A Connection Method for Non-Monotonic and Autoepistemic Logic

by
George K. Tsiknis

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Abstract

In this paper, first we present a connection method for non-monotonic logic, together with its soundness and completeness proof. Then, we extend this to a proof procedure for autoepistemic logic. In the last section, we also discuss some improvements on the method through structure sharing techniques.

1 Introduction

Non monotonic or default reasoning arises naturally in most enterprises dealing with logic. Such reasoning sanctions inferences based on assumptions should not they contradict the current knowledge and beliefs. Deductions of this kind imitate the human reasoning in some respect, and are plausible when the reasoner has incomplete knowledge of the world in concern.

Several approaches to default reasoning have emerged in the last ten years.Most of them have the flavour of an extension to first order logic or bear a resemblance to a modal logic. Among them, we distinguish the non-monotonic logic of McDermott and Doyle [5] (also McDermott's extension in [6]), Reiter's default logic [8], Moore's autoepistemic logic [7], and Delgrande's approach using conditional logic [3].

In this paper, we deal with the non-monotonic and autoepistemic logic only. We believe that these approaches, even though they are at a primitive stage, they seem more interesting and ambitious than the other approaches (except conditional logic) in two respects. First, since the default statements are formulae in the language they can be reasoned about in the logic. In default logic on the other hand, reasoning about the default is not possible. Second, since they use only general inference rules (not depending on the defaults) they provide some kind of semantics that is closer to the classical one.

The first part of the paper, deals with the non-monotonic logic. We briefly discuss its main features and give a proof procedure for the propositional case using the connection calculus developed by Bibel [1]. In the second part we extend this method to obtain a proof procedure for the autoepistemic logic. At the end, we discuss possible improvements as well as possibilities of similar procedures for other default logics. We regret that due to lack of time, we do not consider Delgrande's approach in this paper. We believe that this method is among the most promising ones, but it is far more difficult to find a proof procedure, even for the propositional case. We now proceed with the non-monotonic logic.

2 Non-Monotonic Logic

McDermott and Doyle in [5] in order to achieve non-monotonic reasoning, propose an extension of the classical first order logic by introducing a sentential operator 'M' that informally means "is consistent". As a result, the non-monotonic language NL is $L \cup \{"M"\}$, where L is a first order language. The set of well formed formulas NW is the smallest set that is closed under the first order formation rules together with the rule : "if $w \in NW$ then $Mw \in NW$".

Non-monotonic inferences then, can be sanctioned by using a single (meta)rule : "Mp is derivable if $\neg p$ is not derivable. Unfortunately, this rule is not an inference rule at all because it is circular. For this reason, McDermott and Doyle define the theorems of a theory by means of the fixed points of a non-monotonic operator NM as follows.

Definition 1 *Given a theory (set of formulae) A and any set of formulae S,*

$$NM_A(S) = Th(A \cup As_A(S))$$

where

$$As_(S) = \{Mq | q \in L \wedge \neg q \notin S\} - Th(A)$$

and Th means first order theorems.

The non-monotonic theorems of A then, is the set TH(A) defined by

$$TH(A) = \bigcap(\{L\} \cup \{S | NM_A(S) = S\})$$

that is, the NM-theorems of A is the intersection on all fixed points of NM_A or the entire language if no such fixed points exist.

□

Davis in [2] gives a more comprehensive definition of the non-monotonic theorems of a theory A which resembles the usual maximum consistent extension construction. An equivalent definition (similar to Moore's) of a fixed point, which we will extensively use in this paper is the following.

Definition 2 *Given a theory A, S is a fixed point of A if*

$$T = Th(A \cup \{Mq | \neg q \notin T\})$$

$\square$

Notice that according to the first definition, if a theory A does not have any fixed point, the whole language is assigned as its fixed point. If we think of the fixed point as maximal extension of A (in fact they are similar to Reiter's extensions), it is possible a consistent set A to have an inconsistent extension. For this reason we abandon the previous definition of TH(A); instead, we adopt the following one.

Definition 3 *The non-monotonic theorems NM(A) of a set of formulae A is defined as :*

$$TH(A) = Th(A) \cup \{S | NM_A(S) = S\}$$

$\square$

With this modification, the following lemmata hold (and are given without proof).

Lemma 1 *A set of formulae A has an inconsistent fixed point iff A is inconsistent.*

Lemma 2 *If a set A has an inconsistent fixed point it is its only fixed point.*

Lemma 3 *If a set A is consistent any of its fixed points is consistent.*

In [6] also has been shown that fixed points are really maximal sets, that is, if S,S' are two fixed points of A neither one is a proper subset of the other.

With this modifications and clarifications we are now ready to present a proof procedure for the propositional non-monotonic logic. From now on the modifier "propositional" should implicitly be understood whenever terms like "language", "theory", "fixed points", etc. are being used.

3 Connection Method for Propositional Non-Monotonic Logic

In this section, we will give a proof procedure for the propositional non-monotonic logic based on Bibel's connection method [1]. McDermott and Doyle in [6] outline a similar procedure based on the tableau method but their completeness proof is not quite correct. The connection method gives a procedure which can be improved (see last section) using structure sharing and also it can be easily extended to a proof procedure for autoepistemic logic (next section).

Notation: Given a formula q, [q] denotes the matrix in *normal* form that positively represents q.

Given a finite set of premises A and a formula p, we investigate what it means p to be a theorem of A. First, p has to be in any fixed point S of A. But each such S is the deductive closure of A together with some assumptions Mq_i,$i \geq 0$, and q is consistent with A. Consequently, p is in S iff there is a finite set X of assumptions Mq_i such that $A \cup X \vdash p$, i.e. p is a tautaulogical consequence of $A \cup X$. The latter implies that the matrix $[A \wedge X \rightarrow p]$ is complementary, which in turn suggests that any non-complementary path of $[A \rightarrow p]$ contains some Mq_i such that q_i is consistent with A. But the consistency of q_i can be determined by the complementarity of the matrix $[A \rightarrow \neg q_i]$, which in turn, triggers the generation of more matrices. It is now clear, that our proof procedure will need, instead of a single one, a family of matrices defined as follows.

Definition 4 *A family of matrices FF(A,p) for a goal p and premises A is the smallest set of matrices that satisfies :*

1. $[A \rightarrow p] \in FF(A,p)$.
2. *if* $F \in FF(A,p)$ *and there is a p-noncomplementary (p means propositionally) path in F that contains the formula Mq then* $[A \rightarrow \neg q] \in FF(A,p)$.

□

Now we have to define when a matrix in a family is complementary or not. Since the complementarity of a matrix depends on the status of other matrices in the family, it is necessary to assign labels to matrices which justifies the next definition.

Definition 5 *A labelling of a family FF(A,p) is a function LF : FF(A,p) $\rightarrow$ { 'OPEN','CLOSED' }.*

We embark now to define complementarity.

Definition 6

1. *A* literal *is a formula of the form p or $\neg p$ or Mq or $\neg Mq$, where p is a propositional constant and q is any formula.*
2. *A* connection *is a set $\{L, \neg L\}$ called propositional connection, or $\{Mq\}$ called modal connection, where, L is a literal and q any formula.*
3. *Given a family FF(A,p) and a labelling LF of it, a connection in any matrix F in the family is* complementary *if it is a propositional connection or if it is a modal one { Mq } and $[A \rightarrow \neg q] \in FF(A,p)$ and $LF([A \rightarrow \neg q]) = OPEN$.*
4. *A path through a matrix (defined in the usual way) is complementary if it contains a complementary connection.*
5. *A matrix is complementary if it contains a spanning set of connections.*
6. *A labelling LF of a family FF(A,q) is* consistent *if for any $F \in FF(A,q)$, LF(F)=OPEN iff there is a path through F that is not complementary.*

□

With this definition a procedure that tests if p is a non-monotonic theorem of A ($A \mid\sim p$) is as follows :

Connection procedure for validity of $A \mid\sim p$

1. Generate the family FF(A,p).
   (a) $FF(A,p) \leftarrow [A \rightarrow p]$.

(b) While there is a path r in some matrix of FF(A,p) such that r is not p-complementary and $Mq \in r$ for some formula q, and $[A \rightarrow q] \notin FF(A,p)$ add $[A \rightarrow q]$ to FF(A,p).

2. For any labelling LF of FF(A,p) do : [If LF is consistent and $LF([A \rightarrow p]) = OPEN$ stop and return not valid.]

3. Return valid.

This procedure is inefficient but always terminate. Since A is finite, there are only finite number of matrices in FF(A,p), and hence, finitely many labellings to be considered. The correctness (soundness and completeness) of this procedure is given by the following two theorems.

Lemma 4 *Let X, Y be sets of formulas and q a formula of a propositional language such that $X \subseteq Y$, $Y \vdash q$ and $X \not\vdash q$. Every non-complementary path of $[X \rightarrow q]$ share a connection with every open path of $[\overline{Y - X}]$ where $[\overline{Y - X}]$ is the matrix that represents the disjunction of the negations of formulas in Y-X.*

Proof: Since $Y \vdash q$, by the completeness of the connection calculus, $[Y \rightarrow q]$ is complementary. But any path p through it, is the union of a path p' through $[X \rightarrow q]$ and a path p" through $[\overline{Y - X}]$. Consequently, if p is complementary and p', p" are not, there must exist a connection between a literal in p' and one in p".

□

Theorem 1 *If S is a fixed point of A and p any formula, there is a consistent labelling LF of the family FF(A,p) such that $q \in S$ iff $[A \rightarrow q] \in FF(A,p)$ and $LF([A \rightarrow q]) = CLOSED$.*

Proof: Let S be a fixed point of A. Then

$$S = Th(A \cup \{Mp | \neg p \notin S\})$$

If A is inconsistent, then any matrix $[A \rightarrow r]$ can only be labelled CLOSED and the theorem is true. Assume that A is consistent. Consider a family FF(A,p) and the labelling LF such that $LF([A \rightarrow q]) = OPEN$ if $q \not\in S$, CLOSED otherwise. We claim that LF is a consistent labelling.

Let $[A \rightarrow q]$ be a matrix labelled OPEN. Suppose that it is complementary. If every path in it is p-complementary then $A \vdash q$ and $q \in S$. Otherwise, each p-noncomplementary path should contain a modal literal Mq' such that $LF([A \rightarrow \neg q']) = OPEN$. Since $\neg q' \not\in S$, Mq' is among the assumptions of S. Let $Y = \{q_1, \ldots, q_n\}$ be the set of all such modal literals for the matrix $[A \rightarrow q]$, then by the previous lemma $[A \wedge Y \rightarrow q]$ is p-complementary and $q \in S$.

Now we consider the matrices labelled CLOSED. Let $[A \rightarrow r]$ be one of them. Since $r \in S$ either $A \vdash r$ or there is a minimal set of assumptions $X = \{Mq_1, \ldots, Mq_n\}$ such that $X \neq \emptyset$, $\neg q_i \not\in S$, and $A \cup X \vdash r$. In the first case $[A \rightarrow r]$ is complementary. In the second case by the previous lemma, every p-noncomplementary path of $[A \rightarrow r]$ contains at least one element of X. But for every i, $1 \leq i \leq n$, $\neg q_i \not\in S$ and the matrix $[A \rightarrow \neg q_i]$ is labelled OPEN. Consequently, every path is complementary.

□

Theorem 2 *If there is a consistent labelling LF for a family FF(A,p), there exists a fixed point S of A such that for every matrix $[A \rightarrow q] \in FF(A,p)$, $LF([A \rightarrow q]) = CLOSED$ iff $q \in S$.*

Proof: First, we extend FF(A,p) as following. If Mq occurs in A and there is not any matrix in the family that has a p-noncomplementary path containing Mq, then the matrix $[A \rightarrow \neg q]$ is added. Let FF'(A,p) be this extension of FF(A,p) and LF' a consistent extension of LF. Obviously, such an extension always exists since the labels of the new matrices do not interfere with the old ones.

We will construct S from the labelling. Let

$$R_0 = \{Mq | LF'([A \rightarrow \neg q]) = OPEN\}$$

$$S_0 = Th(A \cup R_0)$$

Let $Mq_1, Mq_2, \ldots$ be an enumeration of all formulas of the form Mq in $L - R_0$, such that if Mq_i is a subformula of Mq_j then $i \leq j$. We define $R_i, S_i, i = 0, 1, \ldots$ as

$R_{i+1} = Ri$ if $\neg q_{i+1} \in S_i$ else $R_i = R_i \cup \{Mq_{i+1}\}$,

$S_{i+1} = Th(A \cup R_{i+1})$

Now let $S = \cup S_i$ and $R = \cup R_i$. Clearly, $S_i \subset S_{i+1}$ and $S = Th(A \cup S)$. To show that S is a fixed point need to show that

$$R = \{Mq | \neg q \notin S\}$$

•

a. Let $\neg q \notin S$. We will show $Mq \in R$. If $Mq \in R_0$ then $Mq \in R$. Otherwise, q must be some q_i. Since $\neg q \notin S, \neg q \notin S_i$. So, $Mq \in R_{i+1}$ and $Mq \in R$.

b. Now, we show that $R \subset \{Mq | \neg q \notin S\}$, that is, if $Mq \in R$, then $\neg q \notin S$.

Case 1. Assume $Mq \in R_0$. There exists a matrix $[A \rightarrow \neg q]$ which is labelled OPEN. Assume $\neg q \in S$. If $A \vdash \neg q$ then, $[A \rightarrow \neg q]$ is complementary, and the labelling is inconsistent. Otherwise, $A \not\vdash \neg q$ and exists an S_k such that $S_{k-1} \not\vdash \neg q$ and $S_k \vdash \neg q$.

In the matrix $[A \rightarrow \neg q]$ there is a p-noncomplementary path that contains Mq_k. Consequently,$[A \rightarrow \neg q_k] \in FF'(A, p)$. If the last matrix is OPEN, then $Mq_k \in R_0$ and $S_{k-1} \vdash \neg q$. If it is CLOSED, let X be the set of Mr such that Mr is in some p-noncomplementary path of it, and $[A \rightarrow \neg r]$ is labelled OPEN. Since the labelling is consistent, $A \cup X \vdash \neg q_k$. But $X \subset R_0$, so $\neg q_k \in S_{k-1}$ and $S_{k-1} \vdash \neg q$.

Case 2. q should be some q_i. So, $Mq \in R_i$ and $\neg q \notin S_{i-1}$. Assume $\neg q \in S$, that is $\neg q \in S_k$ for some $k \geq i$ and $\neg q \notin S_{k-1}$. Then, $A \cup R_{k-1} \not\vdash \neg q$ and $A \cup R_k \vdash q$. Or $A \cup R_{k-1} \cup \{Mq_k\} \vdash \neg q$.

But Mq_k is not a subformula of q neither q_i (since $i \leq k$), so, Mq should occur in A. Consequently, there is a matrix $[A \rightarrow \neg q_k]$ in FF'(A,p). By following the same reasoning as in Case 1, we arrive at a contradiction.

Finally, we have to show that the labels agree with the fixed point. If the matrix

$[A \to \neg q]$ is OPEN, then $Mq \in S$ by construction. If it is CLOSED, then $A \cup R_0 \vdash \neg q$ and $\neg q \in S$.

$\square$

Now we give some examples.

Example 1. Let $A = \{Mc \to \neg d, Md \to \neg e, Me \to \neg f\}$. We show that $A \mid\!\sim \neg f$. The family and the labelling are given below:

$$\begin{pmatrix} Mc & Md & Me & \\ d & e & f & \neg f \end{pmatrix} \quad \mathit{CLOSED}$$

$$\begin{pmatrix} Mc & Md & Me & \\ d & e & f & \neg c \end{pmatrix} \quad \mathit{OPEN}$$

$$\begin{pmatrix} Mc & Md & Me & \\ d & e & f & \neg d \end{pmatrix} \quad \mathit{CLOSED}$$

$$\begin{pmatrix} Mc & Md & Me & \\ d & e & f & \neg e \end{pmatrix} \quad \mathit{OPEN}$$

In this case, there is only one consistent labelling in which, the first matrix is labelled CLOSED and $\neg f$ is a NM-theorem of A.

Example 2. Let $A = \{Mc \to \neg d, Md \to \neg c\}$. We show that $A \not\mid\!\sim \neg c$:

$$\begin{pmatrix} Mc & Md & \\ d & c & \neg c \end{pmatrix} \quad \mathit{OPEN} \qquad \mathit{CLOSED}$$

$$\begin{pmatrix} Mc & Md & \\ d & c & \neg d \end{pmatrix} \quad \mathit{CLOSED} \qquad \mathit{OPEN}$$

There are two consistent labellings in one of which the first matrix is labelled OPEN.

Example 3. As a final example, we show that $\{Mc \to \neg d, Md \to \neg c\} \mid\!\sim Mc \vee Md$:

$$\begin{pmatrix} Mc & Md & & \\ d & c & Mc & Md \end{pmatrix} \quad \mathit{CLOSED} \qquad \mathit{CLOSED}$$

$$\begin{pmatrix} Mc & Md & \\ d & c & \neg c \end{pmatrix} \quad OPEN \quad CLOSED$$

$$\begin{pmatrix} Mc & Md & \\ d & c & \neg d \end{pmatrix} \quad CLOSED \quad OPEN$$

There are two consistent labellings in which the first matrix is labelled CLOSED.

4 Autoepistemic Logic

There are many problems with the nonmonotonic logic as Moore points out in [7]. All of them come from the fact that the logic is very weak to cope with the negation of modal formulas. Thus, if p is in a fixed point S of A $\neg M \neg p$ is not forced in S. This is because McDermott and Doyle want "M" to mean "is consistent". Moore shows that if this semantic is changed and allows Mq to informally mean that $\neg q$ is not believed, then a stronger logic can be obtained. Consequently, Moore defines an autoepistemic extension T of a set A as

$$T = Th(A \cup \{Mq | \neg q \notin T\} \cup \{\neg M \neg q | q \in T\})$$

Since everything else we have said for the nonmonotonic logic applies also here, we now extend the previous connection method for the autoepistemic logic.

In the previous section, when we defined complementary paths for matrices in a family, because of the structure of the nonmonotonic fixed points, we only considered formulas of the form Mq. In the present logic, we have also to consider the formulas of the form $\neg Mq$. As a first consequence of this, clause 2 of definition 4 of FF(A,p) should be replaced by :

(2) If $F \in FF(A,p)$ and there is a p-noncomplementary path in F that contains the formula Mq or $\neg Mq$ then $[A \rightarrow \neg q] \in FF(A,p)$.

In addition, in definition 6.2 a set of the form $\{\neg Mq\}$ should be also considered to be a connection, while definition 6.3 should be replaced by :

Given a family FF(A,p) and a labelling LF, a connection in any matrix F in the family is complementary if either it is a propositional connection or it is of the form { Mq} where $[A \rightarrow \neg q] \in FF(A,p)$ and $LF([A \rightarrow \neg q]) = OPEN$ or it is of the form $\{\neg Mq\}$ where $[A \rightarrow \neg q] \in FF(A,p)$ and $LF([A \rightarrow \neg q]) = CLOSED$.

The connection procedure stated in the previous section can be applied to test for validity in this case also—Step 1 has to be modified to take into consideration the new definition of FF(A,p). The proofs of soundness and completeness of this system are very much similar to the previous proofs and are omitted.

5 Improvements

In this section we consider the connection method for autoepistemic logic only. Some of the improvements we discuss here, apply to nonmonotonic logic as well.

In [9], Wallen has developed a number of connection methods for classical modal logics. It would be beneficial if we could use some of these methods for autoepistemic logic. Alas, the semantics of autoepistemic logic is quite different that any one of the modal logic semantics. The M operator here, does not behave as the $\Diamond$ in modal logics. As an example, in section 3, we have showed that $\{Mc \rightarrow \neg d, Md \rightarrow \neg c\} \mid\sim Mc \vee Md$. While this is also true for the autoepistemic logic, it is false for any modal logic. Consider a modal interpretation I=(W,R,V) where W={ w} , R={ (w,w)} and V(w,c)=false, V(w,d)=false. I is clearly a modal model of $\{Mc \rightarrow \neg d, Md \rightarrow \neg c\}$ but not a model of $Mc \vee Md$. Consequently, Wallen's method can not be applied directly. Whether or not any technique involving prefixes is applicable here, has not been explored yet.

Nevertheless, there are some obvious improvements for the last connection method. First, all of the matrices in the family share the matrix $\overline{[A]}$ which negatively represents A. Therefore, all the matrices in a family $FF(A, q_0)$ can be represented by the single matrix $\{\overline{A}, \{[q_0], [\overline{q_1}], \ldots, [\overline{q_n}]\}\}$ where $[\overline{q_1}]$ is the matrix of the goal to be proved and $[\overline{q_i}]$ is the matrix that represents $\neg q_i$ for any $[A \rightarrow \neg q_i] \in FF(A, q_0)$.

The procedure consists of many passes through the last clause of the matrix. In the first pass, it starts from a q-matrix and tries to find a noncomplementary path through the matrix $\{\overline{A}, \overline{q_i}\}$ that does not contain any modal literals. If such one exists, then

a. if q_i is q_0 it stops and reports "not valid".

b. It marks everywhere in the matrix the literal Mq_i (by a dot as in matrix method or by setting $\beta(Mq_i) = 0$.)

c. It deletes any clause that contains the literal $\neg Mq$.

d. It deletes the whole matrix $[\overline{q_i}]$.

If every path starting from $[\overline{q_i}]$ is p-complementary, then

a. if $[\overline{q_i}]$ is $[q_0]$ it stops and reports "valid".

b. it marks every literal $\neg Mq_i$ in the whole matrix.

c. it deletes any clause that contains Mq_i.

d. it deletes $[\overline{q_i}]$.

After the first pas, a labelling is assigned to the remaining q-matrices, and a similar pass is performed, but this time it checkes for complementary (not p-complementary) paths, only steps b and c are applied and the marking and deletions are not permanent (they are undone before the next step). The consistency test is applied as well. At the end of the step, if the matrix $\{\overline{A}, q_0\}$ is not complementary, it stops; otherwise the same step is repeated with another labelling.

We are sure that further improvements exist, but the shortage of time forces us not to consider anything else.

6 Conclusion

We have shown how the connection method can be used as a proof procedure for non-monotonic logic. An extension of this procedure provides a proof method for autoepistemic logic also. We have discussed some improvements and how structure sharing techiques can be applied to this connection method.

It is interesting to see if similar or even better techniques can be applied to Reiter's normal default logic. Reiter, in [8] gives a resolution type procedure for normal default theories, but the consistency test is left out. We believe that a connection method proof that performs this test during the extension step is possible.

A more challenging direction is the development of connection methods for conditional logics as well as for Delgrande's default logic. In these cases, we believe that Fitting's prefixes can successfully be used and more elegand methods than the one we have presented here are possible.

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7 References

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