Nyström's Method versus Fourier Type Methods for the Numerical Solution of Integral Equations

Manfred R. Trummer

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Manfred R. Trummer

University of British Columbia
Department of Computer Science
Vancouver, B.C., CANADA
V6T 1W5

and

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich, Switzerland

ABSTRACT

It is shown that Nyström's method and Fourier type methods produce the same approximation to a solution of an integral equation at the collocation points for Nyström's method. The quadrature rule for numerical integration must have these collocation points as abscissa.

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1. Description of the methods.

In this note we will prove a result on the equivalence of Nyström's and Fourier type methods for the numerical solution of integral equations. We ran into this question while trying to implement the conformal mapping method described in [KT85] with Fourier methods, like in [BE84,BE85].

Throughout the sequel we will assume that the discretized versions of integral equation (1) below have a unique solution. The restriction to the interval $[0,1]$ is only made for the sake of convenience.

We consider the integral equation

$$\lambda f(t) + \int_0^1 k(t,s) f(s) ds = g(t), \quad t \in [0,1], \tag{1}$$

where $k \in L^2([0,1] \times [0,1])$ is a Hilbert-Schmidt kernel. We will show that Nyström's method and Fourier type methods result in the same numerical approximation under certain conditions on the quadrature rule Q which is used to evaluate definite integrals numerically.

We choose n collocation points t_j, $1 \leq j \leq n$ and weights w_j, and we will use the quadrature rule Q defined by

$$Qh := \sum_{j=1}^{n} w_j h(t_j) \approx \int_0^1 h(s) ds \ . \tag{2}$$

Nyström's method consists of collocation at the points t_i and approximating the integral in (1) using the quadrature formula (2), which leads to a system of equations

$$\lambda x_i + \sum_{j=1}^{n} w_j k(t_i, t_j) x_j = g(t_i), \quad 1 \le i \le n, \tag{3}$$

where the x_i are approximations to $f(t_i)$. For second kind equations $(\lambda \neq 0)$ (1) can then be used to obtain approximations at any point t in the interval by means of

$$f(t) \approx \frac{1}{\lambda}\left[g(t) - \sum_{j=1}^{n} w_j k(t, t_j) x_j \right]. \tag{4}$$

See [AT76] for details.

To explain what we mean by *Fourier type methods* we use the space $L^2[0,1]$ of square integrable functions on $[0,1]$ endowed with the scalar product

$$<f, g> := \int_0^1 f(t) g(t) dt . \tag{5}$$

Let $\phi_1, \ldots, \phi_n$ be an orthonormal Chebychev system (see e.g. [SI70]). We try to approximate the solution in the subspace spanned by the ϕ_i :

$$f(t) \approx \sum_{j=1}^{n} c_j \phi_j(t) \tag{6}$$

This can be achieved by expanding k and g in "Fourier" series:

$$g(t) \approx \sum_{i=1}^{n} d_i \phi_i(t) \tag{7}$$

and

$$k(t,s) \approx \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \phi_i(t) \phi_j(s) , \tag{8}$$

where the "Fourier" coefficients d_i and a_{ij} are given by

$$d_i = \langle g, \phi_i \rangle , \tag{7a}$$

$$a_{ij} := \int_0^1 \int_0^1 k(t,s) \phi_i(t) \phi_j(s) dt ds . \tag{8a}$$

Equation (1) is then approximated by

$$\sum_{i=1}^{n} \left(\lambda c_i + \sum_{j=1}^{n} a_{ij} c_j \right) \phi_i(t) = \sum_{i=1}^{n} d_i \phi_i(t) , \tag{9}$$

or, written in matrix form

$$(\lambda I + A) \mathbf{c} = \mathbf{d} . \tag{10}$$

Computing the Fourier coefficients in (7a) and (8a) numerically using the quadrature rule (2) we obtain

$$a_{ij} = \sum_{p=1}^{n} \sum_{q=1}^{n} w_p \phi_i(t_p) k(t_p, t_q) \phi_j(t_q) w_q \tag{11}$$

and

$$d_i = \sum_{p=1}^{n} w_p \phi_i(t_p) g(t_p) . \tag{12}$$

The system (10) can then be rewritten as

$$\lambda c_i + \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} w_p \phi_i(t_p) k(t_p, t_q) \phi_j(t_q) w_q c_j = \sum_{p=1}^{n} w_p \phi_i(t_p) g(t_p) . \tag{13}$$

2. Equivalence of Nyström's and Fourier methods.

THEOREM: *Let $\{\phi_i\}$, $1 \leq i \leq n$, be an orthonormal Chebychev system, and suppose, the quadrature rule (2) has nonzero weights and integrates the Gramian matrix $<\phi_i, \phi_j>$ exactly, i.e.*

$$\sum_{p=1}^{n} w_p \phi_i(t_p)\phi_j(t_p) = \delta_{ij} , \tag{14}$$

where δ_{ij} denotes the Kronecker symbol. Then the approximate values of the solution at the points t_i obtained by the Fourier type method described above are the same as the values x_j obtained by Nyström's method.

<u>*Proof:*</u> In view of (14) equation (13) can be written as

$$\sum_{p=1}^{n} w_p \phi_i(t_p)\mu_p = 0 , \tag{15}$$

where

$$\begin{aligned} \mu_p &= \lambda \sum_{j=1}^{n} c_j \phi_j(t_p) \\ &\quad + \sum_{j=1}^{n} \sum_{q=1}^{n} w_q k(t_p, t_q)\phi_j(t_q)c_j \\ &\quad - g(t_p) . \end{aligned} \tag{16}$$

The matrix L, defined by $L_{ip} := w_p \phi_i(t_p)$, has independent columns iff the matrix $[\phi_i(t_p)]$ has independent columns, because all weights are different from 0. Since $\{\phi_i\}$ is a Chebychev system, the matrix $[\phi_i(t_p)]$ is regular.

Therefore, (15) implies that $\mu_p = 0$ for all p. From (16) we conclude that

$$z_p := \sum_{j=1}^{n} c_j \phi_j(t_p) \tag{17}$$

is a solution of (3). Since this solution is unique, the proof is complete.

□

Remark: For first kind equations ($\lambda = 0$) the result holds even if the Gramian is not integrated exactly.

Finally, we present two situations where the theorem can be applied:

Example 1: "Classical" Fourier method (see e.g. [HE79,BE84,BE85]). If the kernel and the right hand side of equation (1) are both periodic, Fourier methods are extremely powerful. We approximate the solution of (1) by a trigonometric polynomial.

Let m be a positive integer, and $n = 2m+1$. The orthonormal Chebychev system is given by

$$\begin{aligned} \phi_1(t) &= 1 \\ \phi_{2k+1}(t) &= \sqrt{2}\sin(2\pi k t)\,, \quad \text{for } 1 \le k \le m\,, \end{aligned} \tag{18}$$

$$\phi_{2k}(t) = \sqrt{2}\cos(2\pi kt), \quad \text{for } 1 \le k \le m .$$

The corresponding quadrature rule is the trapezoidal rule (for periodic functions)

$$t_k = \frac{k-1}{n}, \quad w_k = \frac{1}{n}, \quad 1 \le k \le n . \tag{19}$$

This rule integrates the functions $\cos(2\pi kt)$ and $\sin(2\pi kt)$ exactly as long as $k < n$. Using trigonometric identities of the form ([AS65, p.72])

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta)+\cos(\alpha+\beta) \tag{20}$$

it is easy to see that the Gramian is integrated exactly.

If $n=2m$ is even, the choice of the orthonormal system is more delicate, since neither $\cos^2(2\pi mt)$ nor $\sin^2(2\pi mt)$ are integrated exactly by the trapezoidal rule (19). In this case the Chebychev system consists of the functions in (18) with m replaced by $m-1$, and the function

$$\phi_n(t) = \cos(2\pi mt) + \sin(2\pi mt) ; \tag{21}$$

then $<\phi_n, \phi_n>$ will be integrated exactly, thus satisfying the assumptions of the theorem. This choice corresponds to interpolating the solution by an "unbalanced" trigonometric polynomial (see e.g. [HE82, p.335]). Most implementations of the classical Fourier method use the balanced trigonometric polynomial (the Fourier coefficient of $\sin(2\pi mt)$ is equal to 0); however, as pointed out by Berrut ([BE85a]), the difference between these two methods is in general negligible.

This example explains why Nyström's method with the trapezoidal rule performs so well for an integral equation with a periodic kernel and inhomogeneity. It thus justifies the choice of this method for the conformal mapping algorithm presented in [KT85,TR84].

Example 2: Take the Legendre polynomials of degree $\leq n-1$ as the orthonormal Chebychev system, and the n-point Gauss-Legendre quadrature formula for Q (see e.g. [AS65]). The Gramian is integrated exactly, because polynomials of degree $\leq 2n-1$ are integrated exactly.

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