On Lower Bounds for Short Noncontractible Cycles In Embedded Graphs

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Abstract

Let $C_{g,n}$ be a constant such that for each triangulation of a surface of genus g with a graph of n vertices there exists a noncontractible cycle of length at most $C_{g,n}$. Let $C(g,n) = C_{g,n}$. Hutchinson in [H87] conjectures that $C(g,n) = O(\sqrt{n/g})$ for $g > 0$. In this paper, we present a construction of a triangulation which disproves this conjecture.

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1.Introduction

One of the problems in topological graph theory is to find short noncontractible cycles in an embedded graph. Consider the family $T_{g,n}$ of all triangulations of a surface of genus g by a graph with n vertices. Define by $C_{g,n}$ the maximum over all elements $T \in T_{g,n}$ of length of a shortest noncontractible cycle in T. Let $C(g,n) = C_{g,n}$. Hutchinson in [H87] has proved that $C(g,n) = O(\sqrt{(n/g)} \log g)$. This means that in any triangulation of a surface of genus g with a graph of n vertices there exists a noncontractible cycle of length at most $c\sqrt{(n/g)}\log g)$ where c is some constant. In the same paper the author conjectures that $C(g,n) = O(\sqrt{(n/g)})$ and that for $n < g$ $C(g,n) = O(1)$. In this paper we construct a counterexample to this conjecture. This result implies that in general we cannot construct a $O(\sqrt{(n/g)})$ planarizing set (a set of vertices whose removal leaves the graph planar) by removing short noncontractible cycles.

Following the ideas from the construction one can show that $C(g,n) = \Omega(\sqrt{(n/g)} \log^* g)$. It means that $C(n,g)$ grows at least as fast as $\sqrt{(n/g)}\log^* g$. We conjecture however that the result obtained by Hutchinson is tight (up to a constant). In other words we conjecture that $C(g,n) = \Theta(\sqrt{(n/g)}\log g)$.

The idea of the construction is derived from algebraic topology. In the next section we present basic facts and definitions used in the construction. For proofs and more formal presentation see for example [K80]. In the third section we prove topological theorems which suggest the way of constructing counterexample. Next we present the construction. In the appendix we prove few algebraic theorems used in the paper.

2. Basic facts and definitions

Throughout the paper by a surface we will understand a closed, connected, orientable surface without boundary. Informally this describers a sphere with g handles where g is the genus of the surface. By F_2 we denote double torus (sphere with two handles)

A graph is said to be *embedded* in an orientable surface of genus g ($g \geq 0$) if it can be drown on it in such a way that no two edges cross. If the graph G is embedded in a surface S the complement of G relative to S is a collection of open sets called *faces*. If all of the faces are open discs we say that the embedding is a *2-cell embedding*. Throughout the rest of the paper we will consider only a 2-cell embedding.

An embedding is called a *triangulation* if every face is bounded by three edges.

The main notion used in the paper is covering.

Definition 2.1. Let X' and X be two surfaces. A continuous mapping p: X'–> X is said to be a *covering map* if each point $x \in$ X has an open neighborhood U_x such that $p^{-1}(U_x)$ is the disjoint sum of open subsets of X' each of which is mapped homeomorphically onto U_x by p. The surface X' is called *the covering surface,* the surface X is called *the basic surface.*

Definition 2.2. The covering is called a *k-fold covering* if for any $x \in$ X it holds that $|p^{-1}(x)| = k$.

It is sometimes convenient to imagine that in a k-fold covering for each U_x from definition 2.1 there exist k copies of U_x in the covering space X each of them mapped onto U_x by p.

Having chosen a point $x \in X$ we can consider the set of all closed paths from x to x. Point x is called a *base point* for those paths.

Definition 2.3. Two closed paths with a base point x are *equivalent relative to base point* x if they are homotopic relative to the base point x (informally : if one can be continuously transformed to another in such a way that endpoints are not moved).

Theorem 2.4. The set of equivalence classes of closed paths based at $x \in X$ forms a group (denoted by $\pi(X,x)$ and called *the fundamental group* of X with base point x).

In the paper we will use a special kind of covering called *regular covering*. This covering has a number of nice properties which will be useful in the construction. However to define regular covering we need a few more facts:

Theorem 2.5. If p: X'–> X is a covering with $x_0' \in X'$, $x_0 \in X$ such that $p(x_0') = x_0$ then the induced homomorphism $p_* : \pi(X',x_0') \to \pi(X,x_0)$ is a monomorphism.

Note that $p_*(\pi(X',x_0'))$ is a subgroup of $\pi(X,x_0)$. We define regular covering as follows:

Definition 2.6. A covering p' : X'–>X is said to be *regular* if the group $p_*(\pi(X',x_0'))$ is a normal subgroup of $\pi(X,x_0)$ (that is for each $g \in p_*(\pi(X',x_0')$ and $h \in \pi(X,x_0)$ $hgh^{-1} \in p_*(\pi(X',x_0'))$.

We present two examples of regular coverings which will be used later.

Example 1. Let X $= F_2$ and X' be a surface of genus $k+1$ ($k\geq 2$). We construct k-fold regular covering p as on figure 2.1. Covering p transforms each of A_i' onto A and each of B_i' onto B.

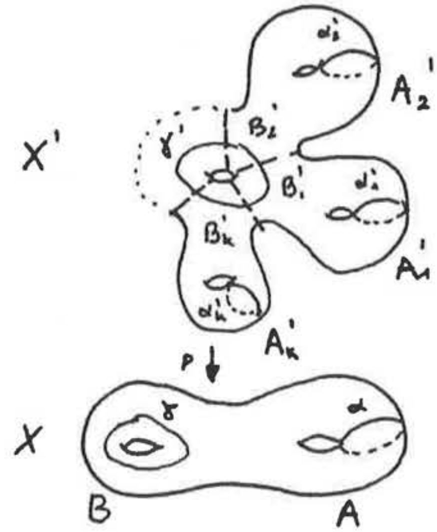

Figure 2.1. A k-fold covering of F_2

It is easy to see that one can generalize the above construction on surface of genus, g, greater then two (see fig 2.2). Then the genus of the covering surface is $k(g-1)+1$.

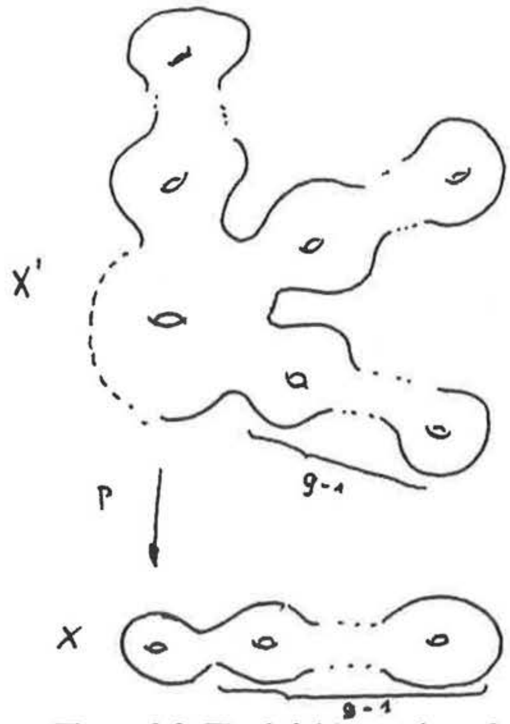

Figure 2.2. The k-fold covering of a surface of genus g

Let γ be a closed path on the surface X which begins at x and ends at y. Let $x' \in p^{-1}(x)$ then γ defines uniquely the path from x' to some y' where $y' \in p^{-1}(y)$. Assume that γ is a closed path with a base point x. Let γ^k denote the closed path composed of the sequence of k paths γ. Consider the smallest number r such that the path in covering space which starts at x' and corresponds to γ^r is closed. The number r is called *developing number of γ with base points x, x'*, in the given covering. The closed path which corresponds to γ^r in the covering space is called a *generalized lift* of γ. We say that γ^r is *r-developing lift* of γ. Note that for a regular covering developing number does not depend on the choice of base points $x \in$ X and $x' \in p^{-1}(x)$.

In the first example γ' is a generalized lift of γ and its developing number is k. The path α has k generalized lifts $\alpha_1',...,\alpha_k'$ each of them having developing number equal to one.

For a regular covering holds:

Property 2.7. In a regular covering generalized lifts of equivalent closed paths have equal developing numbers.

We can divide all simple noncontractible closed paths into two classes : *separating* and *nonseparating* closed paths. A closed path γ is called a separating path if X - γ is disconnected. It is easy to see that each nonseparating closed path can be presented (up to a homeomorphism of the surface) as on figure 2.3a. Similarly a separating noncontractible closed path can be presented as on figure 2.3b. Such a path divides the surface of genus g into two surfaces of genus g_1 and g_2 such that $g_1, g_2 > 0$ and $g_1 + g_2 = g$.

a) a simple nonseparating closed path

b) a simple separating closed path

Figure 2.3

The following example shows that for any separating closed path there exists a 2-fold regular covering such that the preimage of the given path consists of two nonseparating closed paths.

Example 2. Consider the covering p :X'–>F_2 where X' is a surface of genus 3 as shown on figure 2.4. (In particular $p(A_i) = A$, $p(B_i) = B$). In the example the preimage of the simple closed noncontractible seperating path γ is composed of two nonseperating simple paths γ_1', γ_2'.

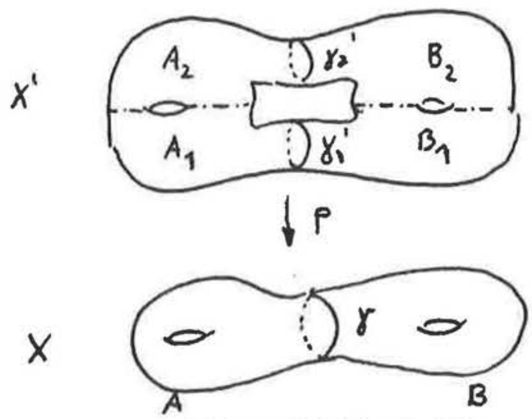

Figure 2.4. The 2-fold covering of F_2

We can generalize the above construction to the case of any simple, seperating, noncontractible closed path, γ, on a surface. In this way we obtain a regular 2-fold covering such that the preimage of γ is composed of two nonseperating simple paths γ_1', γ_2'(see figure2.5).

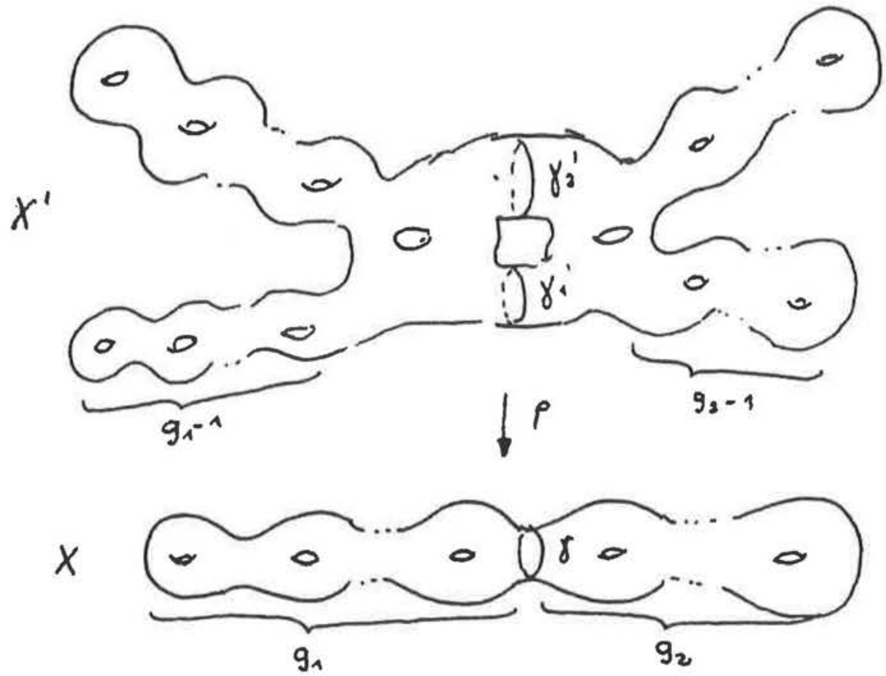

Figure 2.5. The 2-fold covering o a surface of genus g

We observe, in the presented examples, that in a *k*-fold covering there is a certain relationship between the genera of the covering and the base surfaces. This relationship is true in general and can be formulated as follows:

Theorem 2.8. In a *k*-fold covering of a base surface of genus $g > 0$ the covering surface has genus $k(g-1)+1$.

In the definition 2.3 we have introduced an equivalence relation on closed paths with a base point. This was the basic definition needed to define fundamental group and then regular covering. In the rest of the paper we will use another equivalence relation on closed paths. Now we do not fix a base point.

Definition 2.9. Two closed paths are equivalent if they are homotopic (one can be transformed continuously to another).

In the figure 2.6, paths $\gamma_1,\gamma_2,\gamma_3$ are equivalent (we say also that they belong to the same homotopy class) and paths γ_1,γ_4 are not.

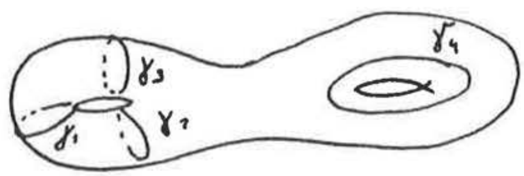

Figure 2.6

3.The topological motivation for the construction

Our goal is to show how to construct a triangulation with "long" noncontractible cycles. As we mentioned before the idea of the construction is derived from algebraic topology. In this section we consider properties of families of closed paths on basic and covering surfaces. For a given family of noncontractible paths on F_2 we construct a regular covering such that all generalized lifts of those paths have big complexity in the sense defined bellow. We can treat the complexity of a path as a measure of its length.

Consider the presentation of F_2 as a regular 8-gon (figure 3.1) , where edges α_i and $\alpha_i{}^{-1}$ are pairwise identified.

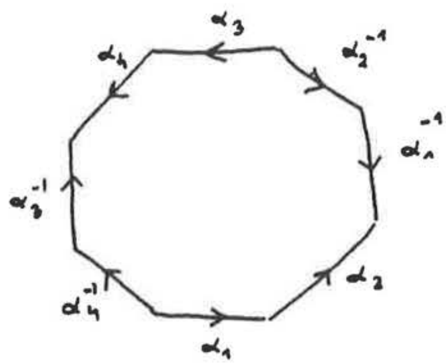

Figure 3.1. The presentation of F_2 as a regular 8-gon

Definition 3.1. Let γ be a closed path on F_2 and $[\gamma]$ its homotopy class. Denote by $l(\gamma,\alpha)$ the number of crossings between closed paths α and γ. The *complexity* of γ is defined as follows:

$$\text{complexity}\ (\gamma) = \min_{\gamma' \in [\gamma]} \sum_{i=1}^{4} l\,(\gamma',\alpha_i\,).$$

Informally if we measure the length of a path by the maximal number of crossings of the given path with edges α_i then complexity of a closed path γ is the length of the shortest path in the homotopy class $[\gamma]$.

Example 3. Consider the surface F_2 and the closed path corresponding to α_1. This is a noncontractible path so its complexity is at least one. Note that α_1 is equivalent to α_1' (see figure 3.2) and therefore its complexity is exactly one.

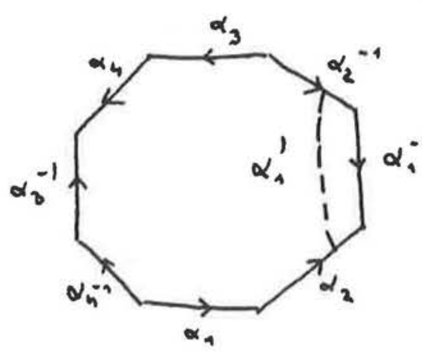

Figure 3.2

Lemma 3.2. There are at most 9^s nonhomotopic paths on F_2 of complexity less or equal to s.

Proof: We can associate with each closed path a word on the alphabet $\{\alpha_1,...,\alpha_4,\alpha_1^{-1},...,\alpha_4^{-1}\}$ in the following way:

Start from any point on the path and travel along the path in any direction. The word associated with the path corresponds to the order in which given path cuts the edges $\alpha_1,...,\alpha_4$ (without loss of generality it is enough to consider only paths which do not contain the common point of edges α_i and assume that each crossing point of the path with α_i looks as on figure 3.3). If an edge α_i is cut as on figure 3.3a then the corresponding symbol in the word is α_i otherwise (figure 3.3b) this symbol is α_i^{-1}.

Figure 3.3

Let $\beta = \alpha_i^{\pm 1}$ .We consider as equal the following pairs of words: $w_1\beta\beta^{-1}w_2$ and w_1w_2 , βw and $w\beta$. It is easy to see that if two paths have associated equal words then they are equivalent.

There is at most 9^s different words of length less or equal to s. So there are at most 9^s nonequivalent closed paths which do not cut edges α_i more then s times. Therefore there are at most 9^s nonequivalent paths of complexity less or equal to s.

Define the complexity of a closed path on a regular covering of F_2 as follows:

Definition 3.3. Let X be a regular covering of F_2. The complexity of a closed path on X is equal to the complexity of its projection on the base surface F_2.

Note that complexity of a closed path has the following properties:

P1: Complexity of a r-developing generalized lift of γ is equal to r times complexity(γ). In particular complexity of a generalized lift of γ is always greater or equal to the complexity of γ.

P2: Complexity of two paths whose projections on F_2 are homotopic are equal. In particular complexity of two homotopic paths are equal.

The main result of this section is stated by the following theorem:

Theorem 3.4. For any constant h there exists a covering surface X of F_2 such that the genus, g, satisfies $c\log^* g \leq h$ for some universal constant c (not depending on the constant h) and all noncontractible closed paths on X have complexity bigger or equal to h.

To prove this theorem we start with a lemma concerning simple closed paths.

Lemma 3.5. Let $\gamma_1,\ldots,\gamma_k$ be a family of simple, closed, noncontractible and nonhomotopic paths on a surface X. Then there exists an m-fold regular covering of X such that each generalized lift of γ_i (i=1,...,k) has developing number at least two and

$$m \leq 16^k$$

Proof: For each γ_i construct a covering space X_i such that at least one generalized lift of γ_i has developing number equal to two. To do so for each nonseparating γ_i use the construction from example 1 and for each separating γ_i use first the construction from example 2 to obtain two nonseparating generalized lifts of γ_i and then apply the construction from example 1 to one of those lifts.

For each i X_i is at most 4-fold covering. By theorem A.4 from the appendix we can construct an m-fold regular covering X' of X such that X' covers also each of X_i and $m \leq (4^4)^k = 16^k$.

Consider a covering $p : X \to F_2$. Let Γ be the family of all closed noncontractible paths on X and let

$$r = \min_{\gamma \in \Gamma} \text{complexity}(\gamma)$$

Then the following holds:

Lemma 3.6. Let γ be a closed path from the family Γ with self-crossings such that it is not homotopic to any path without self-crossings. Then complexity(γ) $\geq$ 2r.

Proof: It follows from the obvious fact that if a closed path γ is composed of two closed noncontractible paths then it cuts edges α_i in at least 2r points (we can again consider without loss of generality only paths which do not contain the common point of edges α_i). Furthermore each path homotopic to γ is composed of two such closed paths.

Proof of theorem 3.4: Construct the covering space in the following inductive way:
Let $X_0 = F_2$. From lemma 3.2 there exist at most 9^h nonhomotopic, noncontractible paths of complexity less or equal to *h*. Consider those homotopy classes of closed paths on X_0 which posess a simple path as a representative. Denote those representatives by $\gamma_{1_0},\ldots,\gamma_{N_0}$. Note that $N_0 \leq 9^h$. By lemma 3.5 we can construct a regular covering X_1 of X_0 in which generalized lifts of all those paths are at least two. The complexity of the lifts of the rest of the paths cannot decrease (comparing complexity of those paths). Note that by lemma 3.6 closed paths which do not have paths without selfcrossing in their homotopy classes have

complexity at least two. Also their generalized lifts have complexity at least two. Therefore complexity of generalized lifts of all noncontractible closed paths is at least two so all closed, noncontractible path in X_1 have complexity at least two. Note that X_1 is a t_1-fold covering where :

$$t_1 \leq 16^{(9^h)}$$

The inductive step is as follows:

Let X_i be a regular t_i-fold covering of F_2, and assume that all closed, noncontractible paths on X_i have complexity at least 2^i. First construct a regular covering X'_{i+1} of X_i. To do this consider those homotopy classes of noncontractible closed paths on F_2 which posess a representative with a simple generalised lift to X_i of complexity less or equal to *h*. *Let* $\gamma_{1_i},\dots,\gamma_{N_i}$ be the simple generalised lifts of those representatives. Observe again that $N_i \leq 9^h$. By lemma 3.5 we can construct a regular at most $16^{(9^h)}$-fold regular covering such that complexity of all generalised lifts of $\gamma_{1_i},\dots,\gamma_{N_i}$ is doubled. Note that by A.1(see appendix) X'_{i+1} is at most $t_i \cdot 16^{(9^h)}$-fold covering of F_2. Let X_{i+1} be a covering of X'_{i+1} which is also a regular covering of F_2. By theorem A.4 from appendix it is at most t_{i+1} -fold covering where

$$t_{i+1} \leq \left(t_i \cdot 16^{(9^h)}\right)^{\left(t_i \cdot 16^{(9^h)}\right)}$$

By a similar argument as in the first step all closed, noncontractible paths in X_{i+1} have complexity at least 2^{i+1}.

Repeat this construction until $2^i \geq h$. We need at most $\lceil \log_2 h \rceil$ steps to do so. Denote by $\widetilde{X}$ the covering space of F_2 we have obtained. By the construction complexity of all paths on $\widetilde{X}$ is bigger or equal to *h*. $\widetilde{X}$ is at most $\tilde{t}$-fold covering where $\tilde{t} \leq t_{\lceil \log h \rceil}$. By lemma 2.8 the genus of $\widetilde{X}$ is less or equal to $\tilde{t} + 1$.

Let $b_1 = 16^{(9^h)}$ and inductively $b_{i+1} = (b_1 b_i)^{(b_1 b_i)}$. By the construction $t_i \leq b_i$.
Let $\tilde{b} = b_j$ where $2^{j-1} < h \leq 2^j$. We will show that $h \leq \log^* b$ where $\log^* n = k$ iff

$$\underbrace{\log \log \ldots \log}_{k \text{ times}} n \le 1 < \underbrace{\log \log \ldots \log}_{k\text{-}1 \text{ times}} n.$$

Throughout the paper by log we understand $\log_2$.

Note that

(i) $\log {b_{i+1}}^3 \le {b_i}^3$ for all i

$$\text{because } \log {b_{i+1}}^3 = 3 \log b_{i+1} = 3 \log(b_1 b_i)^{(b_1 b_i)} = 3 b_1 b_i \log(b_1 b_i) \le$$

$$6 {b_i}^2 \log b_i \le \quad {b_i}^3$$

Note also that

(ii) $\log \log {b_1}^3 = \log \log \left(16^{(9^h)}\right)^3 = \log(3 \cdot 9^h \log 16) = \log 3 + h \log 9 + \log\log 16 \le 4h + 4$

so

(iii) $\log(4h + 4) \le j + 3$

and obviously

(iv) $\underbrace{\log \ldots \log}_{j+2} (j+3) < 1$

From (i) - (iv) we obtain

$$\underbrace{\log \ldots \log}_{2j+4} b_j < 1$$

therefore

$$\underbrace{\log \ldots \log}_{2j+4} \tilde{t} < 1$$

So $\log^* \tilde{t} \le 2j + 4$ and for $h \ge 13$ holds $2j + 4 < h$ so $\log^* \check{t} < h$ and therefore

$$\log^* g \le h \quad \text{for} \quad h \ge 13$$

and the theorem follows.

4. Construction of the triangulation

Let T be a triangulation of F_2 containing edges of the regular 8-gon from figure 3.1. Call such a triangulation a *base triangulation.* Let n(T) denote the number of vertices in the triangulation T. Let X be a covering space of F_2 and p the covering mapping. Triangulation T defines a triangulation T_X of the covering space in such a way that vertices of T_X are preimages of vertices of T and edges of T_X are defined by preimages of edges of T. Triangulation T_X is called *a covering triangulation* of the base triangulation T.

Note that the length of a closed path in T_X is bigger or equal to 1/4 of its complexity. Therefore we can use the ideas from the previous section to construct for any base triangulation T a covering triangulation T_X in which all noncontractible paths are longer then any given constant. To disprove the conjecture we have to consider additionally factor $\sqrt{}((n(T_X)/g)$. However, as we will show in the next lemma, the ratios of the number of vertices to the genus in the covering and the base triangulations do not differ to much.

Lemma 4.1. Let T be a base triangulation and T_X its covering triangulation. Let d = n(T)/2. Then $d \le n(T_X)/g < 2d$.

Proof: Assume that X is a k-fold covering of F_2. Then $n(T_X) = k{\cdot}n(T)$ and by theorem 2.8 genus g of X is equal to k+1.

So $n(T_X)/g = k{\cdot}n(T)/\,(k+1)$

but $k{\cdot}n(T) / (k+1) < k{\cdot}n(T) / k = 2d$

Using this lemma we can prove the following:

Theorem 4.3. For any $h > 0$ there exists a surface X_g of genus g such that for each $n \geq g-1$ there exists a triangulation T_g of the surface X_g with n vertices such that $c\log^* g \leq h$ and T_g do not have noncontractible cycles shorter then $h/4$.

Proof: Let T be the triangulation with only one vertex v_1 as on figure 4.1.

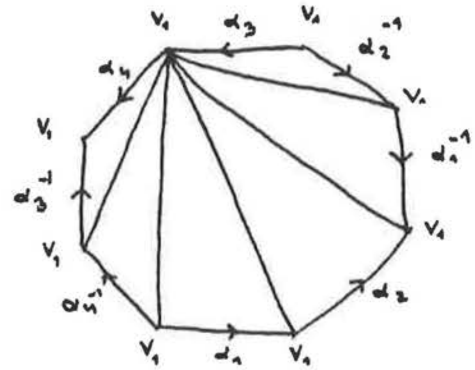

Figure 4.1

Theorem 3.4 implies that there exists a regular $\tilde{t}$-fold covering of F_2 with a covering space of genus $g = \tilde{t} + 1$ such that each noncontractible closed path has complexity at least h and $c\log^* g \leq h$ for some universal constant $c > 0$.

Consider the covering triangulation T_X. Denote by $|\gamma|$ the lenght of the path γ. By definition 3.1 complexity of a closed path γ in T_X is $\leq 4|\gamma|$. So $|\gamma| \geq$ complexity(γ)/4 and therefore $|\gamma| \geq h/4$.The triangulation T_X has $\tilde{t} = g - 1$ vertices. We can add to any triangulation a vertex, say v, without decreasing length of noncontractible cycles. It can be done by the construction given on figure 4.2.

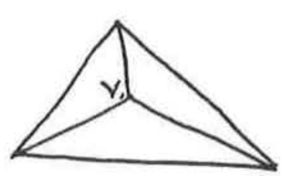

Figure 4.2

This finishes the proof of theorem 4.3.

By the above theorem we have the following corollary:

Corollary 4.4. There exist infinitely many pairs (g,n) such that there exists triangulation T of surface of genus g such that T has n vertices and

$$\sqrt{(n/g)}\, \log^* n \le (8/c)l$$

and all noncontractible cycles in the triangulation T are longer then l.

Proof: It is enough to use theorem 3.4 for pairs (n,g) where $4g \ge n \ge g\text{-}1$.

5. Conclusions

Theorem 4.3 shows that $C(n,g)$ is not $O(\sqrt{(n/g)})$. Corollary 4.4 is the first step to prove that $C(n,g)$ is $\Omega(\sqrt{(n/g)}\log^* n)$. To finish the proof one should extend this corollary to all but finitely many pairs (n,g). There are two main ideas behind such extension:

(i) To extend it over all n one can use the idea of construction of collars of triangulation along each of α_i. They should be constructed in such a way that if the collars are of depth d then if a path γ has complexity l then $|\gamma| \ge dl/4$.

(ii) To extend it over all g one can repeat whole construction for surfaces with boundary and construct collars along boundary as well. Then one can glue two surfaces along boundaries in such a way that the length of the shortest noncontractible cycle will not decrease.

Our construction to disprove the conjecture is elementary. Using more advanced methods ([H67]) one can improve theorem 3.1. In order to keep the paper self-contained we do not present the improvement. On the other hand we hope to prove our conjecture that $C(g,n) = \Theta(\sqrt{(n/g)}\log g)$.

Appendix

In the appendix we prove a theorem concerning existence of a covering space having properties needed in the proof of theorem 4.3. For this proof we need first some lemmas from group theory.

Lemma A.1. Let P,H be subgroups of G of finite index. If the index of P in G is k and the index of H in G is l then the index of $H \cap P$ in G is less or equal to kl.

Proof: Consider right cosets of $P \cap H$ in H. Let $h_1,\dots,h_t$ be represetatives of those cosets. We will show that $h_i P \neq h_j P$ for $i \neq j$. Assume that not. Then there exist h, h' in P such that $h_i h = h_j h'$. So $h_j^{-1} h_i = h' h^{-1}$. Therefore $h' h^{-1} \in P \cap H$ and $h_i = h_j h' h^{-1}$ so h_i and h_j are in the same coset of $H \cap P$ in H which is a contradiction. So the index of $P \cap H$ in H is less or equal to the index of P in G (which is equal to k) and so the index of $P \cap H$ in G is less or equal to lk.

Lemma A.2. Let H be subgroup of G of a finite index k and let $g_1,\dots,g_k$ be representatives of the right cosets. Then $H^* = g_1 H g_1^{-1} \cap \dots \cap g_k H g_k^{-1}$ is a normal subgroup of G.

Proof: Let $a \in G$. Consider the sequence $ag_1 H g_1^{-1} a^{-1},\dots,ag_k H g_k^{-1} a^{-1}$. It is enough to show that the above sequence is a permutation of the sequence $g_1 H g_1^{-1},\dots,g_k H g_k^{-1}$. Consider function f_a: $\{1,\dots,k\} \to \{1,\dots,k\}$ such that $f_a(i) = j$ iff $ag_i \in g_j H$. Note that f_a is a permutation (i.e. it is a bijection) and $f_{a^{-1}}$ gives the inverse function. Furhermore $ag_i H g_i^{-1} a^{-1} = g_{f_a(i)} H g_{f_a(i)}^{-1}$.

Note that H^* is a subgroup of H.

Lemma A.3. Let $H_1,\ldots,H_N$ be subgoups of G with indices $k_1,\ldots,k_N$. There exists a normal subgroup H^* of G such that H^* is also o subgroup of H_i for each i and has index less or equal to $k_1^{k_1}k_2^{k_2}\ldots k_N^{k_N}$

Proof: By lemma A.2 for each i there exists a subgroup H_i^* such that H_i^* is a normal subgroup of G. By lemma A.1 its index is $k_i^{k_i}$. But product of normal subgroups is a normal subgroup so $H^* = H_1^* \cap H_2^* \ldots \cap H_N^*$ is a normal subgroup of G. Note that for each i H^* is a normal subgroup of H_i. By lemma A.2 the index of H^* in G is less or equal to $k_1^{k_1}k_2^{k_2}\ldots k_N^{k_N}$

Let $p_1 : X_1 \to X, \ldots, p_N : X_N \to X$ be k_i-fold coverings. Let $x_1 \in X_1, \ldots, x_N \in X_N$ and $x \in X$ be base points such that $p_i(x_i) = x$. Note that $p_{1*}(\pi(X_1,x_1)),\ldots,p_{N*}(\pi(X_N,x_N))$ are subgroups of $\pi(X,x)$ and index of $p_{i*}(\pi(X_i,x_i))$ in $\pi(X,x)$ is k_i.

Theorem A.4. There exists a k-fold regular covering X' of the base space X which covers also each of X_i ($i = 1,\ldots,N$) such that $k \le k_1^{k_1}k_2^{k_2}\ldots k_N^{k_N}$

Proof: By lemma A.3 we can construct a normal subgroup G of $\pi(X,x)$ of index k where $\le k_1^{k_1}k_2^{k_2}\ldots k_N^{k_N}$ But for any normal subgroup of $\pi(X,x)$ we can construct uniquely corresponding regular covering (see [K80]). If G is a subgroup of p_{i*} $(\pi(X_i,x_i))$ for each i then the constructed regular covering covers also each of X_i.This finishes the proof of theorem A.4.

References

[H67] - M.Hall, "Coset representation in free groups", Trans. Amer. Math. Soc. 49 (1949) pp 422-432.

[H87] - J.Hutchinson, On short noncontractible cycles in embedded graphs,to appear.

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Appendix

In the appendix we prove a theorem concerning existence of a covering space having properties needed in the proof of theorem 4.3. For this proof we need first some lemmas from group theory (see [PP87]).

Lemma A.1. Let P,H be subgroups of G of finite index. If the index of P in G is k and the index of H in G is l then the index of $H \cap P$ in G is less or equal to kl.

Lemma A.2. Let H be subgroup of G of a finite index k and let $g_1,...,g_k$ be representatives of the right cosets. Then $H^* = g_1 H g_1^{-1} \cap ... \cap g_k H g_k^{-1}$ is a normal subgroup of G.

Lemma A.3. Let $H_1,...,H_N$ be subgoups of G with indices $k_1,...,k_N$. There exists a normal subgroup H^* of G such that H^* is also o subgroup of H_i for each i and has index less or equal to $k_1^k k_2^k ... k_N^k$.

Let $p_1 : X_1 \to X, ..., p_N : X_N \to X$ be k_i-fold coverings. Let $x_1 \in X_1, ..., x_N \in X_N$ and $x \in X$ be base points such that $p_i(x_i) = x$. Note that $p_{1*}(\pi(X_1,x_1)),...,p_{N*}(\pi(X_N,x_N))$ are subgroups of $\pi(X,x)$ and index of $p_{i*}(\pi(X_i,x_i))$ in $\pi(X,x)$ is k_i.

Theorem A.4. There exists a k-fold regular covering X' of the base space X which covers also each of X_i (i = 1,...,N) such that $k \leq k_1^{k_1} k_2^{k_2} ... k_N^{k_N}$.

Proof: By lemma A.3 we can construct a normal subgroup G of $\pi(X,x)$ of index k where $\leq k_1^{k_1} k_2^{k_2} ... k_N^{k_N}$. But for any normal subgroup of $\pi(X,x)$ we can construct uniquely corresponding regular covering (see [K80]). If G is a subgroup of p_{i*} $(\pi(X_i,x_i))$ for each i then the constructed regular covering covers also each of X_i.This finishes the proof of theorem A.4.

References

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