NATURAL DEDUCTION BASED SET THEORIES: A NEW RESOLUTION OF THE OLD PARADOXES

Paul C. Gilmore

Department of Computer Science
The University of British Columbia
Vancouver, B.C. V6T 1W5
Canada

Technical Report 84-19

Natural Deduction Based Set Theories:

A new resolution of the old paradoxes

Paul C. Gilmore*

Abstract

The comprehension principle of set theory asserts that a set can be formed from the objects satisfying any given property. The principle leads to immediate contradictions if it is formalized as an axiom scheme within classical first order logic. A resolution of the set paradoxes results if the principle is formalized instead as two rules of deduction in a natural deduction presentation of logic. This presentation of the comprehension principle for sets as semantic rules, instead of as a comprehension axiom scheme, can be viewed as an extension of classical logic, in contrast to the assertion of extra-logical axioms expressing truths about a pre-existing or constructed universe of sets. The paradoxes are disarmed in the extended classical semantics because truth values are only assigned to those sentences that can be grounded in atomic sentences.

*The research reported on in this paper was supported by grants of the Natural Science and Engineering Research Council of Canada.

1. Introduction

The comprehension principle of set theory asserts that a set can be formed from the objects satisfying any given property. The principle leads to immediate contradictions if it is formalized as an axiom scheme within classical first order logic. A resolution of the set paradoxes results if the principle is formalized instead as two rules of deduction in a natural deduction presentation of logic. This presentation of the comprehension principle for sets as semantic rules, instead of as a comprehension axiom scheme, can be viewed as an extension of classical logic, in contrast to the assertion of extra-logical axioms expressing truths about a pre-existing or constructed universe of sets.

The semantics of classical first order logic provides a reductionist view of truth for sentences of the logic. The truth and falsehood of sentences in which the logical connectives and quantifiers appear are reduced to the truth and falsehood of sentences in which fewer of these logical primitives appear, and eventually to the truth and falsehood of atomic sentences in which none of them appear. Atomic sentences are given a priori truth values. The semantic rules of truth and falsehood for the logical primitives provide a precise description of the reductions. The structure of sentences of first order logic is such that every sentence receives a truth value through reductions to atomic sentences.

A natural deduction presentation of the logical syntax of first order logic provides a formalization of logic with an exact correspondence between the semantic rules and the rules of deduction: [Gentzen 1934], [Fitch 1952], [Beth 1955], [Prawitz 1965], or [Smullyan 1968]. In the Gentzen sequent calculus, however, the reductionist view of truth is particularly evident, especially when it is recognized that the axioms $P \rightarrow P$ of the calculus may, without loss, be restricted to atomic formulas.

The elementary syntax of first order logic can be simply extended to include the logical primitives of set abstraction $(v{:}P)$ and set membership $\in$. Additional semantic rules can provide a reductionist view of truth and falsehood for sentences in which these logical primitives appear. As before, atomic sentences are those sentences whose truth values must be given directly since they cannot be reduced by the application of any semantic rule to the truth values of simpler sentences. However unlike with first order logic, not every sentence receives a truth value; some sentences cannot be reduced to atomic sentences.

This is the basis for the consistency of the set theory. The paradoxes are disarmed in the extended classical semantics because the paradoxical assertions cannot be grounded in atomic sentences.

The logical syntax of such a set theory is a simple extension of that of first order logic: Two rules of deduction are added for the logical primitive $(v{:}P)$ to formalize the comprehension principle for sets; the axioms of the theory remain as before $P \rightarrow P$ for atomic formulas P. The consistency of the theory follows simply from its semantics.

This formalization of set theory was first proposed in [Gilmore 1968]. However no semantics for the theory was provided: indeed the theory was quickly discovered to be inconsistent because the axioms were incorrectly stated. When the semantics and the correct axioms for the theory were formulated it was realized that the resulting set theory was barely more powerful than first order logic. It was realized that a second order formulation of the theory was necessary for the development of Peano arithmetic. In [Gilmore 1971] the semantics and logical syntax for such a second order theory was provided. The theory described there is the second order theory NaDSet described again in section 6 below in a simplified and extended form. The first order part of NaDSet is the theory NaDSetI described in sections 2, 3, and 4. It can be seen as a corrected version of the theory of [Gilmore 1968].

In section 5 the elementary syntax of NaDSetI is extended to admit descriptions and a rule for description introduction extends the logical syntax. The extended theory is shown to be consistent. Functional application is defined in the usual manner in terms of descriptions. The possibility of extending the theory further by admitting a rule of extensionality is also briefly discussed in section 5.

In section 6 the elementary syntax, semantics, and logical syntax of NaDSetI are extended to define the second order theory NaDSet. The theory of descriptions of section 5 is available for NaDSet.

Finally a sketch of the development of real analysis is provided in §7. The development of natural numbers, integers, and rational numbers can proceed in the usual way in NaDSet. The classical first order theory of these numbers can be fully developed. The development of real numbers through Dedekind cuts can also proceed in the usual way. But it is apparently not possible to prove within

NaDSet that the set of reals has certain properties; for example that there is no enumeration of all the reals. It is of course possible to prove that for any particular enumeration of real numbers, there does exist a real that has not been enumerated. But the generalization of this argument, necessary to prove that no enumeration of the reals can exist, apparently cannot be formalized within NaDSet.

Because NaDSet is formalized within the Gentzen sequent calculus, it should be possible to develop a formal intuitionistic analysis as well by restricting the calculus to its intuitionistic form. It would be of interest to know, for example, whether the results described in [Troelstra 1969] could be derived within the intuitionistic form of NaDSet.

The origins of the theory NaDSetI can be seen in the logic presented in section 21 of [Church 1941]. There the logical syntax for a first order logic without a universal quantifier is defined within the λ-δ-calculus. In [Fitch 1948] semantics for a logic with a universal quantifier is provided by admitting rules with infinitely many premisses. In [Fitch 1967, 1974] set theories are provided in which sets may be members of themselves, but these theories also depend upon a universal quantification rule with infinitely many premisses. In an appendix to [Prawitz 1965], another set theory of Fitch is described that was first presented in [Fitch 1952].

The paper [Feferman 1984] provides an extensive review of a number of resolutions of the paradoxes. Of these, the one that comes closest to the method described in this paper is that of [Scott 1975]. A theory is developed there that is in many respects similar to NaDSetI. However in that paper the notion of function is taken as primary and the notion of set is defined in terms of it.

The original and continuing motivation for the set theories described in [Gilmore 67] and here is their applications in computer science. Some of those applications were suggested in [McCarthy 63]. A more recent application is described in [Gilmore & Morrison]. A nominalist interpretation of the theories NaDSetI and NaDSet forms the basis for their semantics. Although this interpretation may be repugnant to many mathematicians, it is less so to many computer scientists since computing machines are consummate nominalists.

2. Elementary syntax of NaDSetI.

To assist in the later extension of NaDSetI to NaDSet, the elementary syntax is described in greater detail than might otherwise be necessary.

2.1. Elementary terms are formed by using the following letters with or without numeral subscripts:

1. Individual constants: a, b, d
2. Individual variables: x, y, z
3. Set constants: A, B, D.

Any occurrence of a variable in an elementary term is a free occurrence.

2.2. Elementary formulas are $t = r$ or $t \in r$, where r and t are any terms; any formula $t = r$ is atomic; any formula $t \in r$ is atomic if r is an individual or set constant.

A free occurrence of a variable in t or r is a free occurrence in $t = r$ or $t \in r$.

2.3. Formulas are:

1. Elementary formulas
2. $(P|Q)$, where P and Q are formulas. A free occurrence of a variable in P or Q is a free occurrence in $(P|Q)$.
3. $(v)P$, where v is any variable and P any formula.

A free occurrence in P of a variable other than v is a free occurrence in $(v)P$. No occurrence of v in P is free in $(v)P$.

A sentence is a formula in which no variable has a free occurrence.

Although | is the only propositional connective and (v) the only quantifier introduced here, all of the common connectives and the existential quantifier will be freely used. Recall that $(P|Q)$ is true if and only if both P and Q are false.

2.4. Terms are:

1. Elementary terms
2. Set terms $(t{:}P)$, where t is any term and P any formula. t is the abstracted term of $(t{:}P)$, and the variables occurring free in t are abstracted variables. A free occurrence of a variable in P that is not an abstracted variable is a free occurrence in $(t{:}P)$. No occurrence of an abstracted variable in t or P is a free occurrence in $(t{:}P)$.

Constant terms are terms in which no variable has a free occurrence. The set of all constant terms is denoted by δ.

Set terms represent an important generalization of the usual set abstraction terms in which t is restricted to being a variable. One example will suffice to illustrate the generalization. If (u,v) is the ordered pair of u and v, defined below in the usual way, then $((u,v){:}P)$ is a term of the theory. It is a term with different properties than

$$(w:(Eu)(Ev)(w=(u,v) \wedge P))$$

as it is normally defined. It will be seen that $((u,v):P)$ enjoys a symmetry with respect to truth and falsehood not enjoyed by the other.

An occurrence of a term t in a formula or term P is free in P if each free occurrence of a variable in t is also free in the occurrence of t in P.

Extensively used in the definitions of the semantics and the logical syntax of the theory is a substitution operator $[t/v]$, where v is any variable and t any term. It can be applied to any term or formula and has the effect of replacing each free occurrence of v by t; it is assumed that the operator also makes changes of bound variables where necessary. The changes made are to ensure that each occurrence of t in a term or formula $[t/v]P$, that is not an occurrence in P, is a free occurrence.

The simultaneous substitution operator $[t_1,...,t_k/v_1,...,v_k]$ replaces free occurrences of $v_1,...,v_k$ respectively by $t_1,...,t_k$ simultaneously, not sequentially, so that free occurrences of any of the variables $v_1,...,v_k$ in the terms $t_1,...,t_k$ are not affected by the application of the operator. It is generally written $[\underline{t}/\underline{v}]$, where $\underline{t}$ abbreviates $t_1,...,t_k$ and $\underline{v}$ abbreviates $v_1,...,v_k$.

A bound variable variant of a term or formula is a term or formula obtainable from it by changes of bound variables that do not affect any free occurrence of a variable in the term or formula.

3. Semantics for NaDSetI.

In this section the classical semantics of first order logic is extended by the addition of semantic rules for set abstractions (t:P).

3.1. A base is a set *Bse* of signed atomic sentences satisfying the following conditions:

1. For each atomic sentence P, one and only one of $\pm P$ is in *Bse*;
2. If P' is a bound variable variant of P, then $\pm P'$ is in *Bse* if and only if $\pm P$ respectively is in *Bse*;
3. For all t in δ, $+t = t$ is in *Bse*.

The signs on the atomic sentences of a base indicate the truth value assigned to the sentence by the base: True if + is the prefix and false if - is the prefix. This is a modification of the device of signed formulas of [Smullyan 1968]. Sentences that are not atomic may receive a truth value determined by semantic rules for the connective, quantifier, and for the set terms. These rules are expressed in the following definition:

3.2. The semantic successor *sc*(*Snt*) of a set *Snt* of signed sentences consists of the members of *Snt* together with the following signed sentences:

1. $\pm(P|Q)$, whenever both $-P$ and $-Q$, respectively one of $+P$ or $+Q$ is in *Snt*.
2. $\pm(v)P$, whenever each of $+[t/v]P$ for all t in δ, respectively at least one of $-[t/v]P$ for some t in δ, is in *Snt*.

3. $\pm[\underline{s}/\underline{v}]t \in (t{:}P)$, whenever respectively $\pm[\underline{s}/\underline{v}]P$ is in *Snt*, where $\underline{v}$ is $v_1,\ldots,v_k$, all the abstracted variables of $(t{:}P)$, and $\underline{s}$ is $s_1,\ldots,s_k$, any members of δ.

Note that the rule 3.2.3 assures that a set $(t{:}P)$ is treated symmetrically with respect to truth and falsehood: For example $\pm(r,t)\in((u,v){:}P)$ is in $sc(Snt)$ if $\pm[r,t/u,v]P$ respectively is in *Snt*. The set $(w{:}(Eu)(Ev)(w{=}(u,v) \wedge P))$, where w does not occur free in P, does not enjoy that symmetry.

3.3. The semantic closure $cl(Snt)$ of a set *Snt* of signed sentences is the union of the sets Snt_μ, for ordinals μ, defined as follows:

1. Snt_0 is *Snt*;
2. $Snt_{\mu+1}$ is $sc(Snt_\mu)$;
3. Snt_μ for a limit ordinal μ is the union of Snt_ν for $0 \leq \nu < \mu$.

3.4. Theorem. There is an ordinal μ for which $cl(Snt)$ is Snt_μ.

Proof. There are denumerably many signed sentences. The sets Snt_μ form an increasing chain of sets of signed sentences. Therefore if μ is the first non-denumerable ordinal, then $sc(Snt_\mu) \subseteq Snt_\mu$.

End of proof.

There may, of course, be a much smaller ordinal than the first non-denumerable ordinal for which $cl(Snt)$ is Snt_μ.

The following theorem can be proved directly by induction on the ordinals:

3.5. Theorem. For no base *Bse*, sentence *P*, or bound variable variant *P'* of *P*, is both $+P$ and $-P'$ in $cl(Bse)$.

Not all bases provide an interpretation of NaDSetI, only those satisfying an additional condition:

3.6. A base *Bse* is an interpretation if whenever $+r \in t$ and $-s \in t$ are in $cl(Bse)$ then $-r = s$ is also.

3.7. Theorem. There is a base that is an interpretation.

Proof. Let *Bse* contain all signed atomic sentences $+r = s$ for which s is a bound variable variant of r and all sentences $-r = s$ for which it is not, and contain all signed atomic sentences $-r \in s$. That *Bse* is an interpretation follows easily.

End of proof.

A base *Bse* that is an interpretation, is an interpretation of first order logic with identity. For if clause 3.2.3 is dropped from the definition 3.2 of semantic successor, then $cl(Bse)$ defines the set of first order sentences that is true and the set of first order sentences that is false in the interpretation, and every first order sentence is either true or false. The semantics of NaDSetI is therefore a direct extension of first order semantics. However the addition of the clause 3.2.3 affects sentences that are not first order; there are sentences P that are undecided by the base Bse in the sense that neither $+P$ nor $-P$ is in $cl(Bse)$. The sentence

$$(x{:}\sim x \in x) \in (x{:}\sim x \in x)$$

is an example of a sentence not decided by any interpretation.

Although the classical semantic rules 3.2.1 and 3.2.2 have been used for the propositional connective and the universal quantifier, nevertheless the addition of the rules 3.2.3 has a subtle effect on the meaning of the logical connectives and the quantifiers.

Let P be a formula in which only the variable v occurs free and consider the sentence $(v)P$. Let the range of (v) for P in an interpretation Bse be the set of constant terms t of δ for which $+((v)P \supset [t/v]P)$ is in $cl(Bse)$. For first order logic formulas P, as for all formulas for which $(v)P$ is decided, the range is clearly δ. But the range for some P may be a proper subset of δ. For if $(v)P$ is not decided, then the range of P consists of those t for which $[t/v]P$ is true.

In the definition of base and interpretation, and in the proof of theorem 3.7, no distinction has been drawn between individual and set constants, and it may be asked why the two different kinds of constants are admitted. Individual constants are treated as names of individuals in the usual interpretations of first order logic. This is the way in which such a constant c may be interpreted in a context $c \in r$, or $c = r$, or $r = c$. However the interpretations discussed in this paper all have δ as their universe so that the individual denoted by c in these contexts is c itself. But in a context $t \in c$ it is used like the name of a set, the singleton set with c as its only member. A set constant C on the other hand has a context $t \in C$ as its natural one. In the context $C \in r$ it is treated as a name for itself.

This distinction between the interpretations of individual constants and set constants is only important for section 5 in which descriptions are introduced.

The fact that δ, the set of all constant terms, is the universe for interpretations of NaDSetI means that sentences such as

$$(u{:}u{=}u) \in (u{:}u{=}u),$$

which can easily be shown to be true in all interpretations, receive a nominalist interpretation. The occurrence of '$(u{:}u{=}u)$' to the left of $\in$ is actually a name for itself; the string occurring to the left of $\in$ is therefore mentioned. The occurrence of the string to the right of $\in$ is however being used as a name for a set; namely for the set of elements t of δ for which $t = t$ is true. In this sense no set has another set as member; it can have only names of sets as members. This nominalist interpretation is also used for the second order theory NaDSet. [Gilmore 1971] has a lengthy discussion of this nominalist interpretation.

4. Logical syntax of NaDSetI.

The theory will be presented as a Gentzen sequent calculus. However sequents will be represented as finite sets of signed formulas: The set $\{-P_1,...,-P_m,+Q_1,...,+Q_n\}$ represents the sequent $P_1,...,P_m \rightarrow Q_1,...,Q_m$. The theory can equally well be presented in any system of natural deduction.

4.1. Axioms

1. Base: for any atomic formula *afl*, and bound variable variant *afl'* of it, $\{-afl, +afl'\}$.

These axioms differ from the axioms $\{-P,+P\}$ in the original Gentzen first order logics [Gentzen 1934] in two respects. First P was any formula not just an atomic formula. This respect is unimportant since all first order instances of $\{-P,+P\}$ are derivable from the more restricted axiom set $\{-afl,+afl\}$; the restricted axiom scheme removes redundancies from the Gentzen axiom scheme for first order logic.

However it is essential to note that the elementary formulas $t \in v$, v a variable, are not atomic. The theory proposed in [Gilmore 1968] made the error of allowing $P \rightarrow P$ to be an axiom for any formula P. This can be shown to lead to inconsistency by using the set term $(x{:}(Eu)(u{=}x \wedge \sim x{\in}u))$.

The second respect in which the axioms 2.1 differ from the Gentzen axioms is their reference to bound variable variants. In the Gentzen systems, $\{-P,+P'\}$ is derivable if P' is a bound variable variant of P. In the logic presented here it is not possible to derive sequents like $\{-r{\in}C,+s{\in}C\}$, where s is a bound variable variant of r, from axioms $\{-P,+P\}$. This is an expected result because of the nominalist interpretation given to the assertion of formulas such as $r \in C$.

The remaining axioms are derivable in the second order theory in which $=$ is a defined relation, but are needed in the first order theory:

4.1.2. Identity: for any term r and bound variable variant s of r, $\{+r{=}s\}$.

The rules of deduction are those needed for first order logic with identity, together with rules for set abstraction.

4.2. Rules of deduction. For any sequents Seq and Seq', and formulas P and Q:

1. Propositional

$$\frac{Seq\cup\{+P\} \qquad Seq'\cup\{+Q\}}{Seq\cup Seq'\cup\{-(P|Q)\}}$$

$$\frac{Seq\cup\{-P,-Q\}}{Seq\cup\{+(P|Q)\}}$$

2. Quantificational. For any term t and variables u and v:

$$\frac{Seq\cup\{-[t/v]P\}}{Seq\cup\{-(v)P\}}$$

$$\frac{Seq\cup\{+[u/v]P\}}{Seq\cup\{+(v)P\}}$$

provided u does not occur free in P or any formula of Seq.

3. Abstraction. Let $\underline{v}$ be $v_1,\ldots,v_k$, all the abstracted variables of the set term

$(t{:}P)$, and let $\underline{s}$ be $s_1,\ldots,s_k$, any terms.

$$\frac{Seq\cup\{\pm[\underline{s}/\underline{v}]P\}}{Seq\cup\{\pm[\underline{s}/\underline{v}]t\in(t{:}P)\}}$$

4. Thinning

$$\frac{Seq}{Seq\cup\{\pm P\}}$$

5. Cut

$$\frac{Seq\cup\{-P\} \qquad Seq'\cup\{+P\}}{Seq\cup Seq'}$$

The thinning rule is the only one of the Gentzen structural rules that needs to be maintained.

The last rule is one that can be derived in the second order logic in which $=$ is defined:

6. Identity. For any terms r, s and t:

$$\frac{Seq\cup\{-r\in t\} \qquad Seq'\cup\{+s\in t\}}{Seq\cup Seq'\cup\{-r=s\}}$$

Although there are sentences that are not assigned a truth value by interpretations, there is no difficulty in defining satisfiability and validity for sequents.

4.3. A sequent Seq is satisfied in an interpretation Bse if there are $t_1,\ldots,t_k$ from δ for which

$$[t_1,\ldots,t_k/v_1,\ldots,v_k]Seq\cap cl(Bse)$$

is not empty; here $v_1,...,v_k$ are all the variables with free occurrences in formulas of *Seq*, and the effect of the simultaneous substitution operator being applied to *Seq* is the obvious.

A sequent *Seq* is valid if the above intersection is non-empty for all $t_1,...,t_k$ from δ and all interpretations *Bse*.

4.4. Theorem. All sequences derivable in NaDSetI are valid.

Proof is by induction on the length of derivations.

That any instance of the axiom schemes 4.1 is valid follows immediately from definition 3.1. That the conclusion of a rule of deduction is valid if each of its premisses is valid can be simply demonstrated for each rule.

End of proof.

4.4.1. Corollary. NaDSetI is consistent.

Another proof of consistency is possible:

4.5. Theorem. Cut is a derivable rule of NaDSetI.

Any proof that cut is a derivable rule of first order logic, for example the one in [Gentzen 1934] or [Smullyan 1968], can be easily extended to NaDSetI; it is only necessary to include a case for the abstraction rules. The quantificational rules offer no special difficulties because of the following fact: If *Seq* is an axiom of NaDSetI, then so also is $[t/v]Seq$ for any v and t. Since this is not true for the second order theory NaDSet, a proof that cut is a derived rule is no longer easy for that theory.

Consistency is also a consequence of 4.5 because without cut it is impossible to derive the empty sequent { }.

The fact that an elementary proof of consistency is possible means that NaDSetI cannot adequately formalize arithmetic. Only in NaDSet are all of Peano's axioms derivable. The first order completeness proof of [Smullyan 1968] appears to be extendable to NaDSetI.

The ordered pair is needed for the next section. It is defined in the usual way:

4.6. (s,t) for $(v{:}\, v=(u{:}\, u=s) \lor v=(u{:}\, u=s \lor u=t))$

Two rules of deduction express the essential properties of ordered pair:

4.7. Theorem. The following rules are derivable

$$\frac{Seq \cup \{-s=s', -t=t'\}}{Seq \cup \{-(s,t)=(s',t')\}}$$

$$\frac{Seq \cup \{+s=s'\} \qquad Seq' \cup \{+t=t'\}}{Seq \cup Seq' \cup \{+(s,t)=(s',t')\}}$$

A proof can be adapted from the proof of 417 of [Quine 1952].

5. Functions, Descriptions, and Extensionality.

The customary functional notation of mathematics is not a necessary part of logic since formulas in which they appear can be translated into formulas in which they do not appear. But nevertheless the notation is a convenient shorthand and its introduction into NaDSetI gives insights into the theory's intensional character. Descriptions are used to define functional application in the customary fashion. Their introduction into NaDSetI is made easy by the Gentzen sequent presentation of the logic. The functional notation is used in a brief discussion of the lambda calculus, and of extensionality.

In [Russell 1905] functional application was defined in terms of descriptions:

5.1. (rt) for $(\iota v{:}(v,t) \in r)$

Here the functional application notation of the lambda calculus has been adopted, and the notation for descriptions has been changed slightly from that of Russell; see [Quine 1951] page 149. The notation adopted here keeps iota right side up, and suggests that descriptions are a special case of set abstraction, rather than the application of a special quantifier.

The theory NaDSetI is expanded to accommodate descriptions by extending the elementary syntax to include description terms, and by adding a new rule for the logical syntax. The resulting theory is called NaDSetI with descriptions, or just NaDSetId.

5.2. The elementary syntax of NaDSetId is that of NaDSetI but for the following changes in the definition 2.4 of term:

1. The abstracted term t of a set term $(t{:}P)$ must be a term of NaDSetI; that is, no description term may occur in it.
2. Description terms $(\iota t{:}P)$ are terms where t is any term without descriptions and P is any formula. Free occurrences of variables in $(\iota t{:}P)$ are defined exactly as for $(t{:}P)$.

The restriction 5.2.1 is introduced to ensure that a set term has a precise meaning.

The logical syntax of NaDSetId treats descriptions much in the same manner as [Hilbert and Bernays 1934]. The scope of a description, in the sense of Russell, is always the full context in which it appears.

5.3. The logical syntax of NaDSetId:

1. The axioms of NaDSetI are the axioms of NaDSetId; thus no description can occur in an axiom.
2. The formula P in the conclusion of the thinning rule cannot contain any descriptions.
3. A rule for descriptions is introduced: Let Q be a formula in which no description occurs and let r be any term in which the variable v does not occur free. Let Seq' be obtained from Seq by replacing an occurrence of the term r in a signed formula of Seq by the description $(\iota t{:}Q)$. Then

$$\frac{\{+r\in(\iota t{:}Q)\}\quad \{+(\underline{u})(\underline{v})(Q \wedge [\underline{v}/\underline{u}]Q \supset \underline{u}{=}\underline{v})\}\quad Seq}{Seq'}$$

Here $\underline{u}$ are all the free variables of t, $\underline{v}$ are variables of the same number distinct from $\underline{u}$ and not occurring free in Q, and $\underline{u} = \underline{v}$ is the conjunction $u_1 = v_1 \wedge \cdots \wedge u_k = v_k$. Note that that the occurrence of r replaced by $(\iota t{:}Q)$ is generally not a free occurrence.

Restricting the axioms of NaDSetId to being the axioms of NaDSetI and restricting thinning is necessary if descriptions are to maintain their meaning. To use two examples of Russell, it is not possible to assert: The present king of France is bald or the present king of France is not bald. For France does not presently have any king. However it is possible, for example, to assert that the author of the Waverly novels is the author of the Waverly novels, or that he was or was not bald, since the novels do have a single author Scott.

The rule for descriptions has three premisses. The first assures that r can possibly be the t such that Q, and the second assures that it is the only possible candidate. The first two premisses justify the conclusion being drawn from the third premiss.

An instance of the rule may help clarify its use. The instance involves lambda, or functional abstraction:

5.4. $(\lambda t{:}r)$ for $((v,t){:}v{=}r)$,

where v does not occur free in t or r.

This is a generalization of the usual λ abstraction notation in the same fashion that the set abstraction notation was generalized: The term t replaces a variable, although it may of course be a variable, but may not contain any descriptions.

In the statement of the following theorem, a formula P is said to be derivable if the sequent $\{+P\}$ is derivable.

5.5. Theorem. $((\lambda t{:}r)[\underline{s}/\underline{u}]t)=[\underline{s}/\underline{u}]r$ is derivable for any terms t, r, and $\underline{s}$ without descriptions; here $\underline{u}$ are all the variables occurring free in t.

Proof. From 5.1 and 5.4 it is sufficient to derive

$(\iota v{:}(v,[\underline{s}/\underline{u}]t) \in ((v,t){:}v{=}r)) = [\underline{s}/\underline{u}]r$, that is

$(\iota v{:}Q) = [\underline{s}/\underline{u}]r$, where

Q is $(v,[\underline{s}/\underline{u}]t) \in ((v,t){:}v{=}r)$.

An application of 5.2.3 with this formula as conclusion will have r of 5.2.3 replaced by $[\underline{s}/\underline{u}]t$, and have premisses:

$\{+[\underline{s}/\underline{u}]r \in (v{:}Q)\}$,

$\{+(v)(x)(Q \wedge [x/v]Q \supset v{=}x)\}$, and

$\{+[\underline{s}/\underline{u}]r = [\underline{s}/\underline{u}]r\}$.

The last of these is an axiom. The first follows by abstraction from the last by observing that the last is $[\underline{s},[\underline{s}/\underline{u}]r/\underline{u},v]v{=}r$, and that $([\underline{s}/\underline{u}]r,[\underline{s}/\underline{u}]t)$ is $[\underline{s},[\underline{s}/\underline{u}]r/\underline{u},v](v,t)$. The second of these follows from the axioms $\{-v{=}[\underline{s}/\underline{u}]r,\ +v{=}[\underline{s}/\underline{u}]r\}$ and $\{-v{=}x,\ +v{=}x\}$ by abstraction by observing that $v{=}[\underline{s}/\underline{u}]r$ is $[\underline{s}/\underline{u}]v{=}r$ and $(v,[\underline{s}/\underline{u}]t)$ is $[\underline{s}/\underline{u}](v,t)$.

End of proof.

The rule 5.3.3 for descriptions adds little to the deductive power of NaDSetI:

5.6. Theorem. In a derivation it may be assumed that no application of 5.3.3 precedes the application of any of the rules 4.2.

Proof. Consider any derivation in which the conclusion of an application of 5.3.3 is a premiss of an application of one of the rules 4.2. Consider a first such application; that is, consider an application of a rule 4.2 in which each premiss has the desired property, but at least one premiss is the conclusion of an application of 5.3.3. This application will be referred to as the designated application.

It may be assumed that the designated application is not an application of either of the propositional rules, the $+(v)$ rule, or the thinning rule. For if that were the case then the application of 5.3.3 could just as well follow the application of the other rule.

The rules requiring a special argument are therefore the $-(v)$ rule, both abstraction rules, cut and identity, and the reasons for requiring a special argument are similar. In the case of $-(v)$ one occurrence of t in $-[t/v]P$ of the premiss may have the occurrence of $(\iota t{:}Q)$ introduced by the application of 5.3.3. Clearly that application of 5.3.3 cannot follow the application of $-(v)$. In the case of either of the abstraction rules one occurrence of an s in $\underline{s}$ may have the occurrence of $(\iota t{:}Q)$. In the case of cut one of $-P$ and $+P$ may have the occurrence, and in the case of identity, the t of one of $-r \in t$

and $+s \in t$.

All applications of 5.3.3 in a derivation of a premiss of the designated application follow any application of any rule of 4.2. Because of the restrictions on axioms and on thinning, the first application of 5.3.3 in the derivation of a premiss for the designated application must have a premiss without description terms. By postponing or dropping all applications of 5.3.3, correct premisses are obtained for the designated application.

End of proof.

Since no application of 5.3.3 need precede any application of the rules 4.2, the method used for constructing a derivation without any cut from one with cuts, can equally well be applied to derivations of NaDSetId:

5.6.1. Corollary. Cut is a derivable rule of NaDSetId.

Thus NaDSetId, like NaDSetI, is consistent.

The identities of theorem 5.6 resemble the identities (β) of [Scott 1975] of the axioms of extensional λ-calculus. The full development of the λ-calculus within NaDSetId is beyond the scope of this paper. Nevertheless, it is natural to ask whether identities similar to the other axioms can be derived. The axioms (α) of [Scott 1975] clearly relate to the axioms 4.1.2. Each instance of the remaining axioms (ξ) of [Scott 1975] is the conjunction of the following two formulas:

($\xi 1$) $(\lambda u{:}r)=(\lambda u{:}s) \supset (u)r=s$

($\xi 2$) $(u)r=s \supset (\lambda u{:}r)=(\lambda u{:}s)$

A derivation of ($\xi 1$) for any terms r and s without descriptions is an easy exercise; indeed it is possible to derive a stronger form of it in which the $=$ of the antecedant is replaced by $=_e$, extensional identity, defined

$$r =_e s \quad \text{for} \quad (u)(u{\in}r \equiv u{\in}s),$$

where u does not occur free in r or s.

A derivation of ($\xi 2$) is not possible; it requires an additional rule of deduction for the theory, similar to an extensionality rule:

$$\frac{Seq \cup \{+r{=_e}s\}}{Seq \cup \{+r{=}s\}}$$

Whether this rule can be consistently added to the other rules of NaDSetI or whether NaDSetI, like the theory of [Gilmore 1967] is inconsistent with an extensionality rule, is not known. If it could be added, then descriptions could be defined in the manner of [Quine 1951] and the rule for descriptions would not be needed. But it is not clear that the addition of such a rule to NaDSetI is desirable or necessary for its intended applications.

6. A second order theory NaDSet.

6.1. The elementary syntax of NaDSet is but a slight variation of that of NaDSetI.

1. Elementary terms now include set variables X, Y, Z with or without numeral subscripts.
2. Elementary formulas are $t \in r$, where r is any term and t is any term in which no set variable occurs free; $t \in r$ is atomic if r is an individual constant, a set constant, or a set variable.
3. The definition of formulas is unchanged. However, note that v in 2.3.3 may now be a set variable, as well as an individual variable.
4. The definition of term is unchanged except for a restriction on the abstracted term t of a set term: No set variable may occur free in t.

The definition of base and semantic successor for NaDSetI was greatly simplified by the elements in the range of the individual variables being constant terms and therefore terms of the theory. For NaDSet the range of the individual variables is still the set δ of constant terms, although it is a set that now has terms that are not terms of the first order theory. To provide an equally simple definition of base and semantic successor for the second order theory, however, requires the use of a device of [Robinson 1951] for the range of the set variables. The device consists in extending the concept of constant by admitting a possibly non-denumerable set Δ of constants in the definition of term and formula.

Δ is any set that includes all the individual and set constants of the formal theory, but in addition may include any number of additional constants; these latter will be called special constants. However, just as free occurrences of set variables may not appear in a term t of an elementary formula $t \in r$, or in the abstracted term t of a set term $(t{:}P)$, so also no special constant may appear.

Terms and formulas of the enlarged elementary syntax will be called special terms and formulas. As before, a sentence is a formula without free variables. Thus an atomic special sentence is a sentence $t \in r$ where t is in δ and r is in Δ.

6.2. A base *Bse* on Δ is a set of signed atomic special sentences satisfying the following conditions:

1. For each t in δ and C in Δ, one and only one of $\pm t \in C$ is in *Bse*;
2. If P' is a bound variable variant of P, then $\pm P'$ is in *Bse* if and only if $\pm P$ respectively is in *Bse*.

6.3. The semantic successor *sc*(*Snt*) of a set *Snt* of signed special sentences is defined as in 3.2, except for the restriction of the clause 3.2.2 to individual variables, and the addition of one clause for the set variables:

1. $\pm(v)P$, where v is a set variable, whenever each of $+[C/v]P$ for all C in Δ, respectively at least one of $-[C/v]P$ for some C in Δ, is in *Snt*.

The definition of semantic closure, *cl*(*Snt*), of a set *Snt* of signed special sentences is unchanged from 3.3. Theorem 3.4 is still correct. However since a set

Snt of signed special sentences will be non-denumerable if Δ is non-denumerable, the proof of 3.4 no longer applies. However, since Δ is of fixed cardinality there will be such an ordinal.

As in the case of NaDSetI, not every base provides an interpretation for the theory.

6.4 Covers and Interpretations

1. Let T be any special constant term.

   A special constant C from Δ is said to cover T for a base *Bse* if for all t in δ, $\pm t \in C$ is in *Bse* whenever $\pm t \in T$ respectively is in *cl*(*Bse*).

2. A base *Bse* is an interpretation of NaDSet if for each special constant term T there is a special constant C covering it for *Bse*.

6.5. Theorem. There exists an interpretation of NaDSet.

Proof. Consider the following base *Bse*. Let Δ contain a special constant for each subset of δ that is closed with respect to bound variable variants. For each such special constant C and all t in δ, $\pm t \in C$ is in *Bse* if t is, respectively is not, a member of the set corresponding to C. For the members of C of Δ that are not special constants, $-t \in C$ is taken to be in *Bse* for all t in δ. Clearly each constant term is covered by a special constant for *Bse*.

End of proof.

6.6. The logical syntax of NaDSet is but a slight variation of that of NaDSetI.

1. The axioms include only the base axioms 4.1.1. Note however that the enlarged definition of atomic in 6.1.2 enlarges the base axiom scheme.
2. The quantificational rules 4.2.2 must be altered to reflect that there are both individual and set variables: In the $-(v)$ rule no set variable may occur free in t if v is an individual variable. In the $+(v)$ rule, u and v must both be individual variables, or both set variables.
3. No set variable may occur free in the terms $\underline{s}$ of the abstraction rules 4.2.3.
4. The identity rule 4.2.6 is dropped.

The treatment of descriptions in NaDSet is unchanged from that of NaDSetI with identity defined: $r{=}s$ for $(X)(r{\in}X \supset s{\in}X)$.

The definition 4.3 of satisfiability and validity of sequents carries over for sequents of NaDSet, as does the theorem 4.4. NaDSet is therefore consistent. Whether 4.5, the redundancy of cut, can be proved for NaDSet is not known, although it is plausible. Also plausible is a proof of the completeness of the theory in the sense of [Henkin 1953]. A proof of the redundancy of cut may be a by-product of a completeness proof.

7. Real Analysis in NaDSet

The development of second order arithmetic within NaDSet can proceed without any difficulties. First a zero is chosen:

$$0 \text{ for } (u{:}u \neq u).$$

Then a successor function must be chosen. The function $(\lambda u{:}(x{:}x{=}u))$ that forms the singleton set from an argument will do. An abbreviation for applications of it to an argument is defined:

$$t' \text{ for } ((\lambda u{:}(x{:}x{=}u))t).$$

Finally the set of natural numbers is defined:

$$N \text{ for } (u{:}(X)(0 \in X \wedge (v)(v{\in}X \supset v' \in X) \supset u{\in}X))$$

The definition of the set of natural numbers illustrates the usual role of second order variables in inductive definitions. Because set terms are a part of NaDSet, such inductive definitions have very broad uses in NaDSet.

That the set of natural numbers so defined has all the properties expected of such a set is easy to verify. Second order classical arithmetic can be developed within NaDSet. Inductive definitions of the sum and product predicates can be given in their usual form and a functional notation introduced as well.

The set *Int* of integers, negative and positive, can be defined in the usual way along with an identity and an order $<$. The set of rationals is then

$$Rat \text{ for } ((x,y){:}x{\in}Int \wedge y{\in}Int \wedge 0{<}y)$$

Finally the reals can be defined as Dedekind cuts:

$$Real \text{ for } (\{x,y\}: x \cup y =_e Rat \wedge x \cap y =_e 0$$

$$\wedge (u)(v)(u \in x \wedge v \in y \supset u < v))$$

Here $\cup$ and $\cap$ are defined in the usual way. Again addition, multiplication, identity and ordering for reals can be defined, and a classical real analysis developed. It would be of interest to know how much of this analysis could be given an intuitionistic form by restricting the sequents of NaDSet to containing at most a single +formula.

One kind of classical argument, however, apparently fails within NaDSet. This is the argument establishing for example the non-denumerability of the reals, or that a power-set of a set is of greater cardinality than the set. A formalization of the arguments within NaDSet would require accepting as axioms $\{-t \in var, +t' \in var\}$ for individual variables *var*. It was such axioms that led to the inconsistency of [Gilmore 67].

The failure of these arguments within NaDSet is not important for its intended applications.

BIBLIOGRAPHY

Beth, E.W., Semantic entailment and formal derivability, Proc. Kon. Neder. Akad. van Weten, afd. Letter, Niewe Reeks, 18, 1955.

Church, Alonzo, The Calculi of Lambda-Conversion, Princeton University Press, 1941.

Feferman, Solomon, Toward Useful Type-Free Theories, I, Journal of Symbolic Logic, pp. 75-111, March 1984.

Fitch, Frederick B., An extension of basic logic, Journal of Symbolic Logic, Vol. 3, pp. 95-116, 1948.

Symbolic Logic, New York, 1952. A complete and consistent modal set theory, Journal of Symbolic Logic, Vol. 32, 1967.

Elements of Combinatory Logic, Yale University Press, New Haven, Conn. 1974.

Gentzen, Gerhard, Untersuchungen ü ber das logische Schliessen, Mathematische Zeitschrift, Vol. 39, pp. 176-210, 405-431, 1934-5.

Gilmore, P.C., The consistency of partial set theory without extensionality, a paper presented at the 1967 Institute on Set Theory, UCLA; Proc. Symposia Pure Maths. XIII, Part II, Amer. Math. Soc., pp. 147-153, 1974.

A Formalized Naive Set Theory, a paper delivered to the Summer Conference on Intuitionism and Proof Theory, Buffalo, New York, August 1968.

A Consistent Naive Set Theory: Foundations for a Formal Theory of Computation, IBM Research Report RC 3413, June 22, 1971. A slightly revised form of this report appears as Combining Unrestricted Abstraction with Universal Quantification, pp. 99-124, in To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, editors J.P. Seldin and J.R. Hindley, Academic Press, 1980.

Gilmore, P.C., Morrison, R., Foundations for Self Referential Databases, in preparation.

Henkin, L., Completeness in the Theory of Types, Journal of Symbolic Logic 18, pp. 19-29, 1953.

Hilbert, D. and Bernays, P., Grundlagen der Mathematik I, Springer, 1934.

McCarthy, John, A Basis for a Mathematical Theory of Computation, pp. 33-70, Computer Programming and Formal Systems, editors P. Braffort, D. Hirschberg, North Holland, 1963.

Prawitz, Dag, Natural Deduction, Almqvist and Wiksell, 1965.

Quine, Willard van Orman, Mathematical Logic, Harvard University Press, Revised Edition, 1951.

Robinson, Abraham, On the Metamathematics of Algebra, North Holland, 1951.

Russell, Bertrand, On denoting, Mind, new series 14, pp. 479-493, 1905.

Scott, Dana, Combinators and Classes, pp. 1-26, λ-Calculus and Computer Science Theory, Lecture Notes in Computer Science 37, Springer, 1975.

Smullyan, R., First-Order Logic, Springer-Verlag, 1968.

Troelstra, A.S., Principles of Intuitionism, Lecture Notes in Mathematics No. 95, pp. 111, Springer-Verlag, 1969.