NUMERATION MODELS OF $\lambda\beta$-CALCULUS

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Abstract

Numeration models of extensional λ-calculus have been studied (see [5,7]). In this paper, we study numeration models of $\lambda\beta$-calculus. Engeler's graph algebra construction [3] is applied to the category of numerations and is used as a tool to obtain numeration models of $\lambda\beta$-calculus. Several classes of numeration models are studied and several examples of them are presented.

§1. $\lambda\beta$-calculus

The $\lambda\beta$-calculus developed by Church [2] is the following formal system: Let V be a countable set of variables. A λ-term is either a variable $x \in V$, application (MN) of λ-terms M and N, or abstraction (λx.M) of a λ-term by a variable x. T denotes the set of all λ-terms.

We assume a natural meaning of a λ-term occurring in some other λ-term. An occurrence of a variable x in M is ***bound*** if it is inside a part of M of the form (λx.M). Otherwise it is ***free***. For any terms M, L and a variable x, the result of substituting L for each free occurrence of x in M (and changing bound variables to avoid clashes) is denoted by M[x:=L].

The calculus has the following two reduction rules:

Reduction Rules

$(\alpha):(\lambda x.M) \rightarrow (\lambda y.M[x:=y])$ x is not bound in M and y does not occur in M

$(\beta):((\lambda x.M)L) \rightarrow M[x:=L]$

□

By Gödel numbering variables and λ-terms we can realize constructions of λ-terms as a system of recursive functions. Let $v:N \rightarrow V$ and $\tau:N \rightarrow T$ be computable bijections. The syntax of λ-terms corresponds to the following system of recursive functions:

$is\text{-}var(n) \iff \tau(n) \in V$

$is\text{-}apply(n) \iff \tau(n) = (ML)$ for some $M,L \in T$

$is\text{-}abst(n) \iff \tau(n) = (\lambda x.M)$ for some $x \in V$ and $M \in T$.

$\tau(inc(n)) = v(n)$

$is\text{-}var(n) \Rightarrow v(var(n)) = \tau(n)$

$is\text{-}apply(n) \Rightarrow \tau(apply(rator(n), rand(n))) = \tau(n)$

$is\text{-}abst(n) \Rightarrow \tau(abst(bound(n), body(n))) = \tau(n)$.

§2. NUMERATION MODELS OF λβ-CALCULUS

Definition 2.1. (Ersov [4]).

A *numeration (of a set* X) is a surjection $\gamma: N \to X$. A *morphism* from a numeration $\gamma_1: N \to X_1$ to another $\gamma_2: N \to X_2$ is a function $f: X_1 \to X_2$ such that for some recursive function r_f, $f \cdot \gamma_1 = \gamma_2 \cdot r_f$. Such r_f is called a *realization* of f. In case r_f is primitive recursive, we say f is *primitive*.

□

It can readily be seen that numerations and morphisms form a category. (See Ersov [4]).

Let $\gamma: N \to X$ be a numeration such that for some numeration $\gamma\!\uparrow: N \to Hom(\gamma,\gamma)$, $\gamma \rhd \gamma\!\uparrow$ in the category of numerations. Let $v: N \to V$ be the computable bijection discussed in §1. Furthermore let $(\Phi: \gamma \to \gamma\!\uparrow, \Psi: \gamma\!\uparrow \to \gamma)$ be the retraction pair, i.e. $\Phi(\Psi(f)) = f$.

An *environment* (or *valuation)* is a primitive morphism from ν to γ. We write Env to denote the set of all environments. Using a Gödel numbering $<\psi_i>$

of primitive recursive functions N → N, we can introduce a numeration σ:N → Env as follows:

$$\sigma_i = \rho \quad \text{where} \quad r_\rho = \psi_i$$

It can readily be seen that updating an environment

$$\rho[x := d](z) = \text{if } x = z \text{ then d else } \rho(z)$$

where x∈V and d∈X has a realization, i.e.

$$\sigma_i[v(n) := \gamma(m)] = \sigma_{update(i,n,m)}$$

for some recursive function $update$: $N^3 \rightarrow N$. In other word, updating operation is a morphism from $\sigma \times \nu \times \gamma$ to σ.

Definition 2.2.

Let γ be as above. We say γ is a *numeration model of* $\lambda\beta$*-calculus* iff the following interpretation function ξ:

$$\xi(\tau(n),\sigma_i) := \text{if } is\text{-}var(n) \text{ then } \sigma_i(\tau(n))$$
$$\text{else if } is\text{-}apply(n) \text{ then}$$
$$\Phi(\xi(\tau(rator(n),\sigma_i))(\xi(\tau(rand(n)),\sigma_i))$$
$$\text{else if } is\text{-}abst(n) \text{ then}$$
$$\Psi(\lambda x \in X.\ \xi(\tau(body(n)),\sigma_i[\tau(bound(n)) := x]))$$

is well-defined and it is a morphism from $\tau \times \sigma$ to γ.

□

It is important to notice that since ξ is a morphism from $\tau \times \sigma$ to γ, $\lambda x \in X.\ \xi(\tau(body(n)), \sigma_i[\tau(bound(n)):=x])$ is a morphism from γ to γ realized by $\lambda m \in N. r_\xi(body(n), update(i, bound(n), m))$. Thus

$$\Psi(\lambda x \in X.\ \xi(\tau(body(n)), \sigma_i[\tau(bound(n)):=x]))$$

is defined. Furthermore the next theorem supports the relevance of this definition:

Theorem 2.3.

Let γ be a numeration model of $\lambda\beta$-calculus with an interpretation morphism $\xi: \tau \times \sigma \rightarrow \gamma$, then we have:

$$\tau(n) \xrightarrow{\lambda\beta} \tau(m) \text{ implies for all } i \in N,\ \xi(\tau(n), \sigma_i) = \xi(\tau(m), \sigma_i)$$

where $\tau(n) \xrightarrow{\lambda\beta} \tau(m)$ means that τ(n) can be reduced to τ(m) by one of the reduction rules of $\lambda\beta$-calculus.

□

Definition 2.4.

A numeration model γ is λ-***representable*** iff there is a recursive function ***rep***: N→N such that

$$\gamma(n) = \xi(\tau(rep(n)), \sigma_i) \text{ for all } i \in N.$$

A λ-representable numeration model γ is λ-definable iff there is a recursive function ***def*** such that if a morphism f: $\gamma \rightarrow \gamma$ is realized by a recursive function ϕ_m then

$$f(\gamma(n)) = \xi((\tau(def(m))\tau(rep(n))), \sigma_i) \text{ for all } i \in N$$

where $<\phi_i>$ is a Gödel numbering of partial recursive functions.

□

Note. In a λ-representable numeration model γ: N→X, every element of X can be represented by a closed λ-term. If γ is λ-definable then every morphism $\gamma \to \gamma$ can be defined by some closed λ-term. Outstanding point here is that we can obtain such λ-term from a Gödel number of a recursive function which realizes the morphism.

§3. NUMERATED FUNCTIONAL DOMAINS

This section consists of modification of results in [5,7] for non extensional λ-calculus. Proofs of theorems can easily be obtained by suitably modifying proofs in [5,7], thus they are omitted.

Definition 3.1.

Let γ_1: $N \to X_1$ and γ_2: $N \to X_2$ be numerations. A numberation γ: $N \to Hom(\gamma_1, \gamma_2)$ is ***acceptable*** iff there are recursive functions ***realize, numerate***: N→N such that

(1) $r_{\gamma(n)} = \phi_{realize(n)}$

(2) if ϕ_n realizes $f: \gamma_1 \to \gamma_2$ then $\gamma($***numerate***$(n)) = f$.

□

It can readily be seen that (1) is equivalent to the existence of a (universal) recursive function $U: N^2 \to N$ such that

$$\gamma(n)(\gamma_1(m)) = \gamma_2(U(n,m)).$$

Also it is known that all acceptable numerations of $Hom(\gamma_1,\gamma_2)$ are recursively isomorphic (see [5]). This means that there is at most one acceptable numeration of $Hom(\gamma_1,\gamma_2)$. Thus we write $(\gamma_1 \rightarrow \gamma_2)$ to denote the acceptable numeration of $Hom(\gamma_1,\gamma_2)$, if any.

Definition 3.2.

A ***numerated functional domain*** (NFD) is a numeration γ: N→X satisfying:

(1) The acceptable numeration $(\gamma \rightarrow \gamma)$: N→Hom$(\gamma,\gamma)$ exists.

(2) $\gamma \triangleright (\gamma \rightarrow \gamma)$ in the category of numerations.

□

Proposition 3.3.

If γ: N→X is an NFD then it is a numeration model of $\lambda\beta$-calculus.

□

The converse of 3.3 does not hold. The existence of an interpretation morphism is not strong enough to prove that $\gamma\uparrow$ is acceptable.

We can given an algebraic characterization of NFD's. A ***countable applicative system*** is an algebra $(X,\cdot)$ where $\cdot$ is a binary operation over a countable set X. The set T(X) of ***terms*** (using countably many variables $x_0,x_1,\ldots$) over $(X,\cdot)$ is inductively defined as follows:

$x_i \in T(X)$

$a \in X \Rightarrow a \in T(X)$

$A, B \in T(X) \Rightarrow (A \cdot B) \in T(X).$

We assume that $\cdot$ associates to the left, also we drop $\cdot$ if it does not cause confusion. To denote that a term A has variables $x_0, x_1, \ldots, x_n$, we write $A(x_0, x_1, \ldots, x_n)$. Let ρ: N→T(X) be a Gödel numbering of terms.

Definition 3.4.

A *realizably combinatory algebra* (RCA) is a 5-tuple $(X, \cdot, \theta, \gamma, \rho)$ such that:

(1) $(X, \cdot)$ is a countable applicative system

(2) γ: N→X is a numeration

(3) $\cdot$ is a morphism from $\gamma \times \gamma$ to γ.

(4) There is a recursive function λ such that if $\rho(n) = A(x_1, \ldots, x_n)$ then $\gamma(\lambda(n))$ = f is a unique element of X satisfying:

$$f y_1 \ldots y_n = A(x_1 := y_1, \ldots, x_n := y_n)$$

where $A(x_1 := y_1, \ldots, x_n := y_n)$ is the result of substituting y_i for x_i in A $(1 \leq i \leq n)$.

(5) $\theta x_0 x_1 = x_0 x_1$

$\forall x \in X. (x_0 x = x_1 x) \Rightarrow \theta x_0 = \theta x_1$

$\theta\theta = \theta$

□

Definition 3.5.

An RCA $(X,\cdot,\theta,\gamma,\rho)$ is ***computationally complete*** iff there is a recursive function alg such that if ϕ_n realizes f: $\gamma \rightarrow \gamma$ then $\sigma(alg(\mathrm{n}))$ is a term with a free variable, say x and

$$f(z) = (\sigma(alg(n)))(x := z)$$

$\Box$

Proposition 3.6. (Characterization Theorem I)

(1) If $(X,\cdot,\theta,\gamma,\rho)$ is a computationally complete RCA then γ is a NFD, where $(\gamma \rightarrow \gamma)$:N$\rightarrow$Hom$(\gamma,\gamma)$ is defined by $(\gamma \rightarrow \gamma)(n) = \Phi(\gamma(n))$ where Φ maps elements of X to functions X$\rightarrow$X defined by $\Phi(x)(y) = x \cdot y$.

(2) If γ: N$\rightarrow$X is a NFD with a retraction pair $(\Phi:\gamma \rightarrow (\gamma \rightarrow \gamma), \Psi:(\gamma \rightarrow \gamma) \rightarrow \gamma)$ then $(X,\cdot,\theta,\gamma,\rho)$ is a computationally complete RCA where $\cdot$ is defined by:

$$x \cdot y = \Phi(x)(y).$$

and

$$\theta = \xi((\lambda x\, \lambda y.xy),\sigma_1).$$

$\Box$

This proposition is a numeration version of Barendregt's [1] and Meyer's [9] result. It is very important to notice that the class of computationally complete RCA's (or equivalently NFD's) is not the same as the class of numeration models. This indicates a difference between numeration models and set theoretical models. As shown in Meyer [9], in set theoretical case, models of $\lambda\beta$-calculus are the same as combinatory algebras. This difference is due to the following reasons:

(1) $\gamma \rhd \gamma\uparrow$ being a numeration model is not strong enough to imply

$\gamma\dagger: N \rightarrow Hom(\gamma,\gamma)$ being acceptable.

(2) To obtain the corresponding numerated combinatory algebra from $\gamma\dagger$, it is crucial to have acceptability of $\gamma\dagger$.

(3) To obtain a numeration model from a RCA, it is crucial to assume the computational completeness of the RCA.

§4. CHARACTERIZATION OF λ-DEFINABLE NUMERATION MODELS

Even though we can not show good characterization of numeration models of λβ-calculus, we can nicely characterize λ-definable models as a sub-class of NFD's.

Definition 4.1.

A NFD γ is ***λ-representable*** iff there is a recursive function rep:N→N such that:

$$\gamma(n) = \xi(\tau(rep(n)),\sigma_i) \quad \text{for all } i \in N.$$

where ξ is the interpretation morphism which makes γ a numeration model of λ-calculus.

□

By a slite modification of arguments for extensional λ-calculus (see [7]), we have:

Theorem 4.2.

If a numeration γ is a λ-definable numeration model then it is a λ-representable NFD.

□

Theorem 4.3.

If γ is a λ-representable NFD then it is a $\lambda\beta$-definable numeration model of λ-calculus.

□

Corollary 4.4. (Characterization Theorem II)

A numeration γ is a λ-definable numeration model iff it is a λ-representable NFD.

□

The proofs for these theorems establish the following relationship between acceptability of $\gamma\uparrow$ and λ-definability of a numeration model γ of $\lambda\beta$-calculus:

(1) If γ is λ-definable then $\gamma\uparrow$ is acceptable.

(2) If $\gamma\uparrow$ is acceptable and γ is λ-representable then γ is λ-definable.

This correspondance supports the relevance of the concept of acceptable numerations of morphism spaces discussed in [5].

By adding an extra condition to computationally complete RCA, we can characterize λ-definable numeration models. A computationally complete RCA $(X,\cdot,\theta,\gamma,\rho)$ is λ-*representable* iff there is a recursive function $rep : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\gamma(n) = \xi(\tau(rep(n)), \sigma_i) \quad \text{for all } i \in N.$$

Theorem 4.5. (Characterization Theorem III)

A numeration γ: N→X is a λ-definable numeration model iff (the corresponding) $(X, \cdot, \theta, \gamma, \rho)$ is a λ-representable computationally complete RCA.

□

§5. A NUMERATION MODEL CONSTRUCTION

A set theoretical construction of models of λβ-calculus is known (see Meyer [9] and Engeler [3]). We study a numeration version of this construction.

Before we prove the main result we present another characterization of RCA's.

Theorem 5.1.

$(X, \cdot, \theta, \lambda, \rho)$ is a RCA iff $(X, \cdot)$ is an applicative system such that $\cdot$ is a morphism from $\gamma \times \gamma$ to γ and there exist K,S ∈X satisfying:

$$Kx_0x_1 = x_0 \tag{1}$$

$$Sx_0x_1x_2 = (x_0x_2)(x_1x_2) \tag{2}$$

$$\theta x_0x_1 = x_0x_1 \tag{3}$$

$$\forall x \in X. (x_0x = x_1x) \Rightarrow \theta x_0 = \theta x_1 \tag{4}$$

$$\theta\theta = \theta \tag{5}$$

Proof. Due to the constructiveness of the Curry's proof to establish equivalence between countable applicative system with K,S and combinatory complete applicative systems.

□

Definition 5.2. (Engeler)

For any set X, define G(X) as follows:

$$G(X) = \bigcup_n G_n(X)$$

where

$$G_0(X) = X$$

$$G_{n+1}(X) = G_n(X) \cup \{(\alpha \rightarrow b) \mid b \in G_n(X),\ \alpha: \mathit{finite\ subset\ of}\ \ G_n(X)\}$$

where $(\alpha \rightarrow b) = (\alpha, b)$.

□

It can readily be seen that G(X) is the smallest set satisfying:

$$Y = X \cup \{(\alpha \rightarrow b) \mid b \in Y,\ \alpha: \text{finite subset of } y\}. \tag{5.1}$$

This construct yields a numeration $\gamma_\chi: N \rightarrow G(X)$ for a numeration χ:N→X.

Definition 5.3.

Let χ:N→X be a numeration. Define numerations $\gamma_n: N \rightarrow G_n(X)$ as follows:

$$\gamma_0 = \chi$$

$$\gamma_{n+1}(2m) = \chi(m)$$

$$\gamma_{n+1}(2m+1) = (\gamma_n(\kappa(m_1)) \rightarrow \gamma_n(m_2))$$

where $m = \langle m_1, m_2 \rangle$ and κ is the standard enumeration of finite sets of natural numbers. Finally let $\gamma_\chi: N \rightarrow G(X)$ be the numeration of G(X) obtained by dovetailing $\{\gamma_n\}$.

Definition 5.4.

Let $\gamma_X: N \rightarrow G(X)$ be as above. We say a subset $S \subset G(X)$ is ***computable*** iff $\{n \mid \gamma_X(n) \in S\}$ is a recursively enumerable set. We write $CP(G(X))$ to denote the set of all computable subsets of G(X).

□

Let $\{W_m\}$ be a Gödel numbering of recursively enumerable sets. Using this Gödel numbering, we can introduce a numeration $\pi_X: N \rightarrow CP(G(X))$ as follows:

$$\pi_X(m) = \gamma_X(W_m).$$

Definition 5.5.

For $M, N \in CP(G(X))$, define $M*N$ by:

$$M*N = \{b \mid \exists \beta.\beta: \textit{finite subset of } N, (\beta \rightarrow b) \in M\}.$$

□

Definition 5.6. (Ersov)

For any numeration χ, we define an equivalence relation $=_\chi$ over natural numbers by $i =_\chi j$ iff $\chi(i) = \chi(j)$. We say χ is ***positive*** if $=_\chi$ is semi-decidable.

□

It can readily be seen that γ_X is positive iff χ is positive.

Lemma 5.7.

$*$ is a morphism from $\pi_X \times \pi_X$ to π_X if χ is positive.

Proof. $\pi_X(m) \times \pi_X(n) = \{\gamma_X(j) \mid \exists i.\ \gamma_X(\kappa(i)) \overset{fin}{\subset} \gamma_X(W_n), (\gamma_X(\kappa(i) \rightarrow \gamma_X(j) \in \gamma_X(W_m)\}$ where $A \overset{fin}{\subset} B$ means A is a finite subset of B. Since γ_X is positive

$$\{j \mid \exists i.\ \gamma_X(\kappa(i)) \overset{fin}{\subset} \gamma_X(W_n), (\gamma_X(\kappa(i)) \rightarrow \gamma_X(j)) \in \gamma_X(W_m)\}$$

is r.e. Also we can compute a Gö del number of it from m and n.

□

Let ρ: $N \rightarrow T(CP(G(X)))$ be a Gö del numbering of terms over $CP(G(X))$.

Theorem 5.8.

$(CP(G(X)), *, \theta, \pi_X, \rho)$ is a RCA if χ is positive

where $\theta = \{(\alpha \rightarrow (\beta \rightarrow b)) \mid \alpha, \beta\text{: finite subsets of } G(X), b \in \alpha * \beta\}$.

Proof. First notice that $\theta \in CP(G(X))$. Define K,S by:

$$K = \{(\alpha \rightarrow (\beta \rightarrow b)) \mid \alpha, \beta \overset{fin}{\subset} G(X), b \in \alpha\}$$

$$S = \{(\alpha \rightarrow (\beta \rightarrow (\gamma \rightarrow b))) \mid \alpha, \beta, \gamma \overset{fin}{\subset} G(X), b \in (\alpha\gamma)(\beta\gamma)\}$$

Since γ_X is positive it can readily be seen that K,S are computable subsets of G(X). We have:

$$\begin{aligned} KMN &= \{s \mid \exists \beta \overset{fin}{\subset} N. \exists \alpha \overset{fin}{\subset} M. (\alpha \rightarrow (\beta \rightarrow s)) \in K\} \\ &= \{s \mid \exists \alpha \overset{fin}{\subset} M. s \in \alpha\} \\ &= M. \end{aligned}$$

Similarly we can show:

$$SMNL = (ML)(NL).$$

θ satisfying (3) $\sim$ (5) of 5.1 can easily be checked. Thus by 5.1 we have established the theorem.

□

Theorem 5.9.

Let χ be positive then $(CP(G(X)),*,\theta,\pi_X,\rho)$ is computationally complete.

Proof. Let $f : \pi_X \rightarrow \pi_X$ be a morphism realized by $r_f = \phi_i$. We have:

$$\begin{aligned} f(\pi_X(m)) &= f(\gamma_X(W_m)) \\ &= \pi_X(r_f(W_m)) \\ &= \gamma_X(W_{r_f(m)}) \\ &= \gamma_X(\Phi_{h(i)}(W_m)) \end{aligned}$$

where h is a recursive function and Φ_Z is an enumeration operator with an index Z. Since χ is positive, γ_X is positive and so there is a recursive function g s.t.

$$\pi_X(g \cdot h(i)) * \pi_X(m) = \gamma_X(W_{r_f(m)})$$

□

The language of $\lambda\beta$-calculus is too weak to represent all elements of $CP(G(X))$ by closed terms.

§6. EXAMPLES OF NUMERATION MODELS

(Example 1): Term Models.

It is known that as an immediate consequence of Church-Rosser Theorem, we can construct a countable model of $\lambda\beta$-calculus. The model construction can be sketched as follows: Let $\equiv$ be the smallest equivalence relation over T, containing reduction rules of $\lambda\beta$-calculus. Let $TM = \{[t] \mid t \in T\}$ where [t] is the equivalence class of t with respect to $\equiv$. A term f$\in$T determines a function $\overline{f}: TM \rightarrow TM$ such that

$$\overline{f}([t]) = [(f\,t)].$$

Let $(TM \rightarrow TM) = \{\overline{f} \mid f \in T\}$. Then $\Phi: TM \rightarrow (TM \rightarrow TM)$ and $\Psi: (TM \rightarrow TM) \rightarrow TM$ given by:

$$\Phi([t]) = \overline{t}$$

$$\Psi(\overline{f}) = [f]$$

establish a retraction $TM \triangleright (TM \rightarrow TM)$. This retraction allows us to form a model of $\lambda\beta$-calculus. For details see Barendregt [1].

Now let $\gamma: N \rightarrow TM$ be the following numeration of TM:

$$\gamma(n) = [\tau(n)].$$

It can be shown that $(TM \rightarrow TM) = Hom(\gamma,\gamma)$. Now let $\gamma\uparrow: N \rightarrow Hom(\gamma,\gamma)$ be the following numeration:

$$\gamma\uparrow(n) = \overline{\tau(n)}.$$

It can readily be seen that the following holds:

(1) $\gamma\uparrow: N \rightarrow Hom(\gamma,\gamma)$ is acceptable, thus $\gamma\uparrow = (\gamma \rightarrow \gamma)$.

(2) $\gamma \triangleright (\gamma \rightarrow \gamma)$ is the category of numeration.

Thus by 3.3 $\gamma \triangleright (\gamma \rightarrow \gamma)$ is a numeration model of $\lambda\beta$-calculus.

Notice that this γ is not a λ-definable model, for if t is an open term then [t] can not be represented by a closed term.

Also notice that this numeration model does not follow from the construction of §5.

(Example 2): Graph Models

As observed in the previous section, for any positive numeration $\chi: N \rightarrow X$, $(CP(G(X)), *, \theta, \pi_\chi, \rho)$ is a computationally complete RCA.

(Example 3): RE model.

RE model is a slight variant of numeration version of Engeler's graph models. This model due to Scott [10] is outstanding because it is λ-definable for a suitably expanded λ-terms.

Let RE be the set of all recursively enumerable sets of natural numbers. Let $\gamma: N \rightarrow RE$ be a Gödel numbering of recursively enumerable sets. For each $u \in RE$, let $fun(u): RE \rightarrow RE$ be the following continuous function:

$$fun(u)(x) = \{m \mid \exists n.\ \kappa(n) \subset x,\ (n, m) \in u\}$$

where κ is the standard enumeration of finite subsets of N. Due to u, x being recursively enumerable, $fun(u)(x) \in RE$. In fact $fun(u)$ is an enumeration operator. Define a numeration $\gamma\uparrow: N \rightarrow fun(RE)$ by:

$$\gamma\uparrow(i) = fun(\gamma(i)).$$

It can readily be seen that

$$fun(RE) = Hom(\gamma,\gamma).$$

Furthermore we can show that $\gamma\uparrow$ is acceptable, thus $\gamma\uparrow = (\gamma\rightarrow\gamma)$. Now let $graph : fun(RE)\rightarrow RE$ be the following function:

$$graph(f) = \{(n,m) \mid m \in f(\kappa(n))\}.$$

Notice that $f \in fun(RE)$ implies $graph(f) \in RE$. It can readily be seen that fun and graph are morphisms $\gamma\rightarrow(\gamma\rightarrow\gamma)$ and $(\gamma\rightarrow\gamma)\rightarrow\gamma$ respectively. Furthermore $graph \cdot fun = id_{RE}$. Thus $\gamma\triangleright(\gamma\rightarrow\gamma)$ in the category of numeration. Thus γ is NFD.

Let us add the following constant symbols to the syntax of λ-terms:

$$0, s, p, cond,$$

and let us interpret them as follows:

$$0 = \{0\}$$

$$s(x) = \{n+1 \mid n \in x\}$$

$$p(x) = \{n \mid n+1 \in x\}$$

$$cond(x)(y)(z) = \{n \mid n \in x, 0 \in z\} \cup \{m \mid m \in y, \exists k.k+1 \in z\}.$$

As shown in [10], every element of RE and $Hom(\gamma,\gamma)$ can be denoted by closed λ-terms of this expanded language. We can modify the interpretation morphism for this expansion. The results of §3 and §4 hold for this expanded λβ-calculus.

Notice that to within isomorphism through Gödel numbering of finite subsets of N and pairing, N satisfies the equation (5.1). Furthermore, $CP(N) = RE$.

Thus RE model is essentially the same as the graph model obtained from an initial numeration $id_N : N \rightarrow N$. An intensive study of relations between RE model and graph models can be found in Longo [8].

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