On The Modality of Convex Polygons

by

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Abstract

Under two reasonable definitions of random convex polygons, the expected modality of a random convex polygon grows without bound as the number of vertices grows. This refutes a conjecture of Aggarwall and Melville.

1 Introduction

The *modality* of a vertex of a polygon is the number of local maxima in the sequence of distances from that vertex to the other vertices, in their natural order around the polygon. The modality of a polygon is the maximum of the modalities of its vertices. A polygon is *unimodal* if its modality is 1; otherwise, it is *multimodal.*

Interest in the modalilty of convex polygons was stimulated by Avis et al. [2], who showed that a linear time algorithm proposed by Dobkin and Snyder [4] for computing the diameter of a convex polygon fails for some multimodal convex polygons. Bhattacharya and Toussaint [3] showed that a diameter algorithm of Snyder and Tang [8] also fails for some multimodal convex polygons. Later Toussaint [9] showed that the Dobkin and Snyder algorithm is correct for convex unimodal polygons, although the Snyder and Tang algorithm fails even for some unimodal convex polygons. Toussaint [9] also gives algorithms for other problems which assume unimodality. Aggarwall and Melville [1] give algorithms for determining the modality of convex and nonconvex polygons, and conjecture that under any "reasonable" definition of a random convex polygon, most convex polygons are unimodal.

This paper refutes that conjecture, for two apparently reasonable definitions of random convex polygons. A natural definition of a random convex polygon is the convex hull of a random collection of points. When n points are drawn uniformly from a disk, the expected modality of their convex hull is $\Theta\left(\frac{\log n}{\log \log n}\right)$. When n points are drawn from a two dimensional normal distribution, their convex hull has an expected modality of $\Theta\left(\frac{\log \log n}{\log \log \log n}\right)$.

What is the significance of the preceding results to the study of geometric algorithms? Since a convex polygon can be tested for unimodality in linear time [1], linear time algorithms which are only correct on unimodal convex polygons can be used to design algorithms which are always correct, and run in linear time on unimodal inputs. In the setting of random graphs, Palmer [5] suggests using the probability that an algorithm works on a large random input as a measure of its power. Applying the same principle in this setting, our results suggest that algorithms which work only for unimodal convex polygons are weak.

The proofs of our results apply only to very large n. What of moderate size n? Table 1 was computed using a linear congruential pseudo-random number generator. It indicates that convex hulls of modality 2 (which tend to be somewhat oblong) are prevalent for moderate size point sets. No convex hulls of modality exceeding 3 were generated.

	Modality		
n	1	2	3
4	88	12	0
8	70	29	1
16	54	45	1
32	51	48	1
64	34	64	2
128	34	65	1
256	20	79	1
512	9	89	2
1024	5	94	1

(a) Uniform

	Modality		
n	1	2	3
4	84	16	0
8	64	36	0
16	48	51	1
32	39	60	1
64	28	71	1
128	30	69	1
256	32	66	2
512	29	70	1
1024	29	69	2

(b) Normal

Table 1: Modalities of convex hulls of n random points

2 Lower bounds on expected modality

This section introduces the basic techniques, and establishes lower bounds. The proofs are fairly crude, but relatively simple. All logarithms are base e.

Theorem 1: When n points are drawn at random uniformly from a disk, the expected modality of their convex hull is $\Omega\left(\frac{\log n}{\log\log n}\right)$ and the probability that their convex hull is unimodal approaches 0 as $n \to \infty$.

Proof: Assume that n is very large. Write $x \sim y$ when $\lim_{n\to\infty} y/x = 1$. Let R be the radius of the disk, and assume for convenience that the disk is centered at the origin of a Cartesian coordinate system. The *measure* of a region is the probability that a random point falls in that region.

Let $m \sim n^{\frac{1}{3}}$ be an even integer. (Not coincidentally, the expected number of vertices on the convex hull of the n points is $\Theta(m)$ [6,7].) Inscribe a

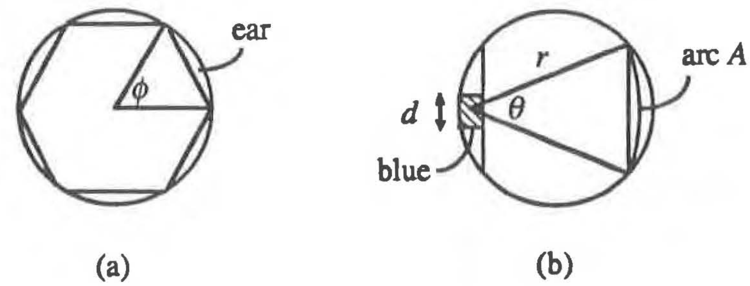


Figure 1: The ears and the blue region.

regular m-gon in the boundary of the disk, as illustrated in figure 1(a). Call each connected region of the disk strictly outside the m-gon an *ear*. If $\phi = 2\pi/m$ is the angle defined by an ear, then each ear has area $A_{\text{ear}} = \frac{\phi R^2}{2} - R^2 \cos\frac{\phi}{2}\sin\frac{\phi}{2} \sim \frac{\phi^3 R^2}{12}$ and measure $P_{\text{ear}} = \Theta(1/n)$. Intuitively, points in an ear will tend to be hull vertices, and with fairly high likelihood will be local maxima for points in the ear on the opposite side of the disk.

We will bound the probability that the modality is $\geq k-1$, for arbitrary $k > 1$. Choose two ears on opposite sides of the circle. In one of them (say, the left hand one), color a region near the center blue, as indicated in figure 1(b). The blue region has depth $d = \frac{\phi R}{24k^3}$, width $w = R - R\cos\frac{\phi}{2} \sim \frac{\phi^2 R}{8}$ and area $\sim wd = \Theta(A_{\text{ear}}/k^3)$.

Cut the other ear with an arc A of radius $r \sim 2R$, as illustrated, in such a way that arc A defines an angle of $\theta = \frac{\phi}{2}$. Cut angle θ into k equal parts, as shown in figure 2(a), each part an angle of $\alpha = \frac{\phi}{2k}$. In figure 2, angle α is greatly exagerated for clarity. Cut each angle α into 6 parts, as illustrated

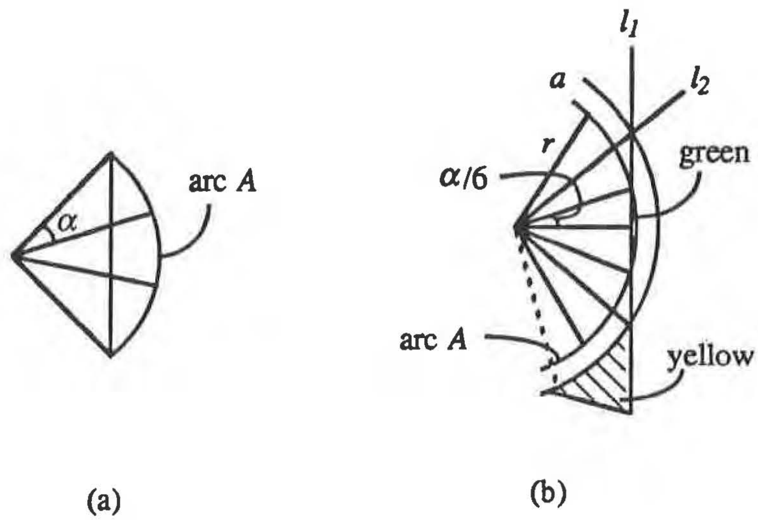


Figure 2: The green and yellow regions.

in figure 2(b). The region marked *green* is delimited by a section of arc A of angle $\alpha/3$ and its chord, and excludes its boundary. In all, there are k green regions, and each has area $\sim \frac{\alpha^3 R^2}{81} = \Theta(A_{\text{ear}}/k^3)$.

The arc drawn outside arc A in figure 2(b) is concentric with A, and passes through the intersection of lines l_1 and l_2. One easily computes the separation of the two arcs as $a \sim \frac{\alpha^2 R}{12}$. There are $k-1$ *yellow* regions, one of which is illustrated in figure 2(b). Each yellow region has area $\sim \frac{\alpha^3 R^2}{6} - \frac{\alpha^3 R^2}{81} = \Theta(A_{\text{ear}}/k^3)$.

Claim 1: Let B, G and Y be points in the blue, green and yellow regions, respectively. Then distance BG is less than distance BY.

Proof: Arc A approximates, up to high order effects, arcs centered on every point along a horizontal midline of the blue region. Let $B = (x, y)$ and

$Q = (x, 0)$. Let r be the distance from Q to arc A. Since the green regions are inside arc A, $BG \le r + \frac{d}{2}\sin\frac{\phi}{2}$. Since the yellow regions are outside the second arc, $BY \ge r + a - \frac{d}{2}\sin\frac{\phi}{2}$. So $BY - BG$ is asymptotically at least $a - \frac{d\phi}{2} > 0$. □

Say that a collection of n points in the disk is *bad* if exactly one point falls in each of the blue, green and yellow regions, and no other points fall in either of the two ears under consideration. Notice that each of the $2k$ points in the green, yellow and blue regions of a bad set of points must be vertices of the convex hull. By Claim 1, each of the yellow points must be local maxima for the blue point, and the convex hull has modality at least $k - 1$.

Since $P_{\text{ear}} = \Theta(1/n)$, the probability that a random set of points in the disk is bad is asymptotically $> n^{2k}\left(\frac{c_1}{nk^3}\right)^{2k}\left(1 - \frac{c_2}{n}\right)^{n-2k} > \left(\frac{c_3}{k^3}\right)^{2k}$ for some positive constants c_1, c_2 and c_3. Let B_i be the event that a random set of n points is bad for the i^{th} pair of ears, $i = 1, \ldots, m/2$, and let $p_k = \Pr(\overline{B}_1 \cap \cdots \cap \overline{B}_{m/2})$. Since $m \ll n$, and k will also be $\ll n$, $B_1, \ldots, B_{m/2}$ are very nearly independent. So $p_k < \left(1 - \left(\frac{c_4}{k^3}\right)^{2k}\right)^{m/2}$, for some constant $c_4 > 0$. For any fixed k, $p_k \to 0$ as $n \to \infty$. In fact, $p_k \to 0$ provided $\left(\frac{m}{2}\right)\left(\frac{c_4}{k^3}\right)^{2k} \to \infty$ as $n \to \infty$. That happens for $k = \frac{g \log m}{\log\log m}$ and $g < \frac{1}{18}$, so Theorem 1 is established. □

In the case of a two dimensional normal distribution, the expected number of points on the convex hull is only $\Theta(\log^{\frac{1}{2}} n)$ [6,7], so the expected modality must be lower than in the uniform case. Nevertheless, a very similar proof applies.

Theorem 2: When n points are drawn from a two dimensional normal distribution, the expected modality of their convex hull is $\Omega\left(\frac{\log\log n}{\log\log\log n}\right)$ and the probability that their convex hull is unimodal approaches 0 as $n \to \infty$.

Proof: Let $\phi(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$ be the normal density function, and $\phi(r) = \frac{1}{2\pi}e^{-\frac{r^2}{2}}$ be its polar version. Let $b \sim (2\log n)^{1/2}$ be the solution to $\frac{1}{b}e^{-\frac{b^2}{2}} = \frac{1}{n}$, and let $P = \int_{-\infty}^{\infty}\int_{b}^{\infty}\phi(x,y)dxdy$ be the measure of the half-plane to the right of the line $x = b$. It is important that P be quite small.

Claim 2: $P = O(1/n)$.

Proof: The value of P can be expressed as an integral in polar coordinates as $P = \int_b^{\infty}\phi(r)2r\theta dr$, where $\theta = \cos^{-1}\frac{b}{r}$. Let $t = r^2/b^2$. Then $t = 1/\cos^2\theta \approx 1+\theta^2$, and the integral becomes

$$P \sim \frac{b^2}{2\pi}\int_1^{\infty} e^{\frac{-b^2t}{2}}(t-1)^{\frac{1}{2}}dt.$$

But $b(t-1)^{\frac{1}{2}} \leq k$ for $t \leq 1+\frac{k^2}{b^2}$, so

$$P < \sum_{k\geq 1}\frac{bk}{2\pi}\int_{1+\frac{(k-1)^2}{b^2}}^{1+\frac{k^2}{b^2}} e^{-\frac{b^2t}{2}}dt$$

$$< \frac{1}{b\pi}e^{-\frac{b^2}{2}}\sum_{k\geq 1} ke^{-\frac{(k-1)^2}{2}}$$

$$= \frac{c}{b}e^{-\frac{b^2}{2}}$$

for some positive constant c. That establishes the claim. □

Now consider a circle C, centered at the origin, of radius $R = b + 1/b$. That part of the interior of C which lies to the right of the line $x = b$ will play the role of an ear, as in the preceding proof. Let P_{ear} be the measure of the ear.

Claim 3: $P_{\text{ear}} = \Omega(1/n)$.

Proof: On the circle of radius R, the probability density is $\phi(R) > \frac{1}{2\pi}e^{-\frac{b^2}{2}-2} = \Theta(b/n)$, by the chooice of b, and the density over the ear is greater than that. The ear subtends an angle of $\theta \sim \sqrt{2}/b$, so the ear has area $\frac{\theta^3 R^2}{12} = \Theta(1/b)$. So $P_{\text{ear}} = \Omega(1/n)$. □

Now choose m to satisfy $2\pi/m = \theta \sim \sqrt{2}/b$, which implies that $m = \Theta(\log^{\frac{1}{2}} n)$. Round m to the nearest even integer.

The remainder of the proof is almost identical to that of theorem 1. If the two ears under consideration are aligned so that their chords are vertical, then in addition to the previous conditions, a bad set of points must have no points to the right of the line $x = b$, or to the left of the line $x = -b$, so that the points in the green, yellow and blue regions must be on the convex hull. But that happens with constant positive probability. The previous proof assumed that the probability density was uniform. Here, the density varies only by a constant factor over an ear, which is sufficient for the proof to carry through.

As before, the result is that the expected modality is $\Omega\left(\frac{\log m}{\log\log m}\right)$. However, in this case that is $\Omega\left(\frac{\log\log n}{\log\log\log n}\right)$. □

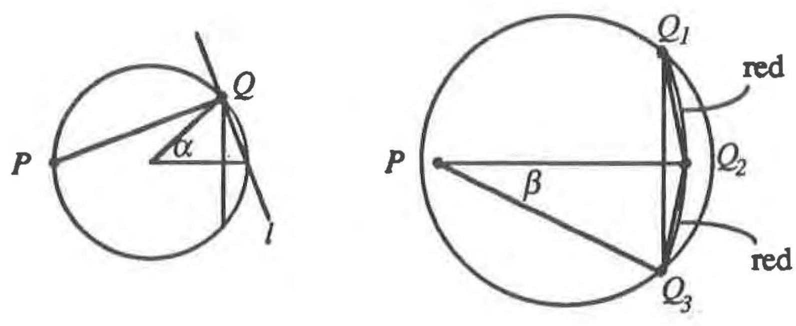
3 Upper bounds on expected modality

Theorem 3: When n points are drawn at random uniformly from a disk, the expected modality of their convex hull is $O\left(\frac{\log n}{\log\log n}\right)$.

Proof: The proof is similar in spirit to that of the lower bound; it shows that the lower bound construction is nearly optimal. Let $m = \left(\frac{\pi^2 n}{4\log n}\right)^{\frac{1}{3}}$ and $\alpha = \frac{2\pi}{m}$, and suppose that the disk has radius R. An α-ear is a region of the disk outside some chord subtending angle α. Figure 3(a) shows an α-ear, to the right of line l.

Claim 4: With probability approaching 1 as $n \to \infty$, no α-ear is empty.

Proof: If any α-ear is empty, then one of the ears associated with some fixed inscribed regular $2m$-gon must also be empty. So the probability that



(a) (b)

Figure 3: The red regions

any α-ear is empty is asymptotically less than $2m\left(1-\frac{\pi^2}{12m^3}\right)^n < 2me^{-\frac{\log n}{3}}$, which approaches 0 as $n \to \infty$. □

Claim 4 implies that, with probability approaching 1, every hull point lies within a band between circles of radius $S = R\cos\alpha$ and radius R. Let $P_{\text{band}} = \Theta\left(\frac{\log^{\frac{2}{3}} n}{n^{\frac{2}{3}}}\right)$ be the measure of that band.

Let P be an arbitrary hull point. Let Q be another hull point which is a local maximum for P, and l be the line through Q perpedicular to line PQ, as shown in figure 3(a). In the figure, P is shown to lie on the boundary, which for the present purposes is accurate. The part of the disk on the opposite side of l from P must be empty, since otherwise Q could not be a local maximum for P. So if Q does not lie in the 2α-ear directly opposite P, then there must be an empty α-ear, as shown in figure 3(a), which is presumed not the case. The measure of a 2α-ear is $P_{\text{ear}} = \Theta(\frac{\log n}{n})$.

Suppose a point P has been chosen, along with k other points $Q_1, \ldots, Q_k$. Those k points can only be local maxima for P if there is a local minimum hull point between adjacent pairs of local maxima. The probability of getting the necessary local minima is maximized when $Q_1, \ldots, Q_k$ lie equally spaced on an arc A centered at P, as shown if figure 3(b) for $k = 3$. Between each adjacent pair Q_i and Q_{i+1} is a red region, where a local minimum point for P should lie. Since $Q_1, \ldots, Q_k$ all lie in the 2α-ear opposite P, and P is near the boundary, the angle β at P subtended by the portion of arc A between adjacent local maxima is $\beta \sim \frac{\alpha}{k-1}$, and the area of each red region is $\sim \frac{\beta^3 R^2}{3}$. The measure of each red region is thus $P_{\text{min}} \sim \frac{\beta^3}{3\pi} = \Theta(m^{-3}(k-1)^{-3})$. Considering all possibilities, the probability p_k that, for some hull point P, there are points $Q_1, \ldots, Q_k$ in the opposite 2α-ear, with a local minimum between each adjacent pair, satisfies

$$\begin{aligned} p_k &< nP_{\text{band}}\binom{n-1}{k} P_{\text{ear}}^k (n-k-1)^{k-1} P_{\text{min}}^{k-1} \\ &< \frac{c^k n^{\frac{1}{3}} (\log n)^{2k}}{(k-1)^{4k-3}} \end{aligned}$$

for some constant $c > 0$. Choosing $k = \frac{c' \log n}{\log \log n}$ for $c' > 1/6$ causes p_k to approach 0 as $n \to \infty$, and to do so at a super-exponential rate in k. It follows that the expected modality is $O\left(\frac{\log n}{\log \log n}\right)$. □

Again, the proof for points drawn from a normal distribution is very similar to the case of uniformly distributed points in a disk.

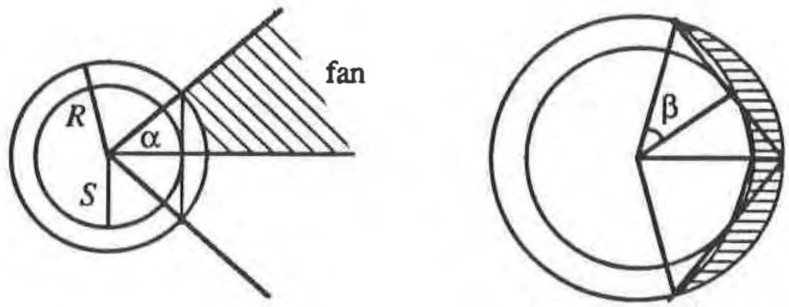
Theorem 4: When n points are drawn at random from a two dimensional normal distribution, the expected modality of their convex hull is $O\left(\frac{\log\log n}{\log\log\log n}\right)$.

Proof: This proof is similar to the preceding one, but is complicated slightly by th fact that the probability density is no longer uniform over an ear. Let $\lambda = 2\pi e \log^{\frac{1}{2}} n \log\log n$ and $S = \left(2\log\frac{n}{\lambda}\right)^{\frac{1}{2}}$. Let C be the circle of radius S, centered at the origin. An S-half-plane is any half-plane not including the origin whose boundary line is tangent to circle C.

Claim 5: With probability approaching 1 as $n \to \infty$, no S-half-plane is empty.

Proof: Let $r = \left(2\log\frac{n}{\lambda} + 2\right)^{\frac{1}{2}}$, and let C' be a circle of radius r, concentric with C. Let α be the angle subtended by a chord of circle C', tangent to circle C, as shown in figure 4(a). Then $\cos\alpha = S/r \approx \left(1 - \frac{\log\lambda}{2\log n}\right)\left(1 + \frac{\log\lambda - 1}{2\log n}\right)$, from which it follows that $\alpha \sim 1/\log^{\frac{1}{2}} n$. One of $\frac{2\pi}{\alpha}$ *fans* is shaded in figure 4(a). The measure of the region outside circle C' is $\int_r^\infty 2\pi s\phi(s)ds = e^{-\frac{r^2}{2}}$. So the measure of each fan is $\frac{\alpha\lambda}{2\pi e n}$, and the probability that *any* fan is empty is $\frac{2\pi}{\alpha}\left(1 - \frac{\alpha\lambda}{2\pi e n}\right)^n < \frac{2\pi}{\alpha}e^{-\frac{\alpha\lambda}{2\pi e}}$, which approaches 0 as $n \to \infty$. □

Claim 6: With probability approaching 1 as $n \to \infty$, all hull points lie in a band whose boundaries are circles C and C'', where C'' has radius



(a) (b)

Figure 4: The ears for Theorem 4

$R = (2\log(\lambda n))^{\frac{1}{2}}$, and is concentric with C. That band has measure $P_{\text{band}} < e^{-\frac{S^2}{2}} = \frac{\lambda}{n}$.

Proof: If any hull point lies inside circle C, then some S-half-plane must be empty, which is presumed not the case. The probability that any point at all lies outside circle C'' is $< ne^{-\frac{R^2}{2}} = 1/\lambda$. □

Let P be an arbitrary hull point, without loss of generality on the x-axis, and let Q be a local maximum hull point for P. Then Q must lie within the shaded area of figure 3(b), which we call an ear. The ear is bounded by circle C'' and the locus of all points U such that, for some point V on the x-axis, outside circle C, angle PUV is a right angle, and line UV is tangent to circle C. As illustrated, the angle subtended by the ear is 4β. Approximately solving $\cos\beta = S/R$ gives $\beta \sim \left(\frac{2\log\lambda}{\log n}\right)^{\frac{1}{2}}$.

Claim 7: The measure of an ear is $P_{\text{ear}} = O\left(\frac{\lambda}{n\log^{\frac{1}{2}} n}\right)$.

Proof: At an angle of α above horizontal, the inner boundary of the ear is at a distance of $S + x$ from the origin, where $\frac{S}{S+x} = \cos\theta$ and $\theta \sim \frac{\alpha}{2}$, from which it follows that $x > \frac{S\alpha^2}{10}$. So

$$
\begin{aligned}
P_{\text{ear}} &< \frac{1}{2\pi}\int_0^{2\beta} e^{-\frac{(S+x)^2}{2}}\,d\alpha \\
&< \frac{\lambda}{2\pi n}\int_0^{\infty} e^{-\frac{S^2\alpha^2}{10}}\,d\alpha \\
&< \frac{\lambda}{\sqrt{2\pi} nS} \\
&= O\left(\frac{\lambda}{n\log^{\frac{1}{2}} n}\right)
\end{aligned}
$$

□

Now we give something away. Suppose that the probability density is uniform over the ear, taking on a value of λ/n. Then the best way to place k points in the ear, in the hope that they might be local maxima for P, is to distribute them evenly on an arc centered on P. Then each region where a local minimum hull point must fall has measure $P_{\min} \sim \left(\frac{2\beta}{k-1}\right)^3\left(\frac{4R^2}{12}\right)\left(\frac{\lambda}{n}\right)$. Substituting $\beta \sim \left(\frac{2\log\lambda}{\log n}\right)^{\frac{1}{2}}$, $\lambda = 2\pi e \log^{\frac{1}{2}} n \log\log n$ and $R \sim (2\log n)^{\frac{1}{2}}$ gives $P_{\min} = O\left(\frac{(\log\log n)^{5/2}}{n(k-1)^3}\right)$. As in the case of a uniform distribution, the probability p_k that the modality is at least k satisfies

$$
\begin{aligned}
p_k &< nP_{\text{band}}\binom{n-1}{k}P_{\text{ear}}^k(n-k-1)^{k-1}P_{\min}^{k-1} \\
&< \frac{(\log n)^{\frac{1}{2}}c^k(\log\log n)^{3.5k-1.5}}{(k-1)^{4k-3}}
\end{aligned}
$$

for some constant c. But $p_k \to 0$ as $n \to \infty$ provided $k \geq \frac{a \log\log n}{\log\log\log n}$, and $a > 1$. Moreover, p_k decreases super-exponentially in k, so the expected value of k is $O\left(\frac{\log\log n}{\log\log\log n}\right)$, and Theorem 4 is established. □

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