An Alternative Characterization of Precomplete Numerations

Akira Kanda
Department of Computer Science
The University of British Columbia
Vancouver, B.C. V6T 1W5
Canada

Technical Report 84-20

An Alternative Characterization of Precomplete Numerations

Akira Kanda

Department of Computer Science
University of British Columbia
Vancouver, B.C. V6T 1W5
Canada

(November 1984)

Abstract

Eršov [1] characterized precomplete numerations as those numerations which satisfy the 2nd recursion theorem. In this short note we show that they are exactly those numerations which satisfy the strongest form of the 2nd recursion theorem.

§1. Preliminaries

Definition 1.1.

A *numeration (of a set A)* is a surjection $\alpha: N \to A$ where N is the set of all natural numbers. If A is a singleton, α is called ***trivial***.

$\square$

Definition 1.2.

A numeration $\alpha: N \to A$ is ***precomplete*** if for every partial recursive function $f: N \to N$ there is a recursive function $g: N \to N$ such that

$$f(i)\downarrow \quad \textit{implies} \quad \alpha(f(i)) = \alpha(g(i)),$$

and we can effectively compute a Gödel number of g from that of f. We say g *makes f total modulo α*.

$\square$

The next result is due to Eršov [1]. However for the sake of the completeness of this paper, we present a proof of this theorem. We also think that this will help English speaking readers, for the original proof of Eršov is written in German. We assume that φ is a Gödel numbering of partial recursive functions $N \to N$.

Proposition 1.3. (Eršov 2nd Recursion Theorem)

A numeration α: $N \rightarrow A$ is precomplete iff there is a recursive function fix satisfying:

$$\varphi_n(fix(n))\downarrow \quad implies \quad \alpha(\varphi_n(fix(n))) = \alpha(fix(n)).$$

Proof. Assume α is precomplete. There is a total recursive function g such that:

$$\varphi_i(i)\downarrow \qquad implies \qquad \alpha(\varphi_i(i)) = \alpha(g(i)).$$

Let $\varphi_m = \varphi_n \cdot g$. Assume $\varphi_m(m)\downarrow$. Then we have:

$$\varphi(g(m)) = \alpha(\varphi_m(m)) = \alpha(\varphi_n(g(m))).$$

Take $fix(n) = g(m)$. Since we can compute a Gödel number of g from n, fix is a recursive function. Conversely assume such fix exists. Let $h: N \rightarrow N$ be a partial recursive function. Define a partial recursive function H: $N^2 \rightarrow N$ by:

$$H(x,y) = \begin{cases} h(x) & \text{if h (x)}\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

By $S-m-n$ theorem there is a total recursive function $f: N \rightarrow N$ such that:

$$H(x,y) = \varphi_{f(x)}(y).$$

Let $h' = fix \cdot f$. Then h' is a recursive function and:

$$\alpha(h'(x)) = \alpha(fix(f(x)))$$

$$= \alpha(\varphi_{f(x)}(fix(f(x))))$$

$$= \alpha(H(x,y))$$

$$= \alpha(h(x))$$

whenever $h(x)\downarrow$. Construction of h' is uniform in h.

$\square$

§2. Another Characterization of Precomplete Numerations

The strongest form of Kleene 2nd recursion theorem states that we can enumerate fix-points of each partial recursive function $N \to N$. In this section we characterize precomplete numerations as those which satisfy this recursion theorem.

Lemma 2.1.

Let π: $N \to X$ be a precomplete numerations. There is a recursive function η: $N \to N$ such that

$$i < \eta(i) < \eta^2(i) < \cdots \qquad \text{and}$$

$$\pi(i) = \pi(\eta(i)) = \pi(\eta^2(i)) = \cdots$$

Proof. In case π is trivial, this is obviously true. We first prove that from $z_0, z_1, \ldots, z_r$ such that $\pi(z_0) = \pi(z_1) = \cdots = \pi(z_r)$, we can compute z_{r+1} such that $z_{r+1} \notin \{z_0, z_1, \ldots, z_r\}$ and $\pi(z_r) = \pi(z_{r+1})$. Let m be a number such that $\pi(m) \neq \pi(z_0)$. Define a recursive function $f : N \to N$ by:

$$f(t) = \begin{cases} z_0 & \text{if } t \notin \{z_0, z_1, \ldots, z_r\} \\ m & \text{otherwise.} \end{cases}$$

Then by the 2nd recursion theorem, there is $n_f \in N$ satisfying:

$$\pi(f(n_f)) = \pi(n_f) = \begin{cases} \pi(z_0) & \text{if } n_f \notin \{z_0, z_1, \ldots, z_r\} \\ \pi(m) & \text{otherwise.} \end{cases}$$

If $n_f \notin \{z_0, z_1, \ldots, z_r\}$ then set $z_{r+1} = n_f$. If $n_f \in \{z_0, z_1, \ldots, z_r\}$ then $n_f = z_j$ for some $j \leq r$. Thus $\pi(n_f) = \pi(z_0) = \pi(m)$.

This contradicts to $\pi(z_0) \neq \pi(m)$. Thus $n_f \notin \{z_0, z_1, \ldots, z_r\}$. Therefore we have a recursive function $\Psi^* : N^2 \to N$ such that

$$x \neq y \quad \textit{implies} \quad \Psi^*(z_0, x) \neq \Psi^*(z_0, y) \quad \text{and}$$

$$\pi(z_0) = \pi(\Psi^*(z_0, 0)) = \pi(\Psi^*(z_0, 1)) = \cdots .$$

In fact $\Psi^*(z_0, x) = z_x$. Now define η: $N \to N$ by:

$$\eta(z) = \Psi^*(z, k)$$

where $k = \mu y. [\Psi^*(z, y) > z]$.

Then η is a recursive function satisfying:

$$i < \eta(i) < \eta^2(i) < \cdots \quad \text{and}$$

$$\pi(i) = \pi(\eta(i)) = \pi(\eta^2(i)) = \cdots$$

$\square$

Theorem 2.2. (The Characterization Theorem)

A numeration χ: $N \to X$ is precomplete iff there is a recursive injection $n: N^2 \to N$ such that

$$\varphi_z(n(z,y))\downarrow \ \textit{implies} \ \ \chi(\varphi_z(n(z,y))) = \chi(n(z,y)).$$

Proof. Assume χ is precomplete. By the previous lemma we have a recursive function η: $N \to N$ such that

$$i < \eta(i) < \eta^2(i) < \cdots \qquad \text{and}$$

$$\chi(i) = \chi(\eta(i)) = \chi(\eta^2(i)) = \cdots$$

Define a recursive function Ψ: $N^2 \to N$ by:

$$\Psi(i,j) = \mu y. \, [y > j \text{ and } y = \eta^k(i) \text{ for } \textit{some } k \in N].$$

Define a function $t: N^2 \to N$ by:

$$t(0,0) = \Psi(0,0)$$

$$t(i,j) = \Psi(i,y)$$

where $y = \mu w. \, [\Psi(i,w) \neq t(\bar{i},\bar{j}) \ \wedge \ \sigma(\bar{i},\bar{j}) < \sigma(i.y)]$ where $\sigma(i,j) = 2^i \cdot 3^j$. This t is defined by induction on the linear ordering $(\{\sigma(i,j) \mid i,j \in N\},<)$. It can readily be seen that t is a recursive function satisfying:

(1) $\chi(t(i,j)) = \chi(i)$.

(2) $i_1 \neq i_2$ or $j_1 \neq j_2$ implies $t(i_1,j_1) \neq t(i_2,j_2)$.

By (2) t is injective. Since χ is precomplete there is a recursive function $g: N \to N$ such that

$$\varphi_u(u)\downarrow \textit{ implies } \chi(\varphi_u(u)) = \chi(g(u)).$$

Define $\tilde{g} : N \to N$ by:

$$\tilde{g}(u) = t(g(u), 2^u).$$

Since t is a recursive injection, $\tilde{g}$ is also a recursive injection. Also we have:

$$\chi(\tilde{g}(u)) = \chi(t(g(u), 2^u)) = \chi(g(u)).$$

Let $\varphi_{v(z)} = \varphi_z \cdot \tilde{g}$. Obviously v is a recursive function. Since φ is a precomplete numeration there is a recursive injection $t' : N^2 \to N$ such that:

$$\varphi_{t'(i,j)} = \varphi_i .$$

Define $n : N^2 \to N$ by:

$$n(z, y) = \tilde{g}(t'(v(z), y)).$$

Assume $\varphi_z(n(z, y))\downarrow$. Then we have

$$\begin{aligned}
\chi(n(z,y)) &= \chi(\tilde{g}(t'(v(z),y))) \\
&= \chi(g(t'(v(z),y))) \\
&= \chi(\varphi_{t'(v(z),y)}(t'(v(z),y))) \\
&= \chi(\varphi_{v(z)}(t'(v(z),y))) \\
&= \chi(\varphi_z(\tilde{g}(t'(v(z),y)))) \\
&= \chi(\varphi_z(n(z,y))).
\end{aligned}$$

Obviously this n is a recursive injection. The converse immediately follows from the proposition 1.3.

$\square$

ACKNOWLEDGEMENT

This research was motivated by very stimulating discussions which the author had with Spreen at Aachen in 1984. Thanks are due to him. E. Nelson told the author some redundancy in the original proof of lemma 2.1. The author thanks T. Fong for typing this difficult manuscript. This research was supported by Canada Natural Science and Engineering Research Council grant A2457.

REFERENCES

[1] Eršov, Ju.L., Theorie der Numerierungen I, Zeitshrift für Math. Logik, Bd 19, Heft 4, 1973.

host	ubc-cs
owner	kanda
printer	Computer Science
Name	/usr/tmp/21165dvi

```
Jobheader: on
language: impress

IMPRINT-10 System Version: v1.9 (Serial packet communications)
Page images processed: 1
Pages printed: 1

Number of job messages: 3
Flushed leftover document bytes:
  17 bytes
Rule off page (2)
```

An Alternative Characterization of Precomplete Numerations

Akira Kanda
Department of Computer Science
The University of British Columbia
Vancouver, B.C. V6T 1W5
Canada

Technical Report 84-20