Evolution Properties of Space Curves

Farzin Mokhtarian

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Department of Computer Science
University of British Columbia
Vancouver, B.C.
Canada V6T 1W5

Abstract

The Curvature Scale Space and Torsion Scale Space Images of a space curve are a multi-scale representation for that curve which satisfies several criteria for shape representation and is therefore a preferred representation method for space curves.

The torsion scale space image of a space curve is computed by convolving a path-based parametric representation of the curve with Gaussian functions of varying widths, extracting the torsion zero-crossings of the convolved curves and combining them in a torsion scale space image of the curve. The curvature scale space image of the curve is computed similarly but curvature level-crossings are extracted instead. An *evolved version* of a space curve Γ is obtained by convolving a parametric representation of that curve with a Gaussian function of variance σ^2 and denoted by Γ_σ. The process of generating the ordered sequence of curves $\{\Gamma_\sigma | \sigma \geq 0\}$ is referred to as the *evolution* of Γ.

A number of evolution properties of space curves are investigated in this paper. It is shown that the evolution of space curves is invariant under rotation, uniform scaling and translation of those curves. This property makes the representation suitable for recognition purposes. It is also shown that properties such as connectedness and closedness of a space curve are preserved during evolution of the curve and that the center of mass of a space curve remains the same as the curve evolves. Among other results is the fact that a space curve contained inside a simple, convex object, remains inside that object during evolution.

The two main theorems of the paper examine a space curve during its evolution just before and just after the formation of a cusp point. It is shown that strong constraints on the shape of the curve in the neighborhood of the cusp point exist just before and just after the formation of that point.

I. Introduction

A multi-scale representation for one-dimensional functions and signals was first proposed by Stansfield [1980] and later developed by Witkin [1983]. The signal was convolved with a Gaussian function as its width varied from a small to a large value. The zero-crossings of the second derivative of each convolved signal were extracted and marked in a coordinate system in which the horizontal axis represents the free variable of the signal and the vertical axis represents the width of the Gaussian function. The result is the *Scale Space Image* of the signal.

Mokhtarian and Mackworth [1986] generalized that concept to planar curves. A planar curve Γ was parametrized by arc-length and represented using its coordinate functions:

$$\Gamma = (x(t), y(t)).$$

An *evolved version* of Γ is computed by convolving each of its coordinate functions with a Gaussian function of variance σ^2 and denoted by Γ_σ. The process of generating the ordered sequence of curves $\{\Gamma_\sigma | \sigma \geq 0\}$ is referred to as the *evolution* of Γ.

The curvature of each Γ_σ can be expressed in terms of the first and second derivatives of convolved versions of functions $x(t)$ and $y(t)$. It is therefore possible to extract the curvature zero-crossings of each Γ_σ as σ varies from a small to a large value and mark them in a coordinate system in which the horizontal axis now represents t, the parameter along the curve and the vertical axis again represents the width of the Gaussian function. The result is referred to as the *Curvature Scale Space Image* of the curve.

Mokhtarian [1987] generalized the above concept further to space curves. The parametrization of a space curve can be expressed as

$$\Gamma = (x(t), y(t), z(t)).$$

Curvature and torsion of an evolved space curve can be expressed in terms of the first three derivatives of convolved versions of functions $x(t)$, $y(t)$ and $z(t)$. A scale space representation for space curves consists of the *Torsion Scale Space Image* which contains the torsion zero-crossings map and the curvature scale space image which contains the curvature level-crossings map of the curve.

Scale space representations for planar and space curves satisfy several useful criteria such as *Efficiency, Invariance, Sensitivity, Uniqueness, Detail* and *Robustness* [Mokhtarian 1987]. These properties make them suitable for recognition purposes.

Mackworth and Mokhtarian [1987] analyzed a number of evolution properties of planar curves. This paper generalizes the results obtained in that paper as far as possible to space curves and explores other evolution properties of those curves. Lemma **1** shows that evolution of a space curve is invariant under rotation, uniform scaling and translation of the curve. This is an essential property without which the representation would not be useful. Lemmas **2** and **3** show that connectedness and closedness of a space curve are preserved during its evolution. Lemma **4** shows that the center of mass of a space curve does not move as the curve evolves. Lemma **5** shows that a space curve contained by a simple, convex object, remains inside that object during evolution. Lemma **6** establishes a link between the evolution properties of planar and space curves.

Theorems **1** and **2** contain the main theoretical results of this paper. Theorem **1** states that if a polynomially represented space curve forms a cusp point at u_0 during its evolution, then it either intersects itself or two of its projections intersect themselves in a neighborhood of u_0 just before the formation of the cusp point. An outline of the proof is given below:

Let Γ_σ be the curve with a cusp point at u_0. To simplify the equations, it is assumed that the singularity occurs at the origin. Since a small neighborhood of u_0 is examined, only the lowest degree terms in the coordinate functions of Γ_σ are looked at. Each of those terms can have an odd or an even power of u. A case analysis considers all the eight possible cases that arise and separates the cusp points from other singularities by examining the amount of change in the tangent direction as u varies infinitesimally around u_0. An analytical expression for $\Gamma_{\sigma-\delta}$ is computed by deblurring each of its coordinate functions using an approximation to a deblurring operator derived in [Hummel *et al.* 1987]. Finally, the statement of the theorem is proven by showing that there are two distinct values of u, u_1 and u_2, such that at least two (and sometimes all three) of the following constraints are satisfied on $\Gamma_{\sigma-\delta}$:

$$x(u_1) = x(u_2), \qquad y(u_1) = y(u_2), \qquad z(u_1) = z(u_2)$$

Theorem **2** states that if a polynomially represented space curve forms a cusp point at u_0 during evolution, then either a torsion zero-crossing point exists at u_0 just before and just after the formation of the cusp point or two new torsion zero-crossing points are created in the neighborhood of u_0 just after the cusp point. The following is an outline of the proof:

A case analysis again separates the cusp points from other singularities. An analytical expression for $\Gamma_{\sigma+\delta}$ is computed by blurring each of its coordinate functions using the Gaussian function as the blurring operator. Expressions for the torsion of curves Γ_σ, $\Gamma_{\sigma-\delta}$ and $\Gamma_{\sigma+\delta}$ are derived and used to prove the statement of the theorem.

Theorems **1** and **2** together show that strong constraints exist on the shape of a space curve in the neighborhood of a cusp point. They completely characterize the behavior of a space curve just before and just after the formation of the cusp point.

II. Evolution properties of space curves

This section contains a number of results on the evolution properties of space curves. Lemmas **1-5** describe fundamental properties of space curves that are preserved during evolution.

Lemma 1. Evolution of a space curve is invariant under rotation, uniform scaling and translation of the curve.

Proof: We will show that evolution of a space curve is invariant under a general affine transform which includes transformations consisting of rotation, uniform scaling and translation.

Let $\Gamma = (x(u), y(u), z(u))$ be a space curve and let $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma), Z(u,\sigma))$ be its evolved version. If Γ_σ undergoes an affine transformation, then its original coordinates and its new coordinates, $x_1(u,\sigma), y_1(u,\sigma)$ and $z_1(u,\sigma)$, will satisfy the following relationship:

$$x_1(u,\sigma) = aX(u,\sigma) + b\,Y(u,\sigma) + c\,Z(u,\sigma) + d$$

$$y_1(u,\sigma) = eX(u,\sigma) + f\,Y(u,\sigma) + g\,Z(u,\sigma) + h$$

$$z_1(u,\sigma) = iX(u,\sigma) + j\,Y(u,\sigma) + k\,Z(u,\sigma) + l$$

If Γ is first transformed according to an affine transform and then evolved, its new coordinates, $x_2(u,\sigma), y_2(u,\sigma)$ and $z_2(u,\sigma)$, will be:

$$x_2(u,\sigma) = (a\,x(u) + b\,y(u) + c\,z(u) + d) \circledast g(u,\sigma)$$

$$y_2(u,\sigma) = (e\,x(u) + fy(u) + gz(u) + h) \circledast g(u,\sigma)$$

$$z_2(u,\sigma) = (i\,x(u) + j\,y(u) + k\,z(u) + l) \circledast g(u,\sigma)$$

It follows from the distributivity of the convolution operator [Kecs 1982] that

$$x_2(u,\sigma) = (a\,x(u)) \circledast g(u,\sigma) + (b\,y(u)) \circledast g(u,\sigma) + (c\,z(u)) \circledast g(u,\sigma) + d \circledast g(u,\sigma)$$

$$y_2(u,\sigma) = (e\,x(u)) \circledast g(u,\sigma) + (fy(u)) \circledast g(u,\sigma) + (g\,z(u)) \circledast g(u,\sigma) + h \circledast g(u,\sigma)$$

$$z_2(u,\sigma) = (i\,x(u)) \circledast g(u,\sigma) + (jy(u)) \circledast g(u,\sigma) + (k\,z(u)) \circledast g(u,\sigma) + l \circledast g(u,\sigma)$$

and

$$x_2(u,\sigma) = a(x(u) \circledast g(u,\sigma)) + b(y(u) \circledast g(u,\sigma)) + c(z(u) \circledast g(u,\sigma)) + d = x_1(u,\sigma)$$

$$y_2(u,\sigma) = e(x(u) \circledast g(u,\sigma)) + f(y(u) \circledast g(u,\sigma)) + g(z(u) \circledast g(u,\sigma)) + h = y_1(u,\sigma)$$

$$z_2(u,\sigma) = i(x(u) \circledast g(u,\sigma)) + j(y(u) \circledast g(u,\sigma)) + k(z(u) \circledast g(u,\sigma)) + l = z_1(u,\sigma)$$

Note that this result also holds for convolution operators other than the Gaussian. □

Lemma 2. A connected space curve remains connected during evolution.

Proof: Let $\Gamma = (x(u), y(u), z(u))$ be a connected space curve and let $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma), Z(u,\sigma))$ be its evolved version. We will show that Γ_σ is also a connected curve.

Since Γ is connected, $x(u), y(u)$ and $z(u)$ are continuous functions. Since these functions remain continuous after convolution with a Gaussian, $X(u,\sigma)$, $Y(u,\sigma)$ and $Z(u,\sigma)$ are also continuous. Let u_0 be any value of parameter u and let x_0, y_0 and z_0 be the values of $X(u,\sigma), Y(u,\sigma)$ and $Z(u,\sigma)$ at u_0 respectively. If there is an infinitesimal change in u

$$u \rightarrow u_0 + \epsilon$$

then there will also be infinitesimal changes in $X(u,\sigma)$, $Y(u,\sigma)$ and $Z(u,\sigma)$:

$$X(u,\sigma) \rightarrow x_0 + \delta$$

$$Y(u,\sigma) \rightarrow y_0 + \xi$$

$$Z(u,\sigma) \rightarrow z_0 + \varsigma$$

Therefore, point $P(x_0, y_0, z_0)$ on Γ_σ will go to point $P'(x_0+\delta, y_0+\xi, z_0+\varsigma)$. Let D be the distance between P and P'. Then

$$D = \sqrt{(x_0+\delta-x_0)^2 + (y_0+\xi-y_0)^2 + (z_0+\varsigma-z_0)^2} = \sqrt{\delta^2 + \xi^2 + \varsigma^2}$$

Let $|\delta|$ be the largest of $|\delta|$, $|\xi|$ and $|\varsigma|$. Then

$$D \leq \sqrt{3\,\delta^2} = \delta\sqrt{3}$$

Therefore an infinitesimal change in parameter u results in an infinitesimal change in the position of point P. It follows that Γ_σ is a connected space curve. □

Lemma 3. A closed space curve remains closed during evolution.

Proof: A closed space curve parametrized by the normalized arc-length parameter has $(x(0), y(0), z(0)) = (x(1), y(1), z(1))$. It follows that $(X(0,\sigma), Y(0,\sigma), Z(0,\sigma)) = (X(1,\sigma), Y(1,\sigma), Z(1,\sigma))$. □

If the curvature function is smoothed [Asada and Brady 1986] then closed planar curves and therefore closed space curves may not remain closed [Horn and Weldon 1986].

Lemma 4. The center of mass of a space curve is invariant during evolution.

Proof: Let M be the center of mass of $\Gamma = (x(u), y(u), z(u))$ with coordinates (x_M, y_M, z_M). Then

$$x_M = \frac{\int_0^1 x(u)\,du}{\int_0^1 du} = \int_0^1 x(u)\,du$$

$$y_M = \frac{\int_0^1 y(u)\,du}{\int_0^1 du} = \int_0^1 y(u)\,du$$

and

$$z_M = \frac{\int_0^1 z(u)\,du}{\int_0^1 du} = \int_0^1 z(u)\,du$$

The evolved curve $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma), Z(u,\sigma))$ where

$$X(u,\sigma) = \int_{-\infty}^{\infty} g(v,\sigma)\, x(u-v)\, dv$$

$$Y(u,\sigma) = \int_{-\infty}^{\infty} g(v,\sigma)\, y(u-v)\, dv$$

and

$$Z(u,\sigma) = \int_{-\infty}^{\infty} g(v,\sigma)\, z(u-v)\, dv$$

has M' with coordinates (X_M, Y_M, Z_M) as its center of mass. Now observe that

$$X_M = \int_0^1 X(u,\sigma)\,du = \int_0^1 \int_{-\infty}^{\infty} g(v,\sigma)\, x(u-v)\, dv du$$

$$= \int_{-\infty}^{\infty} g(v,\sigma) \left[\int_0^1 x(u-v)\, du \right] dv = x_M$$

Similarly, $Y_M = y_M$ and $Z_M = z_M$. It follows that $M = M'$. □

Lemma 5. Let Γ be a closed space curve contained inside a simple, convex object G. Γ remains inside G during evolution.

Proof: Since G is simple and convex, every plane P tangent to G contains that object in the left (or right) half-space it creates. Since Γ is inside G, Γ is also contained in the same half-space. Now rotate P and Γ so that P becomes parallel to the YZ-plane and therefore has equation $x = c$. Since P does not intersect Γ, it follows that $x(t_0) \geq c$ for every point t_0 on Γ. Let Γ_σ be an evolved version of Γ. Every point of Γ_σ is a weighted average of all the points of Γ. Therefore $X(t_0, \sigma) \geq c$ for every point t_0 on Γ_σ and Γ_σ is also contained in the same half-space. This result holds for *every* plane tangent to G therefore Γ_σ is contained inside the intersection of all the left (or right) half-spaces created by the tangent planes of G. It follows that Γ_σ is also inside G. □

Yuille and Poggio [1983] and Babaud *et al.* [1986] showed that no new zero or level-crossings are created at higher scales in the scale space images of almost all one-dimensional signals. Mackworth and Mokhtarian [1987] have shown that this property is shared by those planar curves which remain in C_1 during evolution. This property does not appear to generalize to space curves. Lemma 6 defines a relationship between the evolution properties of planar and space curves.

Lemma 6. Let $\Gamma_1 = (x(u), y(u))$ and $\Gamma_2 = (y(u), z(u))$ be planar curves in C_1 which remain in that class during evolution and let $\Gamma = (x(u), y(u), z(u))$ be the space curve defined by Γ_1 and Γ_2. Let C_1 and C_2 be zero-crossing contours in the curvature scale space images of Γ_1 and Γ_2 respectively. If C_1 and C_2 intersect at $E(u_0, \sigma_0)$, then u_0 is a point of zero curvature of Γ_{σ_0}.

Proof: The curvature of a planar curve $\Gamma_1 = (x(t), y(t))$ with an arbitrary parametrization is given by [Goetz 1970]:

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{((\dot{x})^2 + (\dot{y})^2)^{3/2}}$$

Since E is on C_1, the curvature of Γ_1 is zero at point E and

$$\dot{X}(u_0, \sigma_0)\,\ddot{Y}(u_0, \sigma_0) - \dot{Y}(u_0, \sigma_0)\ddot{X}(u_0, \sigma_0) = 0 \tag{1}$$

or

$$\frac{\dot{X}}{\ddot{X}} = \frac{\dot{Y}}{\ddot{Y}}$$

Similarly, since E is on C_2, we have at point E

$$\dot{Y}\ddot{Z} - \dot{Z}\ddot{Y} = 0 \tag{2}$$

or

$$\frac{\dot{Y}}{\ddot{Y}} = \frac{\dot{Z}}{\ddot{Z}}$$

Since E is on both C_1 and C_2, it follows that at E

$$\frac{\dot{X}}{\ddot{X}} = \frac{\dot{Y}}{\ddot{Y}} = \frac{\dot{Z}}{\ddot{Z}}$$

and

$$\dot{X}\ddot{Z} - \dot{Z}\ddot{X} = 0 \tag{3}$$

The curvature of a space curve with an arbitrary parametrization is given by [Goetz 1970]:

$$\kappa = \frac{\sqrt{(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{z}\ddot{x} - \dot{x}\ddot{z})^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})^2}}{((\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2)^{3/2}} \tag{4}$$

It follows from (1), (2), (3) and (4) that $\kappa=0$ at point E. Therefore E is a point of zero curvature of Γ_{σ_0}. □

Theorem **1** examines a space curve in the neighborhood of a cusp point just before the formation of that point.

Theorem 1. Let $\Gamma = (x(u), y(u), z(u))$ be a space curve in C_1 and let $x(u)$, $y(u)$ and $z(u)$ be polynomial functions of u. Let $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma), Z(u,\sigma))$ be an evolved version of Γ with a cusp point at u_0, then there exists a $\delta>0$ such that either $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of point u_0, or two projections of $\Gamma_{\sigma-\delta}$ intersect themselves in a neighborhood of u_0.

Proof: It has been shown by [Hummel *et al.* 1987] that the class of polynomial functions is closed under convolution with a Gaussian. Therefore $X(u,\sigma)$, $Y(u,\sigma)$ and $Z(u,\sigma)$ are also polynomial functions:

$$X(u,\sigma) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \cdots$$

$$Y(u,\sigma) = b_0 + b_1 u + b_2 u^2 + b_3 u^3 + \cdots$$

$$Z(u,\sigma) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 + \cdots$$

Let Γ_σ go through the origin of the coordinate system at $u_0=0$. It follows that $a_0=b_0=c_0=0$. Every cusp point is also a singular point of the curve. Therefore Γ_σ has a singularity at u_0. Now observe that

$$X_u(u,\sigma) = a_1 + 2a_2 u + 3a_3 u^2 + 4a_4 u^3 + \cdots$$

$$Y_u(u,\sigma) = b_1 + 2b_2 u + 3b_3 u^2 + 4b_4 u^3 + \cdots$$

$$Z_u(u,\sigma) = c_1 + 2c_2 u + 3c_3 u^2 + 4c_4 u^3 + \cdots$$

$X_u(u,\sigma)$, $Y_u(u,\sigma)$ and $Z_u(u,\sigma)$ are zero at a singular point. It follows that $a_1=b_1=c_1=0$. As a result, all powers of u in $X(u,\sigma)$, $Y(u,\sigma)$ and $Z(u,\sigma)$ are larger than one.

The following case analysis identifies the cases in which the singular point at u_0 is also a cusp point. Since Γ_σ is examined in a small neighborhood of point $u_0=0$, it can be approximated using the lowest degree terms in $X(u,\sigma)$, $Y(u,\sigma)$ and $Z(u,\sigma)$:

$$\Gamma_\sigma = (u^m, u^n, u^p)$$

Assume without loss of generality that $p>n>m$. Observe that

$$\mathbf{r}_u(u,\sigma) = (X_u(u,\sigma), Y_u(u,\sigma), Z_u(u,\sigma)) = (m\,u^{m-1}, n\,u^{n-1}, p\,u^{p-1})$$

Therefore

$$\mathbf{r}_u(\epsilon,\sigma) = (m\epsilon^{m-1}, n\epsilon^{n-1}, p\epsilon^{p-1}) = \epsilon^{m-1}(m, n\epsilon^{n-m}, p\epsilon^{p-m})$$

and

$$\mathbf{r}_u(-\epsilon,\sigma) = (m(-\epsilon)^{m-1}, n(-\epsilon)^{n-1}, p(-\epsilon)^{p-1})$$

Since m, n and p can be odd or even, the singular point at u_0 must be analyzed in each of eight possible cases:

1. m, n and p are even.

$m-1$, $n-1$ and $p-1$ are odd. So

$$\mathbf{r}_u(-\epsilon,\sigma) = (-m\epsilon^{m-1}, -n\epsilon^{n-1}, -p\epsilon^{p-1}) = -\epsilon^{m-1}(m, n\epsilon^{n-m}, p\epsilon^{p-m})$$

Comparing $\mathbf{r}_u(\epsilon,\sigma)$ to $\mathbf{r}_u(-\epsilon,\sigma)$ shows that an infinitesimal change in parameter u in a neighborhood of the singular point results in a large change in the direction of the tangent vector. Therefore this singularity is a cusp point.

2. m and n are even, p is odd.

$m-1$ and $n-1$ are odd and $p-1$ is even. Therefore

$$\mathbf{r}_u(-\epsilon,\sigma) = (-m\epsilon^{m-1}, -n\epsilon^{n-1}, p\epsilon^{p-1}) = \epsilon^{m-1}(-m, -n\epsilon^{n-m}, p\epsilon^{p-m})$$

A comparison of $\mathbf{r}_u(\epsilon,\sigma)$ and $\mathbf{r}_u(-\epsilon,\sigma)$ again shows that an infinitesimal change in u causes a large change in the tangent direction. Hence this singular point is also a cusp point.

3. m is even, n is odd and p is even.

$m-1$ is odd, $n-1$ is even and $p-1$ is odd. Hence

$$\mathbf{r}_u(-\epsilon,\sigma) = (-m\epsilon^{m-1}, n\epsilon^{n-1}, -p\epsilon^{p-1}) = \epsilon^{m-1}(-m, n\epsilon^{n-m}, -p\epsilon^{p-m})$$

An infinitesimal change in u again results in a large change in the tangent direction. This singularity is a cusp point as well.

4. m is even, n and p are odd.

$m-1$ is odd, $n-1$ and $p-1$ are even. So

$$\mathbf{r}_u(-\epsilon,\sigma) = (-m\epsilon^{m-1}, n\epsilon^{n-1}, p\epsilon^{p-1}) = \epsilon^{m-1}(-m, n\epsilon^{n-m}, p\epsilon^{p-m})$$

A large change in the tangent direction is caused by an infinitesimal change in u. Therefore this singularity also corresponds to a cusp point.

5. m is odd, n and p are even.

m–1 is even, n–1 and p–1 are odd. Therefore

$$\mathbf{r}_u(-\epsilon,\sigma) = (m\epsilon^{m-1}, -n\epsilon^{n-1}, -p\epsilon^{p-1}) = \epsilon^{m-1}(m, -n\epsilon^{n-m}, -p\epsilon^{p-m})$$

A comparison of $\mathbf{r}_u(\epsilon,\sigma)$ and $\mathbf{r}_u(-\epsilon,\sigma)$ now shows that an infinitesimal change in u in the neighborhood of the singular point also results in an infinitesimal change in the tangent direction. Hence, this singular point is *not* a cusp point.

6. m is odd, n is even, p is odd.

m–1 is even, n–1 is odd and p–1 is even. So

$$\mathbf{r}_u(-\epsilon,\sigma) = (m\epsilon^{m-1}, -n\epsilon^{n-1}, p\epsilon^{p-1}) = \epsilon^{m-1}(m, -n\epsilon^{n-m}, p\epsilon^{p-m})$$

The tangent direction changes only infinitesimally in the neighborhood of the singular point. Therefore this singularity is *not* a cusp point either.

7. m and n are odd, p is even.

m–1 and n–1 are even and p–1 is odd. Hence

$$\mathbf{r}_u(-\epsilon,\sigma) = (m\epsilon^{m-1}, n\epsilon^{n-1}, -p\epsilon^{p-1}) = \epsilon^{m-1}(m, n\epsilon^{n-m}, -p\epsilon^{p-m})$$

This singularity is again *not* a cusp point since the tangent direction changes only infinitesimally in its neighborhood.

8. m, n and p are odd.

m–1, n–1 and p–1 are even. Therefore

$$\mathbf{r}_u(-\epsilon,\sigma) = (m\epsilon^{m-1}, n\epsilon^{n-1}, p\epsilon^{p-1}) = \epsilon^{m-1}(m, n\epsilon^{n-m}, p\epsilon^{p-m})$$

The last singular point is *not* a cusp point either since the changes in the tangent direction are again infinitesimal.

It follows from the case analysis above that only the singular points in cases **1-4** are cusp points. We next derive analytical expressions for the curve $\Gamma_{\sigma-\delta}$ so that it can be analyzed in a small neighborhood of the cusp point.

To deblur function $f(u) = u^k$, a rescaled version of that function is convolved with the function $\frac{2}{\sqrt{\pi}}e^{-u^2}(1-u^2)$. This function is an approximation to the deblurring operator derived in [Hummel *et al.* 1987] and is good for small amounts of deblurring. The convolution can be expressed as

$$(D_t f)(u) = \int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}} e^{-v^2}(1-v^2) f(u+2v\sqrt{t})\, dv$$

or

$$(D_t f)(u) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2}(1-v^2)(u+2v\sqrt{t})^k dv$$

where t is the scale factor and controls the amount of deblurring. Solving the integral above yields

$$(D_t f)(u) = \sum_{\substack{p=0 \\ (p\ even)}}^{k} 1.3.5 \cdots (p-1) \frac{(2t)^{p/2}\ k(k-1) \cdots (k-p+1)}{p!} (1-p)\, u^{k-p} \tag{5}$$

$\Gamma_{\sigma-\delta}$ can now be analyzed in each of the cases **1-4**:

Case 1: Γ_σ is approximated by (u^m, u^n, u^p) where m, n and p are even.

The appendix shows that this kind of cusp point must also exist on Γ itself. This is a contradiction of the assumption that Γ is in C_1. Therefore Γ_σ can not have a cusp point of this kind at u_0.

Case 2: Γ_σ is approximated by (u^m, u^n, u^p) where m and n are even and p is odd.

Γ_σ is obtained by deblurring each of its coordinate functions:

$$(D_t x)(u) = u^m - c_1 t u^{m-2} - \cdots - c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^2 - c_{\frac{m}{2}} t^{\frac{m}{2}}$$

$$(D_t y)(u) = u^n - c'_1 t u^{n-2} - \cdots - c'_{\frac{n-2}{2}} t^{\frac{n-2}{2}} u^2 - c'_{\frac{n}{2}} t^{\frac{n}{2}}$$

$$(D_t z)(u) = u^p - c''_1 t u^{p-2} - \cdots - c''_{\frac{p-1}{2}} t^{\frac{p-1}{2}} u$$

Note that $(D_t x)$ and $(D_t y)$ contain even powers of u only, $(D_t z)$ contains odd powers of u only and all constants are positive.

The deblurred curve intersects itself if there are two values of u, u_1 and u_2, such that

$$x(u_1) = x(u_2)$$

$$y(u_1) = y(u_2)$$

$$z(u_1) = z(u_2)$$

It follows from the first two constraints above that $u_1 = -u_2$. Substituting for u_2 in the third constraint and simplifying yields:

$$u_1{}^p - c''_1 t u_1{}^{p-2} - \cdots - c''_{\frac{p-1}{2}} t^{\frac{p-1}{2}} u_1 = 0$$

Now let $t = \delta$ to obtain

$$u_1{}^p - c''_1 \delta u_1{}^{p-2} - \cdots - c''_{\frac{p-1}{2}} \delta^{\frac{p-1}{2}} u_1 = 0 \tag{6}$$

The LHS of (6) is negative for very small values of u_1 since the first term will be smaller than all other terms, which are negative. As u_1 grows, the first term becomes larger than the sum of all other terms and the LHS of (6) becomes positive. Therefore there is a positive value of u_1 at which (6) is satisfied. Hence $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of u_0.

Case 3: Γ_σ is approximated by (u^m, u^n, u^n) where m is even, n is odd and p is even.

As in the previous case, we obtain analytical expressions for $\Gamma_{\sigma-\delta}$:

$$(D_t x)(u) = u^m - c_1 t u^{m-2} - \cdots - c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^2 - c_{\frac{m}{2}} t^{\frac{m}{2}}$$

$$(D_t y)(u) = u^n - c'_1 t u^{n-2} - \cdots - c'_{\frac{n-1}{2}} t^{\frac{n-1}{2}} u$$

$$(D_t z)(u) = u^p - c''_1 t u^{p-2} - \cdots - c''_{\frac{p-2}{2}} t^{\frac{p-2}{2}} u^2 - c''_{\frac{p}{2}} t^{\frac{p}{2}}$$

In this case, $(D_t x)$ and $(D_t z)$ contain only even powers of u and $(D_t y)$ contains only odd powers of u. Again, $\Gamma_{\sigma-\delta}$ can be shown to intersect itself if there are two values of u, u_1 and u_2, such that

$$x(u_1) = x(u_2)$$

$$y(u_1) = y(u_2)$$

$$z(u_1) = z(u_2)$$

It now follows from the first and third constraints above that $u_1 = -u_2$. Substituting for u_2 in the second constraint, letting $t = \delta$ and simplifying yields

$$u_1{}^n - c'_1 \delta u_1{}^{n-2} - \cdots - c'_{\frac{n-1}{2}} \delta^{\frac{n-1}{2}} u_1 = 0 \tag{7}$$

An argument similar to the one used in the previous case shows that there exists a positive value of u_1 at which (7) is satisfied. Therefore $\Gamma_{\sigma-\delta}$ again intersects itself in a neighborhood of u_0.

Case 4: Γ_σ is approximated by (u^m, u^n, u^p) where m is even and n and p are odd.

An analytical expression for $\Gamma_{\sigma-\delta}$ in a neighborhood of u_0 is given by

$$(D_t x)(u) = u^m - c_1 t u^{m-2} - \cdots - c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^2 - c_{\frac{m}{2}} t^{\frac{m}{2}}$$

$$(D_t y)(u) = u^n - c'_1 t u^{n-2} - \cdots - c'_{\frac{n-1}{2}} t^{\frac{n-1}{2}} u$$

$$(D_t z)(u) = u^p - c''_1 t u^{p-2} - \cdots - c''_{\frac{p-1}{2}} t^{\frac{p-1}{2}} u$$

All powers of u in $(D_t x)$ are even and all powers of u in $(D_t y)$ and $(D_t z)$ are odd. As before, $\Gamma_{\sigma-\delta}$ intersects itself if the three constraints

$$x(u_1) = x(u_2)$$

$$y(u_1) = y(u_2)$$

$$z(u_1) = z(u_2)$$

are satisfied simultaneously. It follows from the first constraint that $u_1 = -u_2$. Now substitute for u_2 in the second and third constraints, let $t = \delta$ and simplify:

$$u_1{}^n - c'_1 \delta u_1{}^{n-2} - \cdots - c'_{\frac{n-1}{2}} \delta^{\frac{n-1}{2}} u_1 = 0 \tag{8}$$

$$u_1{}^p - c''_1 \delta u_1{}^{p-2} - \cdots - c''_{\frac{p-1}{2}} \delta^{\frac{p-1}{2}} u_1 = 0 \tag{9}$$

Each of the equations (8) and (9) is satisfied at a positive value of u_1 but, in general, the same value of u_1 will *not* satisfy both. It follows that, in this case, $\Gamma_{\sigma-\delta}$ does *not* intersect itself. However, an argument similar to the ones in the previous two cases shows that the planar curves defined by $(D_t x)$ and $(D_t y)$ and by $(D_t x)$ and $(D_t z)$, that is, the projections of $\Gamma_{\sigma-\delta}$ on the XY and XZ planes respectively, do intersect themselves in a neighborhood of u_0.

This completes the proof of theorem **1**. □

Theorem **2** examines a space curve in the neighborhood of a cusp point just after the formation of that point.

Theorem 2. Let $\Gamma = (x(u), y(u), z(u))$ be a space curve in C_1 and let $x(u)$, $y(u)$ and $z(u)$ be polynomial functions of u. Let $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma), Z(u,\sigma))$ be an evolved version of Γ with a cusp point at u_0, then either a torsion zero-crossing point exists at u_0 on curves $\Gamma_{\sigma-\delta}$ and $\Gamma_{\sigma+\delta}$ or $\Gamma_{\sigma+\delta}$ has two new torsion zero-crossings in a neighborhood of u_0.

Proof: Using a case analysis similar to the one in the proof of theorem **1** to characterize all the possible singularities of Γ_σ at u_0, we again conclude that only the singular points in cases **1-4** are cusp points.

We now derive analytical expressions for $\Gamma_{\sigma+\delta}$ so that it can be analyzed in a neighborhood of u_0. To blur function $f(u) = u^k$, we convolve a rescaled version of that function with the function $\frac{1}{\sqrt{\pi}}e^{-u^2}$, the blurring operator, as follows:

$$F(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-v^2} f(u + 2v\sqrt{t})\, dv$$

or

$$F(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} (u + 2v\sqrt{t})^k dv$$

where t is the scale factor and controls the amount of blurring. Solving the integral above yields

$$F(u) = \sum_{\substack{p=0 \\ (p\ even)}}^{k} 1.3.5 \cdots (p-1) \frac{(2t)^{p/2} k(k-1) \cdots (k-p+1)}{p!} u^{k-p} \tag{10}$$

An expression for $\Gamma_{\sigma+\delta}$ in a neighborhood of the cusp point can be obtained by blurring each of its coordinate functions. Furthermore, expressions for $\Gamma_{\sigma-\delta}$ in a neighborhood of the cusp point can be obtained by deblurring each of its coordinate functions according to (5).

Each of the cases **1-4** can now be analyzed in turn:

Case 1: Γ_σ is approximated by (u^m, u^n, u^p) where m, n and p are even.

The Appendix shows that this type of cusp point must also exist on Γ. This is a contradiction of the assumption that Γ is in C_1. It follows that this kind of cusp point can not arise during evolution.

Case 2: Γ_σ is approximated by (u^m, u^n, u^p) where m and n are even and p is odd.

Observe that

$$\dot{x}(u) = mu^{m-1} \qquad \ddot{x}(u) = m(m-1)\,u^{m-2} \qquad \dddot{x}(u) = m(m-1)(m-2)\,u^{m-3}$$

$$\dot{y}(u) = nu^{n-1} \qquad \ddot{y}(u) = n(n-1)\,u^{n-2} \qquad \dddot{y}(u) = n(n-1)(n-2)\,u^{n-3}$$

$$\dot{z}(u) = pu^{p-1} \qquad \ddot{z}(u) = p(p-1)\,u^{p-2} \qquad \dddot{z}(u) = p(p-1)(p-2)\,u^{p-3}$$

Torsion on Γ_σ is given by [Goetz 1970]:

$$\tau(u) = \frac{\dddot{z}\,\dot{x}\,\ddot{y} - \dddot{z}\,\dot{y}\,\ddot{x} + \dddot{y}\,\dot{z}\,\ddot{x} - \dddot{y}\,\dot{x}\,\ddot{z} + \dddot{x}\,\dot{y}\,\ddot{z} - \dddot{x}\,\dot{z}\,\ddot{y}}{(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{z}\ddot{x} - \dot{x}\ddot{z})^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})^2}$$

or

$$\tau(u) = \frac{mnp((p-1)(p-2)(n-m) + (n-1)(n-2)(m-p) + (m-1)(m-2)(p-n))\,u^{p+n+m-6}}{A + B + C} \qquad (11)$$

where

$$A = ((np(p-1) - pn(n-1))\,u^{p+n-3})^2$$

$$B = ((pm(m-1) - mp(p-1))\,u^{p+m-3})^2$$

$$C = ((mn(n-1) - nm(m-1))\,u^{m+n-3})^2$$

At $u = 0$ (cusp point), τ is undefined. When u is positive or negative, the sign of $\tau(u)$ depends on the sign of the coefficient of the numerator. Let K be that coefficient divided by mnp. Observe that

$$\begin{aligned}
K &= (p-1)(p-2)(n-m) + (n-1)(n-2)(m-p) + (m-1)(m-2)(p-n) \\
&= (p^2-3p+2)(n-m) + (n^2-3n+2)(m-p) + (m^2-3m+2)(p-n) \\
&= np^2 - mp^2 - 3pn + 3pm + 2n - 2m + mn^2 - 3mn + 2m - pn^2 \\
&\quad + 3pn - 2p + pm^2 - 3pm + 2p - nm^2 + 3mn - 2n \\
&= (n-m)p^2 + (m^2-n^2)p + mn^2 - nm^2 \\
&= (n-m)p^2 + (m+n)(m-n)p + mn(n-m) \\
&= (n-m)\,(p^2 - (m+n)p + mn) \\
&= (n-m)\,(p-m)\,(p-n)
\end{aligned}$$

which is positive because of the assumption that $p > n > m$. Since $p+n+m-6$, the power of u in the numerator, is odd, it follows that $\tau(u)$ is positive for positive u and negative for negative u.

We now investigate $\tau(u)$ on $\Gamma_{\sigma+\delta}$. It follows from (10) that $\Gamma_{\sigma+\delta}$ is given by:

$$X(u) = u^m + c_1 t u^{m-2} + \cdots + c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^2 + c_{\frac{m}{2}} t^{\frac{m}{2}}$$

$$Y(u) = u^n + c'_1 t u^{n-2} + \cdots + c'_{\frac{n-2}{2}} t^{\frac{n-2}{2}} u^2 + c'_{\frac{n}{2}} t^{\frac{n}{2}}$$

$$Z(u) = u^p + c''_1 t u^{p-2} + \cdots + c''_{\frac{p-1}{2}} t^{\frac{p-1}{2}} u$$

where all constants are positive, all powers of u in $X(u)$ and $Y(u)$ are even, all powers of u in $Z(u)$ are odd and t equals δ, a small constant. Note also that the last terms in $X(u)$ and $Y(u)$ do not contain any positive powers of u but all terms in $Z(u)$ contain positive powers of u. It follows that the last terms in $\ddot{X}(u)$, $\ddot{Y}(u)$, $\dot{Z}(u)$ and $\dddot{Z}(u)$ do not contain positive powers of u whereas all terms in $\dot{X}(u)$, $\dddot{X}(u)$, $\dot{Y}(u)$, $\dddot{Y}(u)$ and $\ddot{Z}(u)$ contain positive powers of u. Therefore, at $u = 0$, $\dot{X}(u) = \dddot{X}(u) = \dot{Y}(u) = \dddot{Y}(u) = \ddot{Z}(u) = 0$ and $\tau = 0$. As u grows, the terms in $\dot{X}(u)$, $\ddot{X}(u)$, $\dddot{X}(u)$, $\dot{Y}(u)$, $\ddot{Y}(u)$, $\dddot{Y}(u)$, $\dot{Z}(u)$, $\ddot{Z}(u)$ and $\dddot{Z}(u)$ with the largest power of u (which are also the only terms without δ) become dominant and torsion is again given by (11). It follows that $\tau(u)$ is positive for positive u and negative for negative u on $\Gamma_{\sigma+\delta}$. Since τ is zero at $u = 0$, $\Gamma_{\sigma+\delta}$ has a torsion zero-crossing point at $u = 0$.

We next investigate $\tau(u)$ on $\Gamma_{\sigma-\delta}$. From (5) it follows that $\Gamma_{\sigma-\delta}$ is given by:

$$(D_t x)(u) = u^m - d_1 t u^{m-2} - \cdots - d_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^2 - d_{\frac{m}{2}} t^{\frac{m}{2}}$$

$$(D_t y)(u) = u^n - d'_1 t u^{n-2} - \cdots - d'_{\frac{n-2}{2}} t^{\frac{n-2}{2}} u^2 - d'_{\frac{n}{2}} t^{\frac{n}{2}}$$

$$(D_t z)(u) = u^p - d''_1 t u^{p-2} - \cdots - d''_{\frac{p-1}{2}} t^{\frac{p-1}{2}} u$$

where all constants are positive, all powers of u in $D_t x$ and $D_t y$ are even, all powers of u in $D_t z$ are odd and t equals δ, a small constant. It again follows that $\tau = 0$ at $u = 0$, τ is positive for positive u and negative for negative u. Therefore there is also a torsion zero-crossing point at $u = 0$ on $\Gamma_{\sigma-\delta}$. It follows that there is a torsion zero-crossing point at u_0 on $\Gamma_{\sigma-\delta}$ before the formation of the cusp point and on $\Gamma_{\sigma+\delta}$ after the formation of the cusp point.

Case 3: Γ_σ is approximated by (u^m, u^n, u^p) where m is even, n is odd and p is even.

The proof is analogous to that of case **2**, and the same result follows.

Case 4: Γ_σ is approximated by (u^m, u^n, u^p) where m is even, and n and p are odd.

At $u = 0$, the cusp point, τ is undefined. At all other points, $\tau(u)$ is given by (11). Since the coefficient of the numerator of (11) is positive (as shown in the proof of case 2) and $p+n+m-6$, the power of u in the numerator, is even, $\tau(u)$ is positive for positive *and* negative values of u in the neighborhood of u_0 on Γ_σ. Therefore there are *no* torsion zero-crossing points in the neighborhood of u_0 on Γ_σ.

We now investigate $\tau(u)$ on $\Gamma_{\sigma+\delta}$. It follows from (10) that $\Gamma_{\sigma+\delta}$ is given by:

$$X(u) = u^m + c_1 t u^{m-2} + \cdots + c_{\frac{m-2}{2}} t^{\frac{m-2}{2}} u^2 + c_{\frac{m}{2}} t^{\frac{m}{2}}$$

$$Y(u) = u^n + c'_1 t u^{n-2} + \cdots + c'_{\frac{n-1}{2}} t^{\frac{n-1}{2}} u$$

$$Z(u) = u^p + c''_1 t u^{p-2} + \cdots + c''_{\frac{p-1}{2}} t^{\frac{p-1}{2}} u$$

where all constants are positive, all powers of u in $X(u)$ are even, all powers of u in $Y(u)$ and $Z(u)$ are odd and t equals δ, a small constant. Furthermore, note that the last term in $X(u)$ does not contain a positive power of u but all terms in $Y(u)$ and $Z(u)$ contain positive powers of u. Therefore the last terms in $\ddot{X}(u)$, $\dot{Y}(u)$, $\dddot{Y}(u)$, $\dot{Z}(u)$ and $\dddot{Z}(u)$ do not contain positive powers of u whereas all terms in $\dot{X}(u)$, $\dddot{X}(u)$, $\ddot{Y}(u)$ and $\ddot{Z}(u)$ contain positive powers of u. Hence at $u=0$, $\dot{X}(u) = \dddot{X}(u) = \ddot{Y}(u) = \ddot{Z}(u) = 0$ and

$$\tau(u) = \frac{\dddot{Y}(u)\dot{Z}(u)\ddot{X}(u) - \dddot{Z}(u)\dot{Y}(u)\ddot{X}(u)}{(\dot{Z}(u)\ddot{X}(u))^2 + (\dot{Y}(u)\ddot{X}(u))^2} = \frac{\ddot{X}(u)(\dddot{Y}(u)\dot{Z}(u) - \dddot{Z}(u)\dot{Y}(u))}{(\dot{Z}(u)\ddot{X}(u))^2 + (\dot{Y}(u)\ddot{X}(u))^2}$$

Since the denominator is positive and $\ddot{X}(u)$ is positive, to determine the sign of $\tau(u)$, we must determine the sign of the expression: $\dddot{Y}(u)\dot{Z}(u) - \dddot{Z}(u)\dot{Y}(u)$. At $u=0$, using (10) we conclude that the non-zero term of $\dot{Y}(u)$ is:

$$c'_{\frac{n-1}{2}} t^{\frac{n-1}{2}} = 1.3.5 \cdots (n-2) \frac{(2t)^{\frac{n-1}{2}} n!}{(n-1)!} = 1.3.5 \cdots n\, 2^{\frac{n-1}{2}} t^{\frac{n-1}{2}}$$

Similarly, at $u=0$, the non-zero term of $\dot{Z}(u)$ is:

$$c''_{\frac{p-1}{2}} t^{\frac{p-1}{2}} = 1.3.5 \cdots p\, 2^{\frac{p-1}{2}} t^{\frac{p-1}{2}}$$

Using (10), it follows that at $u=0$, the non-zero term of $\dddot{Y}(u)$ is:

$$6c'_{\frac{n-3}{2}}t^{\frac{n-3}{2}} = 6(1.3.5\cdots(n-4))\frac{(2t)^{\frac{n-3}{2}}n!}{6(n-3)!} = (1.3.5\cdots n)(n-1)2^{\frac{n-3}{2}}t^{\frac{n-3}{2}}$$

Similarly, at $u=0$, the non-zero term of $\dddot{Z}(u)$ is:

$$6c''_{\frac{p-3}{2}}t^{\frac{p-3}{2}} = (1.3.5\cdots p)(p-1)2^{\frac{p-3}{2}}t^{\frac{p-3}{2}}$$

Therefore

$$\dddot{Y}(u)\dot{Z}(u) - \dddot{Z}(u)\dot{Y}(u) = (1.3.5\cdots n)(n-1)2^{\frac{n-3}{2}}t^{\frac{n-3}{2}}(1.3.5\cdots p)2^{\frac{p-1}{2}}t^{\frac{p-1}{2}}$$

$$-(1.3.5\cdots n)2^{\frac{n-1}{2}}t^{\frac{n-1}{2}}(1.3.5\cdots p)(p-1)2^{\frac{p-3}{2}}t^{\frac{p-3}{2}}$$

$$=(2t)^{\frac{p+n-4}{2}}(1.3.5\cdots n)(1.3.5\cdots p)(n-p)$$

and it follows that $\dddot{Y}(u)\dot{Z}(u) - \dddot{Z}(u)\dot{Y}(u) < 0$ since $n<p$. Therefore $\tau(u)$ is negative at $u=0$ on $\Gamma_{\sigma+\delta}$. As u grows the terms in $\dot{X}(u)$, $\ddot{X}(u)$, $\dddot{X}(u)$, $\dot{Y}(u)$, $\ddot{Y}(u)$, $\dddot{Y}(u)$, $\dot{Z}(u)$, $\ddot{Z}(u)$ and $\dddot{Z}(u)$ with the largest power of u (which are also the only terms without δ) become dominant and $\tau(u)$ is again given by (11). Since $p+n+m-6$, the power of u in the numerator, in now even, $\tau(u)$ becomes positive as u grows in absolute value. Therefore there exist two new torsion zero-crossings in a neighborhood of u_0 on $\Gamma_{\sigma+\delta}$.

This completes the proof of theorem **2**. □

III. Conclusions

This paper presented several results on the evolution properties of space curves which provide theoretical underpinning for the curvature and torsion scale space representation for space curves that has been proposed before.

It was shown that properties such as connectedness and closedness are preserved during evolution, that the center of mass does not move as the curve evolves, that evolution is invariant under affine transformations of the curve such as uniform scaling, rotation and translation, and that a space curve contained inside a simple, convex object remains inside that object during evolution. The two main theorems of the paper showed that there are strong constraints on the shape of a space curve in the neighborhood of a cusp point just before and just after the formation of that point.

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References

Asada, H. and M. Brady, "The curvature primal sketch," *IEEE-PAMI*, vol. 8, pp. 2-14, 1986.

Babaud, J., A. P. Witkin, M. Baudin, and R. O. Duda, "Uniqueness of the Gaussian Kernel for Scale-Space Filtering," *IEEE-PAMI*, vol. 8, pp. 26-33, 1986.

Goetz, A., *Introduction to differential geometry,* Addison Wesley, Reading, MA, 1970.

Horn, B. K. P. and E. J. Weldon, "Filtering closed curves," *IEEE-PAMI*, vol. 8, pp. 665-668, 1986 .

Hummel, R. A., B. Kimia, and S. W. Zucker, "Deblurring Gaussian Blur," *Computer Vision, Graphics, and Image Processing*, vol. 38, pp. 66-80, 1987.

Kecs, W., *The Convolution Product and Some Applications,* D. Reidel, Boston, U.S.A., 1982.

Mackworth, A. K. and F. Mokhtarian, "The renormalized curvature scale space and the evolution properties of planar curves," Technical Report 87-37, Department of Computer Science, University of British Columbia, Vancouver, B.C., 1987.

Mokhtarian, F. and A. K. Mackworth, "Scale-based description and recognition of planar curves and two-dimensional shapes," *IEEE-PAMI*, vol. 8, pp. 34-43, 1986.

Mokhtarian, F., "Multi-scale description of space curves and three-dimensional objects," Technical Report 87-38, Department of Computer Science, University of British Columbia, Vancouver, B.C., 1987.

Stansfield, J.L., "Conclusions from the commodity expert project," AI Memo 601, MIT AI Lab., Cambridge, MA, 1980.

Witkin, A.P., "Scale space filtering," *IJCAI-83*, pp. 1019-1023, Karlsruhe, W. Germany, 1983.

Yuille, A. L. and T. Poggio, "Scaling Theorems for Zero Crossings," *IEEE-PAMI*, vol. 8, pp. 15-25, 1986.

Appendix

Note that the deblurring operator that was used in the proof of theorems **1** and **2** is good only for small values of t. The exact deblurring operator derived in [Hummel *et al.* 1987] is given by

$$D_N(u) = e^{-u^2} \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{(-1)^k}{\sqrt{\pi}\; k!\; 2^k} H_{2k}(u)$$

where

$$H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n}\left(e^{-u^2} \right)$$

The deblurring operator we have used corresponds to N=3. Observe that for any value of N, $D_N(u)$ will be of the form $\frac{e^{-u^2}}{\sqrt{\pi}}P(u)$ where $P(u)$ is a polynomial with only even powers of u.

Recall that in case **1**, the curve is approximated by (u^m, u^n, u^p) in a neighborhood of the cusp point where m, n and p are all even numbers. Now deblur the curve using $D_N(u)$ with arbitrarily large N as the deblurring operator:

$$(D_N x)(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} P(v)(u + 2v\sqrt{t}\,)^m dv$$

$$(D_N y)(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} P(v)(u + 2v\sqrt{t}\,)^n dv$$

$$(D_N z)(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} P(z)(u + 2z\sqrt{t}\,)^p dz$$

Since m, n and p are even, all the terms in $(u + 2v\sqrt{t}\,)^m$, $(u + 2v\sqrt{t}\,)^n$ and $(u + 2v\sqrt{t}\,)^p$ consist of either even powers of v and u or odd powers of v and u. Since $P(v)$ contains only even powers of v, this property is preserved when each of those expressions is multiplied by $P(v)$.

When the integrals are evaluated, terms with odd powers of v vanish. Terms that remain have only even powers of u. Therefore $(D_N\dot{x})(u)$, $(D_N\dot{y})(u)$ and $(D_N\dot{z})(u)$ have only odd powers of u and $(D_N\dot{\mathbf{r}})(\epsilon) = -(D_N\dot{\mathbf{r}})(-\epsilon)$. Since the value of N used is arbitrarily large, $D_N(u)$ is good for any value of t. In particular, let $t = \sigma^2/2$. It follows that there also exists a cusp point at $u_0 = 0$ on the original curve.