Recursion Theorems and Effective Domains

Akira Kanda
Department of Computer Science
The University of British Columbia
Vancouver, B.C. V6T 1W5
Canada

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ABSTRACT

Every acceptable numbering of an effective domain is complete. Hence every effective domain admits the 2nd recursion theorem of Eršov [1]. On the other hand for every effective domain, the 1st recursion theorem holds. In this note, we establish that for effective domains, the 2nd recursion theorem is strictly more general than the 1st recursion theorem, a generalization of an important result in recursive function theory.

1. Numerations

In this section, we briefly overview a small part of numeration theory which is relevant to our discussion. For details see Eršov [1] and Malcév [4].

Definition 1.1.

A *numeration* (of a set A) is a surjective map $\alpha\colon N \to A$ where N is the set of all natural numbers. A morphism from a numeration $\alpha\colon N \to A$ to another $\beta\colon N \to B$ is a function $h : A \to B$ which can be *realized* by a recursive function, i.e. for which there is a recursive function $r_h : N \to N$ satisfying:

$$h \cdot \alpha = \beta \cdot r_h .$$

For each numeration $\alpha\colon N \to A$, we define an equivalence relation $=_\alpha$ by:

$$n =_\alpha m \qquad iff \quad \alpha(n) = \alpha(m).$$

$\square$

Throughout, we assume that φ is a Gödel numbering of partial recursive functions $N \to N$.

Definition 1.2.

A numeration $\alpha\colon N \to N$ is *precomplete* if for every partial recursive function $f : N \to N$ there is a recursive function $g : N \to N$ s.t. $f(i)\downarrow$ implies $f(i) =_\alpha g(i)$ and we can compute Gödel number of g from that of f. We say *g makes f total modulo α*. Such α is *complete* if there is an element $e \in A$, called a *special element*, s.t. $f(i)\uparrow$ implies $\alpha(g(i)) = e$.

$\square$

Proposition 1.3. (Eršov [1])

A numeration $\alpha\colon N \to A$ is precomplete iff there is a recursive function fix satisfying:

$$\varphi_n(fix(n))\downarrow \qquad \textit{implies} \qquad \varphi_n(fix(n)) =_\alpha fix(n).$$

Proof. Assume α is precomplete. There is a total recursive function g s.t.

$$(\lambda x.\, \varphi_x(x))(i) = \varphi_i(i)\downarrow \quad \textit{implies} \quad \varphi_i(i) =_\alpha g(i).$$

Thus $\varphi_n \cdot g$ is partial recursive. Let $\varphi_m = \varphi_n \cdot g$. Assume $\varphi_m(m)\downarrow$. Then we have:

$$\alpha(g(m)) = \alpha(\varphi_m(m)) = \alpha(\varphi_n(g(m))).$$

Take $fix(n) = g(m)$. Since we can compute a Gödel number of g from n, m can be computed from n. Thus fix is a total recursive function. Conversely assume such fix exists. Let $h\colon N \to N$ be a partial recursive function. Define a partial recursive function $H\colon N^2 \to N$ by:

$$H(x,y) = \begin{cases} h(x) & \text{if } h(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

By s-m-n theorem, there is a total recursive function f s.t.

$$H(x,y) = \varphi_{f(x)}(y).$$

Let $h' = fix \cdot f$. Then h' is a total recursive function and:

$$\alpha(h'(x)) = \alpha(f\, ix \cdot f\,(x))$$

$$= \alpha(\varphi_{f\,(x)}(f\, ix \cdot f\,(x)))$$

$$= \alpha(H(x,y))$$

$$= \alpha(h(x))$$

whenever $h(x)\downarrow$. Thus h' makes h total modulo α. Obviously the construction of h' is uniform in h.

□

Definition 1.4.

Let $\alpha: N \to A$ be a numeration. A subset $B \subset A$ is *α–r.e.* if $\alpha^{-1}(B)$ is r.e. The *Malc̓ev-Eršov topology* M_A over A is a topology defined by the following basis B_A:

$$B_A = \text{the collection of all } \alpha\text{–r.e. sets.}$$

□

Proposition 1.5. (Malc̓ev-Eršov [4,1]).

All morphisms are continuous wrt Malc̓ev-Eršov topology.

□

For any numeration $\alpha: N \to A$, we can introduce a pre-ordering $<_\alpha$ of A by:

$a <_\alpha b$ iff for every α–r.e. set X, $a \in X$ implies $b \in X$.

It is easy to observe that $<_\alpha$ becomes a partial ordering iff M_A is a T_0-space.

Proposition 1.6. (Malćev [4])

Let $\alpha\colon N \to A$ be a complete numeration with a special element $e \in A$. Then e is an $<_\alpha$-smallest element. Thus if M_A is a T_0-space then e is the only special element of A.

$\square$

2. Effective Domains

Definition 2.1.

A *domain* is a partially ordered set $(X, <)$ such that

(1) For every subset $Z \subset X$, if Z is directed then the least upper bound (lub) $\bigsqcup Z$ exists.

(2) The set B_X of compact elements of X is countable.

(3) For every element $x \in X$, $B_x = \{b \in B_X \mid b < x\}$ is directed and $x = \bigsqcup B_x$.

(4) Every bounded subset $Z \subset X$ has a lub $\bigsqcup Z$.

$\square$

Definition 2.2.

Let X be a domain and ϵ: $N \rightarrow B_X$ be a numeration. (X, ϵ) is an ***effective given domain*** if there is a pair (b, l) of recursive predicates satisfying:

$$b(x) \longleftrightarrow E(f_s(x)) \textit{ has an upper bound}$$

$$l(x,k) \longleftrightarrow \epsilon(k) = \bigsqcup \epsilon(f_s(x)) ,$$

where f_s is the standard enumeration of finite subsets of N. An element $x \in X$ is ***computable*** w.r.t. ϵ if for some recursively enumerable (r.e.) set R, $\epsilon(R)$ is directed and $x = \bigsqcup \epsilon(R)$. $Comp(X, \epsilon)$ is the set of all computable elements of (X, ϵ) and is called an ***effective domain (generated by*** ϵ***).***

$\square$

Throughout, W is a Gödel numbering of the set RE of all r.e. sets.

Definition 2.3. (Weihrauch [9])

Let (X,ϵ) be an effectively given domain. A numeration $\chi : N \to Comp(X,\epsilon)$ is *acceptable* if

(1) $\{(m,n) \mid \epsilon(m) < \chi(n)\}$ is r.e.

(2) there is a recursive function d s.t. if $\epsilon(W_n)$ is directed then

$\chi(d(n)) = \bigsqcup \epsilon(W_n)$.

$\square$

This concept can be characterized in terms of a universal function and $s-m-n$ property, as shown in [2]. Also we can easily show that an acceptable numeration of $Comp(X,\epsilon)$ exists and all acceptable numerations of $Comp(X,\epsilon)$ are recursively isomorphic (see [2] and [9]).

Lemma 2.4.

Let $\chi : N \to Comp(X,\epsilon)$ be acceptabe. Then the inclusion map $\subsetneq : B_X \to \epsilon$ is a morphism $\epsilon \to \chi$.

Proof. There is a recursive function $b : N \to N$ s.t. $W_{b(n)} = \{m \mid \epsilon(m) < \epsilon(n)\}$. Also $\epsilon(W_{b(n)})$ is directed. Thus we have

$$\chi(d(b(n))) = \bigsqcup \epsilon(W_{b(n)}) = \epsilon(n).$$

$\square$

Proposition 2.5. (Weihrauch [9]).

Let χ: $N \rightarrow Comp(X, \epsilon)$ be acceptable, then it is complete with a special element $\perp = \bigsqcup \phi$.

Proof. Let $f : N \rightarrow N$ be a partial recursive function. Since χ is acceptable,

$$\{(n, m) \mid \epsilon(n) < \chi(f(m))\} \cup \{i\} \quad where \quad \chi(i) = \perp$$

is r.e. regardless of $f(m)$'s termination. Thus for some recursive function $g : N \rightarrow N$

$$W_{g(m)} = \{n \mid \epsilon(n) < \chi(f(m))\} \cup \{i\}.$$

Obviously $d \cdot g$ is a recursive function. Also $\epsilon(W_{g(m)})$ is directed. Therefore we have:

$$\chi(d \cdot g(m)) = \bigsqcup \epsilon(W_{g(m)}) = \chi(f(m))$$

whenever $f(m)\downarrow$. If $f(m)\uparrow$ then

$$\chi(d \cdot g(m)) = \bigsqcup \epsilon(W_{g(m)}) = \bigsqcup \{\epsilon(i)\} = \perp.$$

Construction of $d \cdot g$ is uniform in f.

$\square$

It can readily be seen that $(PF, <)$ is a domain where PF is the set of all partial functions from $N \rightarrow N$, and $<$ is the set inclusion. In fact B_{PF} = the set of all finite functions. Let ϵ: $N \rightarrow B_{PF}$ be the standard enumeration of finite functions $N \rightarrow N$. Then (PF, ϵ) is an effectively given domain and

$PRF = Comp(PF, \epsilon)$ is the set of all partial recursive functions. Furthermore $\varphi: N \to PRF$ is acceptable. (see [2]). Thus ϕ is the only special element of PRF.

Theorem 2.6. *(The 2nd Recursion Theorem)*

Let $\chi: N \to Comp(X, \epsilon)$ be acceptable. Then there is a recursive function fix s.t.

$$\varphi_n(fix(n))\downarrow \quad \textit{implies} \quad \varphi_n(fix(n)) =_\chi fix(n).$$

Proof. χ is complete thus by 1.3.

$\square$

Since $\varphi: N \to PRF$ is acceptable, this theorem is a generalization of Kleene 2nd recursion theorem.

Definition 2.7.

Let $(X, <)$ be a domain. A function $f: X \to X$ is ***continuous*** if for every directed subset $\theta \subset X$, $\bigsqcup f(\theta)$ exists and $\bigsqcup f(\theta) = f(\bigsqcup \theta)$.

$\square$

It can readily be seen that $f: X \to X$ is continuous iff $\forall b \in B_x . \forall x \in X . b < f(x) \Rightarrow \exists b' \in B_x . b < f(b')$.

Definition 2.8.

Let (X,ϵ) be an effectively given domain. A continuous function $f : X \rightarrow X$ is *computable* iff

$$graph(f) = \{(m,n) \mid \epsilon(m) < f(\epsilon(n))\}$$

is an r.e. set. When $graph(f) = W_n$ we say f has a *g-index* n.

□

Since the restriction h' to $Comp(X,\epsilon)$ of a continuous function $X \rightarrow X$ has a unique continuous extension $h: X \rightarrow X$, we identify h and h'.

In [3] it was shown that a continuous function $f : X \rightarrow X$ is computable iff it is a morphism from χ to χ where χ: $N \rightarrow Comp(X,\epsilon)$ is acceptable. Streicher [8] showed that any morphism $f : \chi \rightarrow \chi$ is continuous. More precisely f has a unique continuous extension to X. This leads to the following generalization of Myhill-Shepherdson theorem [5]:

Theorem 2.9. *(Myhill-Shepherdson Theorem)*

Let (X,ϵ) be an effectively given domain and χ: $N \rightarrow Comp(X,\epsilon)$ be acceptable. $f : X \rightarrow X$ is computable iff it is a morphism from χ to χ. This equivalence is constructive, i.e., from a g-index of f we can compute a Gödel number of a recursive function r_f which realizes f and vice versa.

□

From 2.3, it immediately follows that there is a recursive function $\bar{d}$ s.t. $W_{\bar{d}(n)}$ is directed and $\chi(n) = \bigsqcup \epsilon (W_{\bar{d}(n)})$. Thus $d \cdot \bar{d}(n) =_{\chi} n$ and $\bar{d} \cdot d(n) =_{\chi} n$. In fact

$$W_{\bar{d}(n)} = \{m \mid \epsilon(m) < \chi(n)\}.$$

The next lemma due to [3] states that effective domains are effectively complete:

Lemma 2.10.

There is a recursive function *lim* s.t. $\chi(W_n)$ is directed then $\bigsqcup \chi(W_n) = \chi(lim(n))$.

Proof.

$$\begin{aligned} \bigsqcup \chi(W_n) &= \bigsqcup \{ \bigsqcup \epsilon(W_{\bar{d}(x)}) \mid x \in W_n \} \\ &= \bigsqcup \epsilon(\{i \mid i \in W_{\bar{d}(x)} \text{ and } x \in W_n\}) \\ &= \bigsqcup \epsilon(W_{uni(n)}) \end{aligned}$$

where *uni* is a recursive function s.t.

$$W_{uni(y)} = \{k \mid k \in W_{\bar{d}(j)} \text{ and } j \in W_y\}.$$

Also $\epsilon(W_{uni(y)})$ is directed. Thus

$$\bigsqcup \chi(W_n) = \chi(d(uni(n))).$$

$\square$

Theorem 2.11. *(The 1st Recursion Theorem)*

Let (X, ϵ) be an effectively given domain. There is a recursive function lf s.t. if $f : X \to X$ is a morphism from χ to χ with $\varphi_n = r_f$, then $\chi(lf(n))$ is the least fix point of f.

Proof. It is known that $\bigsqcup\{f^i(\perp) \mid i \in N\}$ is the least fix point of f. (See Scott [7]). Note $f^0(\perp) = \perp$ and $f^i(\perp)$ is the i-fold applications of f to $\perp$. Let $c \in N$ s.t. $\chi(c) = \perp$. Define a recursive function $u : N \to N$ by:

$$u(0) = c$$

$$u(i+1) = r_f(u(i)).$$

Then $\chi(u(i)) = f^i(\perp)$ for all $i \in N$. Also $\{\chi(u(i)) \mid i \in N\}$ is directed. Since the construction of u is uniform in r_f, there is a recursive function ite s.t.

$$W_{ite(n)} = \{u(i) \mid i \in N\}.$$

Thus $\chi(lim(ite(n))) = \bigsqcup f^i(\perp)$.

□

It can readily be seen that all recursive operators are computable functions from PRF to PRF. Thus theorem 2.11 generalizes the 1st recursion theorem of Kleene.

3. The 2nd Recursion Theorem vs. The 1st Recursion Theorem

In recursive function theory, it is known that the 2nd recursion theorem is strictly more general than the 1st recursion theorem.

First, the 2nd recursion theorem in recursive function theory does not require φ_n to realize a morphism from φ to φ. Such requirement is called *extensionality*. On the contrary, the 1st recursion theorem holds only for morphisms from φ to φ.

More importantly, the 2nd recursion theorem is still more general than the 1st recursion theorem even when we are restricting our discussion to morphisms from φ to φ.

Proposition 3.1.

There is an extensional recursive function $h = \varphi_m$ s.t. $\varphi_{fix(m)}$ is not the least fix point of the morphism $\varphi \to \varphi$ determined by φ_m.

$\Box$

Proposition 3.2.

There is a recursive function $tr : N \to N$ s.t. if φ_m is an extensional recursive function which realizes a morphism $f : \varphi \to \varphi$ then $\varphi_{tr(m)}$ realizes f and $\varphi_{fix(tr(m))}$ is the least fix point of f.

$\Box$

These two propositions state that in recursive function theory, the 1st recursion theorem is a special case of the 2nd recursion theorem. For details readers are refered to Rogers [6].

The question is "Can we generalize these observations to effective domains?".

First, the theorem 2.6 does not require φ_n to realize a morphism from χ to χ *(extensionality)*. But the theorem 2.11 is concerned only about morphisms from χ to χ.

Since φ: $N \rightarrow PRF$ is acceptable, 3.1 can be used as an example of what the 2nd recursion theorem 2.6 can do but the 1st recursion theorem (2.11) can not do.

In the following we show that we can generalize 3.2.

Lemma 3.3.

Let (X,ϵ) be an effectively given domain and χ: $N \rightarrow Comp(X,\epsilon)$ be acceptable. There are recursive functions *apr* and *gr* s.t.

(1) $W_{apr(n)} = \{m \mid \epsilon(m) < \chi(n)\}$

(2) $W_{gr(n)} = \{(k,m) \mid \epsilon(k) < \chi(\varphi_n(d \cdot b(m)))\}$

$\square$

Lemma 3.4.

Let $Apply: RE \times RE \to RE$ be the following function:

$$Apply(X,Y) = \{x \mid \exists k . k \in Y \text{ and } (x,k) \in X\}.$$

Then $Apply$ is a morphism $W \times W \to W$.

Proof. There is a recursive function $apply: N^2 \to N$ s.t.

$$W_{apply(i,j)} = \{x \mid \exists k . k \in W_j \text{ and } (x,k) \in W_i\}.$$

$\square$

Theorem 3.5. (Main Theorem)

Let $\chi: N \to Comp(X,\epsilon)$ be acceptable. There is a recursive function Tr s.t. φ_m realizes a morphism $\Phi: \chi \to \chi$ then $\varphi_{Tr(m)}$ realizes Φ and $\chi(fix(T_r(m)))$ is the least fix point of Φ.

Proof. It can readily be seen that $\epsilon\,(Apply(W_{gr(m)}, W_{apr(n)}))$ is directed and we have

$\chi(d \cdot apply(gr(m), apr(n)))$

$= \bigsqcup \epsilon\,(Apply(W_{gr(m)}, W_{apr(n)}))$

$= \Phi(\chi(n))$

$= \chi(\varphi_m(n))$.

Define a recursive function $h_m: N \to N$ by:

$$h_m(n) = d(apply(gr(m), apr(n))).$$

Then $\chi(h_m(n)) = \Phi(\chi(n)) = \chi(\varphi_m(n))$. Since χ is complete, there is a recursive function g such that

$$\chi(g(x)) = \begin{cases} \chi(\varphi_x(x)) & \text{if } \varphi_x(x)\downarrow \\ \perp & \text{otherwise.} \end{cases}$$

Thus $\varphi_r = h_m \cdot g$ is a recursive funciton. Now we have

$$\chi(h_m \cdot g(r)) = \chi(\varphi_r(r)) = \chi(g(r)) = \Phi(\chi(g(r))). \tag{1}$$

Thus $\chi(g(r))$ is a fix point of Φ. Now let $n_0, n_1, n_2, \ldots$ be an effective enumeration of $W_{apr(g(r))}$ where $\epsilon(n_0) = \perp$. Define $n_j{}'$ by:

$$n_0{}' = n_0$$

$$n_{j+1}{}' = n_{j+1} \vee n_j{}'$$

where $\vee$ is a recursive function $N^2 \rightarrow N$ s.t.

$$\epsilon(x \vee y) = \epsilon(x) \sqcup \epsilon(y), \quad \text{if } \epsilon(x) \sqcup \epsilon(y) \text{ exists}$$

Since $\epsilon(W_{apr(g(r))})$ is directed, $n_j{}'$ is well-defined. Then $\epsilon(n_0{}')$, $\epsilon(n_1{}')$,... is a chain and

$$\bigsqcup_j \epsilon(n_j{}') = \chi(g(r)).$$

We can consider $\lambda Y.Apply(X, Y)$ as a process which transforms a stream of Y into a stream of $Apply(X, Y)$ in the obvious way. Now let $a_0, a_1, a_2, \ldots$ be the stream obtained by letting $n_0{}', n_1{}', n_2{}', \ldots$ through $\lambda Y.Apply(W_{gr(m)}, Y)$. Then we have:

$$\epsilon(a_0) < \Phi(\perp)$$

$$\epsilon(a_{k+1}) < \Phi(\epsilon(a_0 \vee \cdots \vee a_k))$$

Therefore for each k,

$$\epsilon(a_k) < \Phi^k(\perp).$$

Thus $\chi(\varphi_m(g(r)) = \bigsqcup \epsilon(a_k) < \bigsqcup_k \Phi^k(\perp)$. Thus $\chi(g(r)) < \bigsqcup_k \Phi^k(\perp)$. But $\chi(g(n))$ is a fix point of Φ and $\bigsqcup_k \Phi^k(\perp)$ is the least fix point of Φ. Therefore

$$\chi(g(r)) = \bigsqcup_k \Phi^k(\perp).$$

The construction of h_m is uniform in m, thus there is a recursive function $Tr: N \to N$ s.t. $\varphi_{Tr(m)} = h_m$. Obviously $g(r) = fix(Tr(m))$.

$\square$

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