are equal to zero. Note that equation (2) is a quadratic equation in the first four moments of functions $f(\omega)$ and $f'(\omega)$ and equation (3) is a quadratic equation in the first six moments of those functions. In general, the k+1st equation, $\frac{d^k}{d\xi^k}\alpha(u,t) = 0$ is a quadratic equation in the first $2k$+2 moments of each of the functions $f(\omega)$ and $f'(\omega)$ or a total of $4k$+4 moments. Our axes are chosen such that $u_0 = 0$. The next section shows that the moments of $f(\omega)$ and $f'(\omega)$ are the coefficients a_k and b_k in the expression of functions $\dot{x}(u)$ and $\dot{y}(u)$ in functions related to the Hermite polynomials. Therefore, having computed the first n derivatives of α at (u_0,t_0), we have n+1 homogeneous equations in the first $4n$+4 coefficients a_k and b_k. To determine the a_k and b_k, we need $3n$+3 additional and independent equations which can be provided by considering three neighboring curvature zero-crossing contours at (u_1,t_0), (u_2,t_0), and (u_3,t_0).

II.B. The moments and the coefficients of expansion of $\dot{x}(u)$ and $\dot{y}(u)$

This section shows that the moments and the moment-pairs in equations $\frac{d^k}{d\xi^k}\alpha(u,t)$ are related respectively to the coefficients of the expression of the functions $\dot{x}(u)$, $\dot{y}(u)$ and the curvature function of Γ, $\kappa(u)$, in functions related to the Hermite polynomials. Expand function

$$\dot{x}(u) = \frac{d}{du}x(u)$$

in terms of the functions $\phi_k(u,\sigma)$ related to the Hermite polynomials $H_k(u)$ by

$$\phi_k(u,\sigma) = (-1)^k \frac{\sigma^{k-1}}{(\sqrt{2}\,)^{k+1}\sqrt{\pi}} H_k(\frac{u}{\sigma\sqrt{2}})$$

$$H_k(u) = (-1)^k e^{u^2} \frac{d^k}{du^k} e^{-u^2}$$

$$\dot{x}(u) = \sum a_k(\sigma)\phi_k(u,\sigma)$$

The coefficients $a_k(\sigma)$ of the expansion are given by

$$a_k(\sigma) = < w_k(u,\sigma), \dot{x}(u) >$$

where $<,>$ denotes inner product in L^2 and $\{w_k(u,\sigma)\}$ is the set of functions biorthogonal to $\{\phi_k(u,\sigma)\}$. The $\{\phi_k(u,\sigma)\}$ are given explicitly by

$$\phi_k(u,\sigma) = \frac{\sigma^{2k-1}}{k!\sqrt{2\pi}} e^{\frac{u^2}{2\sigma^2}} \frac{d^k}{du^k} e^{-\frac{u^2}{2\sigma^2}}$$

and the $w_k(u,\sigma)$ by

$$w_k(u,\sigma) = (-1)^k \frac{d^k}{du^k} e^{-\frac{u^2}{2\sigma^2}}$$

Since

$$\dot{x}(u) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega u}(i\omega)\tilde{x}(\omega)\, d\omega$$

the a_k are given by

$$a_k(\sigma) = \frac{1}{\sqrt{2\pi}} (-1)^k \int < \frac{d^k}{du^k} e^{-\frac{u^2}{2\sigma^2}}, e^{i\omega u} > (i\omega)\tilde{x}(\omega)\, d\omega$$

The inner product is just the inverse Fourier transform of $w_k(u,\sigma)$. Therefore

$$a_k(\sigma) = \int (i\omega)^k e^{\frac{-\omega^2\sigma^2}{2}} (i\omega)\, \tilde{x}(\omega)\, d\omega$$

which is equal to M_k modulus a factor $e^{i\omega u}$, since $t = \frac{\sigma^2}{2}$.

Similarly, the function

$$\dot{y}(u) = \frac{d}{du}y(u)$$

can be expanded in terms of the functions $\phi_k(u,\sigma)$ by

$$\dot{y}(u) = \sum b_k(\sigma)\,\phi_k(u,\sigma)$$

and it again follows that

$$b_k(\sigma) = \int (i\omega)^k e^{\frac{-\omega^2\sigma^2}{2}}\,(i\omega)\,\tilde{y}(\omega)\,d\omega$$

which is equal to M_k' modulus a factor $e^{i\omega u}$.

Furthermore, $a_k'(\sigma)$ and $b_k'(\sigma)$, the coefficients of expansion of functions $\ddot{x}(u)$ and $\ddot{y}(u)$ in terms of the functions $\phi_k(u,\sigma)$, can be seen to be related to $a_k(\sigma)$ and $b_k(\sigma)$ according to the following relationships:

$$a_{k-1}' = a_k(\sigma)$$

$$b_{k-1}' = b_k(\sigma)$$

Therefore $\kappa(u)$, the curvature function of Γ can be expressed as:

$$\begin{aligned}\kappa(u) &= \dot{x}(u)\ddot{y}(u) - \dot{y}(u)\ddot{x}(u)\\ &= \sum a_k(\sigma)\phi_k(u,\sigma) \sum b_k'(\sigma)\phi_k(u,\sigma) - \sum b_k(\sigma)\phi_k(u,\sigma) \sum a_k'(\sigma)\phi_k(u,\sigma)\\ &= \sum\sum a_j(\sigma)\,b_k'(\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma) - \sum\sum b_j(\sigma)a_k'(\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)\\ &= \sum\sum \left(a_j(\sigma)\,b_{k+1}(\sigma) - b_j(\sigma)\,a_{k+1}(\sigma)\right)\phi_j(u,\sigma)\phi_k(u,\sigma)\end{aligned}$$

Therefore if the pairs $a_j(\sigma)b_k(\sigma)$, $j,k=0,\cdots,2n+1$, are all known, the curvature function of Γ can be reconstructed.

II.C. Combining information from more than one contours

To solve the system of equations obtained in section **II.A**, we need to obtain additional equations from other points of the curvature scale space image and relate them to the equations obtained from the first point. Suppose additional equations are obtained in the moments of functions $e^{-\omega^2 t}e^{i\omega u'}(i\omega)\tilde{x}(\omega)$ and $e^{-\omega^2 t}e^{i\omega u'}(i\omega)\tilde{y}(\omega)$ at point (u',t_0). We have

$$\dot{x}(u+u') = \int e^{i\omega u}e^{i\omega u'}(i\omega)\tilde{x}(\omega)\,d\omega = \sum c_k\phi_k(u)$$

and

$$\dot{y}(u+u') = \int e^{i\omega u}e^{i\omega u'}(i\omega)\tilde{y}(\omega)\,d\omega = \sum d_k\phi_k(u)$$

Now observe that

$$\sum c_k \phi_k(u) = \sum a_k \phi_k(u+u')$$

and

$$\sum d_k \phi_k(u) = \sum b_k \phi_k(u+u')$$

That is, $\phi_k(u+u')$ can be expressed as a linear combination of $\phi_j(u)$ with $j \leq k$ as has been shown in [Yuille and Poggio 1983].

II.D. Reconstructing the curvature function

It was shown in section **II.A** that four points from four curvature scale space contours give us $4n+4$ equations in the first $2n+2$ moments of each of the functions $f(\omega)$ and $f'(\omega)$. The first $n+1$ equations form a system of homogeneous quadratic equations in the unknowns: $M_0(u), \cdots, M_{2n+1}(u)$ and $M'_0(u), \cdots, M'_{2n+1}(u)$. The other points, $u+u_k$, $1 \leq k \leq 3$, provide additional equations in the unknowns: $M_0(u+u_k), \cdots, M_{2n+1}(u+u_k)$ and $M'_0(u+u_k), \cdots, M'_{2n+1}(u+u_k)$. As shown in section **II.C**, the moments at $u+u_k$ can be expressed as a linear combination of the moments at u. Therefore it is possible to express all the equations in terms of the moments at u. The result is a system of $4n+4$ homogeneous quadratic equations in $4n+4$ unknowns. That system has at least one solution since the moments of order higher than $2n+1$ of $f(\omega)$ and $f'(\omega)$ are assumed to be zero. However, the solution obtained from a quadratic system of equations is in general not unique.

Equations (2) and (3) can be converted into homogeneous linear equations by assuming that each moment-pair appearing in those equations is a new variable. Table **1** shows the moment-pairs in equations (2) and (3). The + signs designate the moment-pairs in equation (2) and the + and x signs together designate the moment-pairs in equation (3).

	M'_0	M'_1	M'_2	M'_3	M'_4	M'_5
M_0			+	+	x	x
M_1			+		x	
M_2	+	+		x		
M_3	+		x			
M_4	x	x				
M_5	x					

Table **1**

Note that all other moment-pairs in table **1** can be computed from the existing ones using the following relationships:

$$M_iM_j' = \frac{M_{i-1}M_j' . M_iM_{j+1}'}{M_{i-1}M_{j+1}'} = \frac{M_{i+1}M_j' . M_iM_{j+1}'}{M_{i+1}M_{j+1}'} = \frac{M_iM_{j-1}' . M_{i-1}M_j'}{M_{i-1}M_{j-1}'} = \frac{M_iM_{j-1}' . M_{i+1}M_j'}{M_{i+1}M_{j-1}'}$$

As before, we proceed to compute the first n derivatives at point (u_0, t_0) on one of the curvature zero-crossing contours. We now obtain n+1 homogeneous linear equations in some of the moment-pairs M_iM_j' by assuming that each moment-pair is a new variable.

Since this system is in terms of the first $2n+2$ moments of functions $f(\omega)$ and $f'(\omega)$, it will contain $O(n^2)$ moment-pairs. Therefore additional equations are required to constrain the system. To obtain those equations, we proceed as follows:

Assume that moments of order higher than $2n+2$ are zero. Compute derivatives of order higher than n at (u_0, t_0) but set moments of order higher than $2n+2$ to zero in the resulting equations. If a sufficient number of derivatives are computed at (u_0, t_0), the number of equations obtained will be equal to the number of moment-pairs and our linear system will be constrained.

It follows from an assumption of generality that the system will have a unique zero eigenvector and therefore a unique solution modulus scaling. Once the moment-pairs in the system are known, all other moment-pairs can be computed from the known ones using the relationships given above. Since all the moment-pairs M_iM_j' together determine the curvature function of the curve, it follows that the curve can be determined modulus a rigid motion and constant scaling.

Yuille and Poggio [1983] have shown that a 1-D signal can be reconstructed using two points from its scale space image. Note that our result implies that only one point is sufficient for the reconstruction of that signal.

III. Reconstructing a space curve from its torsion scale space

This section contains a proof of theorem 2. Section III.A shows that the derivatives at a point on a torsion zero-crossing contour provide homogeneous equations in the moments of the coordinate functions of the curve. Section III.B shows that the moments are related to the coefficients of expansion of the coordinate functions of the curve in functions related to the Hermite polynomials. Section III.C shows that the cubic equations obtained in section III.A can be converted into a homogeneous linear system of equations which can be solved uniquely for function $\tau(u)\,\kappa^2(u)$.

III.A. Constraints from the torsion zero-crossing contours

Let $\Gamma = (x(u), y(u), z(u))$ be the arc-length parametrization of the curve with Fourier transform $\tilde{\Gamma} = (\tilde{x}(\omega), \tilde{y}(\omega), \tilde{z}(\omega))$. The Fourier transform of the Gaussian filter $G(u,t) = \frac{1}{\sqrt{2t}} e^{-u^2/4t}$ is $\tilde{G}(\omega) = e^{-\omega^2 t}$.

Let $\Gamma_{t_0} = (x(u,t_0), y(u,t_0), z(u,t_0))$ be a curve obtained by convolving $x(u)$, $y(u)$ and $z(u)$ with $G(u,t_0)$. Assume that Γ_{t_0} is in C_∞. Such a t_0 exists since Γ is in C_1. Assume that $\kappa(u,t) \neq 0$ on the torsion zero-crossing contours in a neighborhood of t_0. It follows that the torsion zero-crossings are given by solutions of $\beta(u,t) = 0$ where [Goetz 1970]

$$\beta(u,t) = \dot{x}(\ddot{y}\,\dddot{z} - \dddot{y}\,\ddot{z}) - \dot{y}(\ddot{x}\,\dddot{z} - \dddot{x}\,\ddot{z}) + \dot{z}(\ddot{x}\,\dddot{y} - \dddot{x}\,\ddot{y}) \tag{4}$$

where . represents derivative with respect to u. Note that on Γ $(t=0)$, $\beta(u,t)$ is given by

$$\beta(u,t) = \tau(u,t)\,\kappa^2(u,t) \tag{5}$$

Using the convolution theorem, $\dot{x}(u,t)$, $\dot{y}(u,t)$ and $\dot{z}(u,t)$ can be expressed as following:

$$\dot{x}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(i\omega)\, \tilde{x}(\omega)\, d\omega$$

$$\dot{y}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(i\omega)\, \tilde{y}(\omega)\, d\omega$$

$$\dot{z}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(i\omega)\, \tilde{z}(\omega)\, d\omega$$

and therefore

$$\ddot{x}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(-\omega^2)\, \tilde{x}(\omega)\, d\omega$$

$$\ddot{y}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(-\omega^2)\, \tilde{y}(\omega)\, d\omega$$

$$\ddot{z}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(-\omega^2)\, \tilde{z}(\omega)\, d\omega$$

$$\dddot{x}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(-i\omega^3)\, \tilde{x}(\omega)\, d\omega$$

$$\dddot{y}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(-i\omega^3)\, \tilde{y}(\omega)\, d\omega$$

$$\dddot{z}(u,t) = \int e^{-\omega^2 t} e^{i\omega u}(-i\omega^3)\, \tilde{z}(\omega)\, d\omega$$

Note that the moment of order k of the function $f(\omega) = e^{-\omega^2 t} e^{i\omega u}(i\omega)\, \tilde{x}(\omega)$ is defined by:

$$M_k = \int_{-\infty}^{\infty} (i\omega)^k e^{-\omega^2 t} e^{i\omega u} (i\omega)\, \tilde{x}(\omega)\, d\omega$$

the moment of order k of the function $f'(\omega) = e^{-\omega^2 t} e^{i\omega u} (i\omega)\, \tilde{y}(\omega)$ is defined by:

$$M'_k = \int_{-\infty}^{\infty} (i\omega)^k e^{-\omega^2 t} e^{i\omega u} (i\omega)\, \tilde{y}(\omega)\, d\omega$$

and the moment of order k of the function $f''(\omega) = e^{-\omega^2 t} e^{i\omega u} (i\omega)\, \tilde{z}(\omega)$ is defined by:

$$M''_k = \int_{-\infty}^{\infty} (i\omega)^k e^{-\omega^2 t} e^{i\omega u} (i\omega)\, \tilde{z}(\omega)\, d\omega$$

Therefore equation (5) can be written as:

$$\beta(u,t) = M_0 M'_1 M''_2 - M_0 M''_1 M'_2 - M'_0 M_1 M''_2 + M'_0 M_2 M''_1 + M''_0 M_1 M'_2 - M''_0 M_2 M'_1$$

The Implicit Function Theorem guarantees that the contours $u(t)$ are C_∞ in a neighborhood of t_0. Let ξ be a parameter of the torsion zero-crossing contour. Then

$$\frac{d}{d\xi} = \frac{du}{d\xi}\frac{\partial}{\partial u} + \frac{dt}{d\xi}\frac{\partial}{\partial t}$$

On the torsion zero-crossing contour, $\beta = 0$ and $\frac{d^k}{d\xi^k}\beta = 0$ for all integers k. Furthermore, since the torsion zero-crossing contour is known, all the derivatives of u and t with respect to ξ are known as well. We now compute the derivatives of β with respect to ξ at (u_0, t_0). The first derivative is given by:

$$\frac{d}{d\xi}\beta(u_0,t_0) = \frac{du}{d\xi}\frac{\partial\beta(u_0,t_0)}{\partial u} + \frac{dt}{d\xi}\frac{\partial\beta(u_0,t_0)}{\partial t} \tag{6}$$

where

$$\frac{\partial\beta(u_0,t_0)}{\partial u} = M''_3 M_0 M'_1 - M'_3 M_0 M''_1 - M''_3 M'_0 M_1 + M_3 M'_0 M''_1 + M'_3 M''_0 M_1 - M_3 M''_0 M'_1$$

and

$$\frac{\partial\beta(u_0,t_0)}{\partial t} = M'_3 M_0 M''_2 + M''_4 M_0 M'_1 - M''_3 M_0 M'_2 - M'_4 M_0 M''_1 - M_3 M'_0 M''_2 - M''_4 M'_0 M_1$$

$$+ M_4 M'_0 M''_1 + M''_3 M'_0 M_2 + M_3 M''_0 M'_2 + M'_4 M''_0 M_1 - M_4 M''_0 M'_1 - M'_3 M''_0 M_2$$

and the second derivative is given by:

$$\frac{\partial^2\beta}{\partial\xi^2} = \frac{d^2u}{d\xi^2}\frac{\partial\beta}{\partial u} + \frac{d^2t}{d\xi^2}\frac{\partial\beta}{\partial t} + \left(\frac{du}{d\xi}\right)^2\frac{\partial^2\beta}{\partial u^2} + 2\frac{du}{d\xi}\frac{dt}{d\xi}\frac{\partial^2\beta}{\partial u\partial t} + \left(\frac{dt}{d\xi}\right)^2\frac{\partial^2\beta}{\partial t^2} \tag{7}$$

where

$$\frac{\partial^2\beta}{\partial u^2} = M_4''M_0M_1' + M_2'M_3''M_0 - M_4'M_0M_1'' - M_2''M_3'M_0 - M_4''M_0'M_1 - M_2M_3''M_0'$$
$$+ M_4M_0'M_1'' + M_2''M_3M_0' + M_4'M_0''M_1 + M_2M_3'M_0'' - M_4M_0''M_1' - M_2'M_3M_0''$$

$$\frac{\partial^2\beta}{\partial u\partial t} = M_5''M_0M_1' + M_2M_3''M_1' - M_5'M_0M_1'' - M_2M_3'M_1'' - M_5''M_0'M_1 - M_2'M_3''M_1$$
$$+ M_5M_0'M_1'' + M_2'M_3M_1'' + M_5'M_0''M_1 + M_2''M_3'M_1 - M_5M_0''M_1' - M_2''M_3M_1'$$

and

$$\frac{\partial^2\beta}{\partial t^2} = M_5'M_0M_2'' + 2M_4''M_3'M_0 + M_6''M_0M_1' + M_2M_4''M_1' - M_5''M_0M_2' - 2M_4'M_3''M_0$$
$$- M_6'M_0M_1'' - M_2M_4'M_1'' - M_5M_0'M_2'' - 2M_4''M_3M_0' - M_6''M_0'M_1 - M_2'M_4''M_1$$
$$+ M_6M_0'M_1'' + M_2'M_4M_1'' + 2M_3''M_4M_0' + M_5''M_0'M_2 + M_5M_0''M_2' + 2M_4'M_3M_0''$$
$$+ M_6'M_0''M_1 + M_2''M_4'M_1 - M_6M_0''M_1' - M_2''M_4M_1' - 2M_3'M_4M_0'' - M_5'M_0''M_2$$

Since the parametric derivatives along the torsion zero-crossing contours are zero, equation (6) is equal to zero. Note that equation (6) is in the first five moments of functions $f(\omega)$, $f'(\omega)$ and $f''(\omega)$ and equation (7) is in the first seven moments of those functions. In general, the k+1st equation, $\frac{d^k}{d\xi^k}\beta(u,t) = 0$ is a cubic equation in the first $2k$+3 moments of each of the functions $f(\omega)$, $f'(\omega)$ and $f''(\omega)$.

It follows that the first n+1 equations at (u_0, t_0) are in a total of $3(2n+3) = 6n+9$ moments. Our axes are again chosen such that $u_0 = 0$. Section **II.B** showed that the moments of $f(\omega)$ and $f'(\omega)$ are the coefficients a_k and b_k in the expression of functions $\dot{x}(u)$ and $\dot{y}(u)$ in functions $\phi_k(u,\sigma)$ related to Hermite polynomials. Similarly, it can be seen that the moments of $f''(\omega)$ are the coefficients c_k in the expression of function $\dot{z}(u)$ in functions $\phi_k(u,\sigma)$. Therefore we have n+1 equations in the first $6n$+9 coefficients a_k, b_k and c_k. To determine the a_k, b_k and c_k, we need $5n$+8 additional and independent equations which can be provided by considering six neighboring torsion zero-crossing contours at (u_1, t_0), (u_2, t_0), (u_3, t_0), (u_4, t_0), (u_5, t_0) and (u_6, t_0).

III.B. The moments and the coefficients of expansion of $\dot{x}(u)$, $\dot{y}(u)$ and $\dot{z}(u)$

This section shows that the moments and the moment-triples in equations $\frac{d^k}{d\xi^k}\beta(u,t)$ are related respectively to the coefficients of the expression of the functions $\dot{x}(u)$, $\dot{y}(u)$ and $\dot{z}(u)$ and function $\beta(u)$ in functions related to the Hermite polynomials.

As shown in section **II.B**, $a_k(\sigma)$ and $b_k(\sigma)$, the coefficients of expansion of $\dot{x}(u)$ and $\dot{y}(u)$ in functions $\phi_k(u,\sigma)$, can be shown to be equal to M_k and M_k' modulus a factor $e^{i\omega u}$. Similarly, $c_k(\sigma)$, the coefficients of expansion of $\dot{z}(u)$ can be shown to be equal to M_k'' modulus a factor $e^{i\omega u}$. Furthermore, $a_k'(\sigma)$, $b_k'(\sigma)$ and $c_k'(\sigma)$, the coefficients of expansion of functions $\ddot{x}(u)$, $\ddot{y}(u)$ and $\ddot{z}(u)$ in $\phi_k(u,\sigma)$ respectively, can be seen to be related to $a_k(\sigma)$, $b_k(\sigma)$ and $c_k(\sigma)$ according to the following relationships:

$$\begin{aligned} a_{k-1}'(\sigma) &= a_k(\sigma) \\ b_{k-1}'(\sigma) &= b_k(\sigma) \\ c_{k-1}'(\sigma) &= c_k(\sigma) \end{aligned} \tag{8}$$

and $a_k''(\sigma)$, $b_k''(\sigma)$ and $c_k''(\sigma)$, the coefficients of expansion of functions $\dddot{x}(u)$, $\dddot{y}(u)$ and $\dddot{z}(u)$ in $\phi_k(u,\sigma)$ respectively, can be seen to be related to $a_k(\sigma)$, $b_k(\sigma)$ and $c_k(\sigma)$ by the following relationships:

$$\begin{aligned} a_{k-2}''(\sigma) &= a_k(\sigma) \\ b_{k-2}''(\sigma) &= b_k(\sigma) \\ c_{k-2}''(\sigma) &= c_k(\sigma) \end{aligned} \tag{9}$$

Now observe that the function $\tau(u)\,\kappa^2(u)$ can be expressed as:

$$\begin{aligned} \tau(u)\,\kappa^2(u) &= \dot{x}(u)\big(\ddot{y}(u)\dddot{z}(u) - \dddot{y}(u)\ddot{z}(u)\big) \\ &\quad - \dot{y}(u)\big(\ddot{x}(u)\dddot{z}(u) - \dddot{x}(u)\ddot{z}(u)\big) \\ &\quad + \dot{z}(u)\big(\ddot{x}(u)\dddot{y}(u) - \dddot{x}(u)\ddot{y}(u)\big) \\ &= \dot{x}(u)\ddot{y}(u)\dddot{z}(u) - \dot{x}(u)\dddot{y}(u)\ddot{z}(u) - \dot{y}(u)\ddot{x}(u)\dddot{z}(u) \\ &\quad + \dot{y}(u)\dddot{x}(u)\ddot{z}(u) + \dot{z}(u)\ddot{x}(u)\dddot{y}(u) - \dot{z}(u)\dddot{x}(u)\ddot{y}(u) \\ &= \sum a_i(\sigma)\phi_i(u,\sigma)\sum b_i'(\sigma)\phi_i(u,\sigma)\sum c_i''(\sigma)\phi_i(u,\sigma) \end{aligned}$$

$$-\sum a_i(\sigma)\phi_i(u,\sigma)\sum b_i''(\sigma)\phi_i(u,\sigma)\sum c_i'(\sigma)\phi_i(u,\sigma)$$

$$-\sum b_i(\sigma)\phi_i(u,\sigma)\sum a_i'(\sigma)\phi_i(u,\sigma)\sum c_i''(\sigma)\phi_i(u,\sigma)$$

$$+\sum b_i(\sigma)\phi_i(u,\sigma)\sum a_i''(\sigma)\phi_i(u,\sigma)\sum c_i'(\sigma)\phi_i(u,\sigma)$$

$$+\sum c_i(\sigma)\phi_i(u,\sigma)\sum a_i'(\sigma)\phi_i(u,\sigma)\sum b_i''(\sigma)\phi_i(u,\sigma)$$

$$-\sum c_i(\sigma)\phi_i(u,\sigma)\sum a_i''(\sigma)\phi_i(u,\sigma)\sum b_i'(\sigma)\phi_i(u,\sigma)$$

$$=\sum\sum\sum a_i(\sigma)b_j'(\sigma)c_k''(\sigma)\phi_i(u,\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)$$

$$-\sum\sum\sum a_i(\sigma)b_j''(\sigma)c_k'(\sigma)\phi_i(u,\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)$$

$$-\sum\sum\sum b_i(\sigma)a_j'(\sigma)c_k''(\sigma)\phi_i(u,\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)$$

$$+\sum\sum\sum b_i(\sigma)a_j''(\sigma)c_k'(\sigma)\phi_i(u,\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)$$

$$+\sum\sum\sum c_i(\sigma)a_j'(\sigma)b_k''(\sigma)\phi_i(u,\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)$$

$$-\sum\sum\sum c_i(\sigma)a_j''(\sigma)b_k'(\sigma)\phi_i(u,\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)$$

$$=\sum\sum\sum (a_i(\sigma)b_j'(\sigma)c_k''(\sigma) + b_i(\sigma)a_j''(\sigma)c_k'(\sigma) + c_i(\sigma)a_j'(\sigma)b_k''(\sigma)$$

$$-a_i(\sigma)b_j''(\sigma)c_k'(\sigma) - b_i(\sigma)a_j'(\sigma)c_k''(\sigma) - c_i(\sigma)a_j''(\sigma)b_k'(\sigma))$$

$$\phi_i(u,\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)$$

Using (8) and (9) we obtain

$$\tau(u)\,\kappa^2(u) = \sum\sum\sum (a_i(\sigma)b_{j+1}(\sigma)c_{k+2}(\sigma) + b_i(\sigma)a_{j+2}(\sigma)c_{k+1}(\sigma) + c_i(\sigma)a_{j+1}(\sigma)b_{k+2}(\sigma)$$

$$- a_i(\sigma)b_{j+2}(\sigma)c_{k+1}(\sigma) - b_i(\sigma)a_{j+1}(\sigma)c_{k+2}(\sigma) - c_i(\sigma)a_{j+2}(\sigma)b_{k+1}(\sigma))$$

$$\phi_i(u,\sigma)\phi_j(u,\sigma)\phi_k(u,\sigma)$$

It follows that if the triples $a_i(\sigma)b_j(\sigma)c_k(\sigma)$ in the equation above are all known, the function $\beta(u) = \tau(u)\,\kappa^2(u)$ can be reconstructed.

III.C. Reconstructing the function $\tau(u)\,\kappa^2(u)$

It was shown in section III.C that seven points from seven torsion scale space contours give us $6n+9$ cubic equations in the first $2n+3$ moments of each of the functions $f(\omega)$, $f'(\omega)$ and $f''(\omega)$. Section II.C showed that the moments of order k of any function at $u+u'$ can be expressed as a linear combination of the moments of order less than or equal to k of that function at u. Therefore we obtain a system of homogeneous cubic equations in the first $6n+9$ coefficients of functions $\dot{x}(u)$, $\dot{y}(u)$ and $\dot{z}(u)$ using seven points from the torsion scale space image of Γ (Note that only three equations from the seventh point need be used). That system has at least one solution since the moments of order higher than $2n+2$ of $f(\omega)$, $f'(\omega)$ and $f''(\omega)$ are assumed to be zero. However, the solution obtained from a cubic system of equations is in general not unique.

Equations (6) and (7) can be converted into homogeneous linear equations by assuming that each moment-triple appearing in those equations is a new (7). The + signs designate the moment-triples in equation (6) and the + and the x signs together designate the moment-triples in equation (7). Each table shows those moment-triples which share the same M_k'', $0 \leq k \leq 6$.

	M_0'	M_1'	M_2'	M_3'	M_4'	M_5'	M_6'
M_0							
M_1				+	+	x	x
M_2				+		x	
M_3		+	+		x		
M_4		+		x			
M_5		x	x				
M_6		x					

Table 2. Moment-triples sharing M_0''.

	M_0'	M_1'	M_2'	M_3'	M_4'	M_5'	M_6'
M_0				+	+	x	x
M_1							
M_2				x	x		
M_3	+		x				
M_4	+		x				
M_5	x						
M_6	x						

Table 3. Moment-triples sharing M_1''.

	M_0'	M_1'	M_2'	M_3'	M_4'	M_5'	M_6'
M_0				+		x	
M_1				x	x		
M_2							
M_3	+	x					
M_4		x					
M_5	x						
M_6							

Table 4. Moment-triples sharing M_2''.

	M_0'	M_1'	M_2'	M_3'	M_4'	M_5'	M_6'
M_0		+	+		x		
M_1	+		x				
M_2	+	x					
M_3							
M_4	x						
M_5							
M_6							

Table 5. Moment-triples sharing M_3''.

	M'_0	M'_1	M'_2	M'_3	M'_4	M'_5	M'_6
M_0		+		x			
M_1	+		x				
M_2		x					
M_3	x						
M_4							
M_5							
M_6							

Table 6. Moment-triples sharing M''_4.

	M'_0	M'_1	M'_2	M'_3	M'_4	M'_5	M'_6
M_0		x	x				
M_1	x						
M_2	x						
M_3							
M_4							
M_5							
M_6							

Table 7. Moment-triples sharing M''_5.

	M'_0	M'_1	M'_2	M'_3	M'_4	M'_5	M'_6
M_0		x					
M_1	x						
M_2							
M_3							
M_4							
M_5							
M_6							

Table 8. Moment-triples sharing M''_6.

Note that all other moment-triples in tables 2-8 can be computed from the existing ones using the following relationships:

$$
\begin{aligned}
M_iM'_jM''_k &= \frac{M_iM'_{j-1}M''_k \,.\, M_{i+1}M'_jM''_k}{M_{i+1}M'_{j-1}M''_k} = \frac{M_iM'_{j-1}M''_k \,.\, M_{i-1}M'_jM''_k}{M_{i-1}M'_{j-1}M''_k} \\
&= \frac{M_{i-1}M'_jM''_k \,.\, M_iM'_{j+1}M''_k}{M_{i-1}M'_{j+1}M''_k} = \frac{M_{i+1}M'_jM''_k \,.\, M_iM'_{j+1}M''_k}{M_{i+1}M'_{j+1}M''_k} \\
&= \frac{M_iM'_{j-1}M''_k \,.\, M_iM'_jM''_{k-1}}{M_iM'_{j-1}M''_{k-1}} = \frac{M_iM'_{j+1}M''_k \,.\, M_iM'_jM''_{k-1}}{M_iM'_{j+1}M''_{k-1}} \\
&= \frac{M_iM'_{j-1}M''_k \,.\, M_iM'_jM''_{k+1}}{M_iM'_{j-1}M''_{k+1}} = \frac{M_iM'_jM''_{k+1} \,.\, M_iM'_{j+1}M''_k}{M_iM'_{j+1}M''_{k+1}} \\
&= \frac{M_iM'_jM''_{k-1} \,.\, M_{i-1}M'_jM''_k}{M_{i-1}M'_jM''_{k-1}} = \frac{M_{i-1}M'_jM''_k \,.\, M_iM'_jM''_{k+1}}{M_{i-1}M'_jM''_{k+1}}
\end{aligned}
$$

$$= \frac{M_i M_j' M_{k+1}'' \cdot M_{i+1} M_j' M_k''}{M_{i+1} M_j' M_{k+1}''} = \frac{M_{i+1} M_j' M_k'' \cdot M_i M_j' M_{k-1}''}{M_{i+1} M_j' M_{k-1}''}$$

Again we proceed to compute the first n derivatives at point (u_0, t_0) on one of the torsion zero-crossing contours. We now obtain $n+1$ homogeneous linear equations in some of the moment-triples $M_i M_j' M_k''$ by assuming that each moment-triple is a new variable.

Since this system is in terms of the first $2n+3$ moments of functions $f(\omega)$, $f'(\omega)$ and $f''(\omega)$, it will contain $O(n^3)$ moment-triples. Therefore additional equations are required to constrain the system. To obtain those equations, we proceed as follows:

Assume that moments of order higher than $2n+3$ are zero. Compute derivatives of order higher than n at (u_0, t_0) but set moments of order higher than $2n+3$ to zero in the resulting equations. If a sufficient number of derivatives are computed at (u_0, t_0), the number of equations obtained will be equal to the number of moment-triples and our linear system will be constrained.

It follows from an assumption of generality that the system will have a unique zero eigenvector and therefore a unique solution modulus scaling. Once the moment-triples in the system are known, all other moment-triples can be computed from the known ones using the relationships given above. Since all the moment-triples $M_i M_j' M_k''$ together determine function of $\beta(u)$, it follows that function $\beta(u)$ can be determined modulus constant scaling.

IV. Conclusions

It was shown that the curvature scale space descriptions of planar curves represent those curves uniquely up to constant scaling, rotation and translation and therefore satisfy one of the necessary criteria for any shape representation technique. It was also shown that the torsion scale space description of a space curve Γ represents that curve up to a class represented modulus a scale factor by the function $\beta(u) = \tau(u)\,\kappa^2(u)$ where $\tau(u)$ and $\kappa(u)$ are the torsion and curvature functions of Γ respectively.

Our results indicate that a polynomially represented planar curve in C_1 can be reconstructed using four points of its curvature scale space image at one scale and a polynomially represented space curve in C_1 can be reconstructed modulus the class represented by $\beta(u)$ using seven points of its torsion scale space image at one scale.

Finally, note that our results also apply to the renormalized scale space representation [Mackworth and Mokhtarian 1988] since the zero-crossing contours in those representations can also be seen to be in C_∞.

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