ACCEPTABLE NUMERATIONS OF FUNCTION SPACES

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ABSTRACT

We study when a numeration of the set of morphisms from a numeration to the other is well-behaved. We call well-behaved numerations "acceptable numerations". We characterize acceptable numerations by two axioms and show that acceptable numerations are recursively isomorphic to each other. We also show that for each acceptable numeration a fixed point theorem holds. Relation between Cartesian closedness and S-m-n property is discussed in terms of acceptable numerations. As an example of acceptable numerations, we study directed indexings of effective domains.

1. Numerations

The theory of numerations developed by Ersov [1,2] is a very useful general theory of computation which is based on very simple concepts. In this section, we briefly overview some basic concepts and results of this theory.

Definition 1.1

A numeration (of a set X) is a surjective map $\chi:N\to X$ where N is the set of all natural numbers. A morphism from a numeration $\alpha:N\to A$ to the other $\beta:N\to B$ is a function $h:A\to B$ which can be realized by a recursive function, i.e. for which there is a recursive function $r_h:N\to N$ satisfying:

$$h\cdot\alpha = \beta\cdot r_h \ .$$

For each numeration $\chi:N\to X$ we define an equivalence relation $=_\chi$ by:

$$n =_\chi m \quad \text{iff } \chi(n) = \chi(m) \ .$$

$\square$

It can readily be seen that numerations and morphisms form a category. We denote this category by Num. Notice that for any numeration $\chi:N\to X$, the identity map $id_X:X\to X$ is a morphism from χ to χ, for $id_N:N\to N$ realizes id_X.

Throughout, let ϕ be a Gödel numbering of partial recursive functions.

Definition 1.2

A numeration $\alpha:N\to A$ is precomplete if for every partial recursive function f, there is a recursive function g s.t. $f(n)\downarrow$ implies $f(n) =_\alpha g(n)$, and we can compute a Gödel number of g from that of f. We say g makes f total modulo α.

$\square$

Proposition 1.3 (Ersov)

Let $\alpha:N\to A$ be precomplete, then there is a total recursive function $fix_\alpha:N\to N$ s.t.

$$\phi_n(fix_\alpha(n))\downarrow \text{ implies } \phi_n(fix_\alpha(n)) =_\alpha fix_\alpha(n).$$

Proof Let ϕ_r make $\lambda x.\phi_n(\phi_x(x))$ total modulo α. Let r^* be $\phi_r(r)$. Assume $\phi_n(r^*)\downarrow$ then

$$\phi_n(r^*) = \phi_n(\phi_r(r)) =_\alpha \phi_r(r) = r^*.$$

Since α is precomplete, we can compute r and thus r^* from a Gödel number n of ϕ_n.

□

Proposition 1.4 (Ersov)

The Gödel numbering ϕ is precomplete.

□

Let $\alpha,\beta:N\to S$ be numerations. We say β is reducible to α, in symbols $\alpha<\beta$, iff there is a recursive function f satisfying $\alpha = \beta\cdot f$. Intuitively $\alpha<\beta$ iff we can compute a β-index of an element of S from an α-index of it.

Proposition 1.5 (Ersov)

Let $\alpha,\beta:N\to S$ be precomplete numerations satisfying $\alpha<\beta$ and $\beta<\alpha$, then there is a recursive isomorphism $h:N\to N$ s.t. $\alpha = \beta\cdot h$

□

Definition 1.6

Let $\alpha:N\to A$ and $\beta:N\to B$ be numerations. We define a numberation $\alpha x\beta:N\to AxB$ by:

$$\alpha x\beta(<n,m>) = (\alpha(n),\beta(m))$$

where $<-,->$ is the pairing function.

□

2. Acceptable Numerations of Function Spaces

Let $\alpha:N\to A$ and $\beta:N\to B$ be numerations. In this section we study a concept of "acceptable" numeration of the set $Hom(\alpha,\beta)$ of all morphisms from α to β. Essentially "acceptable" numerations are "well-behaved" numerations.

Definition 2.1 (acceptable numeration)

Let $\alpha:N\to A$ and $\beta:N\to B$ be numerations. A numeration $\tau:N\to Hom(\alpha,\beta)$ is semi-effective iff there is a recursive function $Eval_\tau:N^2\to N$ s.t.

$$\tau(m)(\alpha(n)) = \beta(Eval_\tau(m,n))$$

A semi-effective numeration $\tau:N\to Hom(\alpha,\beta)$ is acceptable iff there is a recursive function $Enum_\tau:N\to N$ s.t. if f is a morphism from α to β which is realized by $r_f = \phi_m$ then $f = \tau(Enum_\tau(m))$.

□

Lemma 2.2

Let $\tau:N\to Hom(\alpha,\beta)$ be a semi-effective numeration, then there is a recursive function $Real_\tau:N\to N$ s.t. $\phi_{Real_\tau(m)}$ realizes $\tau(m)$.

Proof By S-m-n theorem

□

The next theorem states that acceptable numerations are maximum numerations.

Theorem 2.3

Let $\tau:N\to Hom(\alpha,\beta)$ be semi-effective and $\delta:N\to Hom(\alpha,\beta)$ be acceptable then $\tau<\delta$.

Proof Since τ is semi-effective $r_{\tau(m)} = \phi_{Real_\tau(m)}$. Therefore $\tau(m) = \delta(Enum_\delta\cdot Real_\tau(m))$, for δ is acceptable.

□

Theorem 2.4

Let $\tau:N\to\mathrm{Hom}(\alpha,\beta)$ be acceptable, then τ is precomplete.

Proof Let f be a partial recursive function. Assume $f(x)\downarrow$.

$$\tau(f(x))(\alpha(y)) = \beta(r_{\tau(f(x))}(y))$$
$$= \beta(\phi_{\mathrm{Real}_\tau(f(x))}(y)).$$

Since ϕ is precomplete, there is a recursive function g s.t. if $f(x)\downarrow$ then $\phi_{\mathrm{Real}_\tau(f(x))} = \phi_{g(x)}$. Thus we have:

$$\tau(f(x))(\alpha(y)) = \beta(\phi_{g(x)}(y))$$
$$= \tau(\mathrm{Enum}_\tau(g(x)))(\alpha(y)).$$

Therefore $\tau(f(x)) = \tau(\mathrm{Enum}_\tau\cdot g(x))$. Obviously we can compute a Gödel number of $\mathrm{Enum}_\tau\cdot g$ from that of f.

□

The next theorem states that there is only one acceptable numeration of $\mathrm{Hom}(\alpha,\beta)$.

Theorem 2.5 (Recursive Isomorphism Theorem)

Let $\tau,\tau':N\to\mathrm{Hom}(\alpha,\beta)$ be acceptable numerations, then there is a recursive isomorphism $h:N\to N$ s.t. $\tau = \tau'\cdot h$.

Proof By 2.3, 2.4 and 1.5

□

Notation Since there is only one acceptable numeration of $\mathrm{Hom}(\alpha,\beta)$, we denote it by $(\alpha\to\beta)$ if any.

It should be noticed that the acceptability of a numeration corresponds to a generalization of Myhill-Shepherdson theorem [5] in recursive function theory. Thus the main result of this section can be stated roughly as: "numerations satisfying the Myhill-Shepherdson property form the maximam recursive isomorphism class".

The following theorems will explain why we say acceptable numerations are "well-behaved" numerations.

Theorem 2.6

If $\tau:N\to Hom(\alpha,\beta)$ is an acceptable numeration then there is a total recursive function $fix_\tau:N\to N$ s.t.

$$\phi_n(fix_\tau(n))\downarrow \text{ implies } \phi_n(fix_\tau(n)) =_\tau fix_\tau(n).$$

Proof Immediate from 2.4.

□

Theorem 2.7

Let $\tau:N\to Hom(\alpha,\beta)$, $\rho:N\to Hom(\beta,\gamma)$ and $\delta:N\to Hom(\alpha,\gamma)$ be acceptable, then there is a recursive function $Comp_{(\alpha,\beta,\gamma)}:N\times N\to N$ s.t.

$$\delta(Comp_{(\alpha,\beta,\gamma)}(m,n)) = \rho(n)\cdot\tau(m).$$

□

Theorem 2.8

Let $\alpha:N\to A$, $\beta:N\to B$ and $\gamma:N\to C$ be numerations s.t. $(\alpha\times\beta\to\gamma):N\to Hom(\alpha\times\beta,\gamma)$ and $(\alpha\to(\beta\to\gamma)):N\to Hom(\alpha,(\beta\to\gamma))$ are acceptable. Then $(\alpha\times\beta\to\gamma)\cong(\alpha\to(\beta\to\gamma))$ in Num.

Proof Define $Curry:Hom(\alpha\times\beta,\gamma)\to Hom(\alpha,(\beta\to\gamma))$ and

Apply: $Hom(\alpha,(\beta\to\gamma))\to Hom(\alpha\times\beta,\gamma)$ by:

$$Curry(f)(a)(b) = f(a,b)$$

$$Apply(g)(a,b) = g(a)(b)$$

Then $Apply((\alpha\to(\beta\to\gamma)(k))(\alpha(m),\beta(n)))$

$= (\alpha\to(\beta\to\gamma))(k)(\alpha(m),\beta(n))$

$= (\beta\to\gamma)(Eval_{(\alpha\to(\beta\to\gamma))}(k,m))(\beta(n))$

$= \gamma(Eval_{(\beta\to\gamma)}(Eval_{(\alpha\to(\beta\to\gamma))}(k,m),n))$

$= \gamma(\phi_{u(k)}(<m,n>))$

where u is some recursive function due to S-m-n theorem. Therefore we have:

$$\mathrm{Apply}((\alpha\to(\beta\to\gamma))(k))$$
$$= (\alpha\times\beta\to\gamma)(\mathrm{Enum}_{(\alpha\times\beta\to\gamma)}(u(k))).$$

Thus $\mathrm{Apply}\in\mathrm{Hom}((\alpha\to(\beta\to\gamma)),(\alpha\times\beta\to\gamma))$.

Also
$$\mathrm{Curry}((\alpha\times\beta\to\gamma)(k))(\alpha(m))(\beta(n))$$
$$= (\alpha\times\beta\to\gamma)(k)(\alpha\times\beta(<m,n>))$$
$$= \gamma(\mathrm{Eval}_{(\alpha\times\beta\to\gamma)}(k,<m,n>))$$
$$= \gamma(\phi_{v(k)}(<m,n>))$$

where v is some recursive function due to S-m-n theorem.

Thus we have:

$$\mathrm{Curry}\ ((\alpha\times\beta\to\gamma)(k))(\alpha(m))$$
$$= \lambda\beta(n).\gamma(\phi_{v(k)}(<m,n>))$$
$$= \lambda\beta(n).\gamma(\phi_{v'(k,m)}(n))$$
$$= (\beta\to\gamma)(\mathrm{Enum}_{(\beta\to\gamma)}(v'(k,m))$$
$$= (\beta\to\gamma)(\phi_{v''(k)}(m))$$

where v' and v" are recursive functions due to S-m-n theorem.

Thus we have:

$$\mathrm{Curry}((\alpha\times\beta\to\gamma)(k)) = (\alpha\to(\beta\to\gamma))(\mathrm{Enum}_{(\alpha\to(\beta\to\gamma))}(v''(k))).$$

Thus

$$\mathrm{Curry}\in\mathrm{Hom}((\alpha\times\beta\to\gamma),(\alpha\to(\beta\to\gamma))).$$

Obviously
$$\mathrm{Curry}\cdot\mathrm{Apply} = \mathrm{id}$$
$$\mathrm{Apply}\cdot\mathrm{Curry} = \mathrm{id}.$$

$\square$

Corollary 2.9

Let K be a full subcategory of the category of numerations s.t.

(1) For every object $\alpha:N\to A$, $\beta:N\to B$ in $\underline{K}$, there is a unique, up to recursive isomorphism, acceptable numeration $(\alpha\to\beta):N\to Hom(\alpha,\beta)$ in $\underline{K}$. We call such category an <u>acceptable subcategory</u> of <u>Num</u>.

(2) $\alpha x\beta:N\to AxB$ is in $\underline{K}$

(3) $\underline{K}$ has a final object

Then $\underline{K}$ is Cartesian closed.

□

The next theorem relates Cartesian closedness to a generalized S-m-n property.

<u>Theory 2.10</u>

Let $\alpha:N\to A$, $\beta:N\to B$ and $\gamma:N\to C$ be numerations s.t. $(\alpha x\beta\to\gamma):N\to Hom(\alpha x\beta,\gamma)$ and $(\alpha\to(\beta\to\gamma)):N\to Hom(\alpha,(\beta\to\gamma))$ are acceptable. Then there is a recursive function S s.t. $(\alpha x\beta\to\gamma)(m)(\alpha(n),\beta(k)) = (\beta\to\gamma)(S(m,n))(\beta(k))$

<u>Proof</u> By 2.8, $(\alpha x\beta\to\gamma)\cong(\alpha\to(\beta\to\gamma))$. Therefore we have:

$$\begin{aligned}&(\alpha x\beta\to\gamma)(m)(\alpha(n),\beta(k))\\ &= (\alpha\to(\beta\to\gamma))(r_{Curry}(m))(\alpha(n))(\beta(k))\\ &= (\beta\to\gamma)(Eval_{(\alpha\to(\beta\to\gamma))}(r_{Curry}(m),n))(\beta(k)).\end{aligned}$$

Thus a recursive function S s.t.

$$S(m,n) = Eval_{(\alpha\to(\beta\to\gamma))}(r_{Curry}(m),n)$$

satisfies the theorem.

□

3. Effective Domains and Directed Indexings

As an example of acceptable subcategory of <u>Num</u>, we study the category of directed indexings of effective domains. Since the purpose of this section is neither to give an exposition on effective domains nor to present some new results on this theory, we make explicit reference only to that literature which are relevant to acceptable numerations.

A domain is a partially ordered set $(X,<)$ such that

(1) For every subset $Z\subset X$, if Z has an upper bound then the least upper bound (lub) $\bigsqcup Z$ exists.

(2) The set B_X of compact elements of X is countable.

(3) For every element $x\in X$, $B_x = \{b\in B_X \mid b<x\}$ is directed and $x = \bigsqcup B_x$.

Let $\varepsilon: N\to B_X$ be a numeration. $(X,<)$ is an effectively given domain if there is a pair (b,l) of recursive predicates satisfying:

$b(x) \leftrightarrow \varepsilon(f_p(x))$ has an upper bound

$l(x,k) \leftrightarrow \varepsilon(k) = \bigsqcup \varepsilon(f_p(x))$

where f_p is the standard enumeration of finite subsets of N.

An element $x\in X$ is computable w.r.t. ε if for some recursively enumerable set W, $\varepsilon(W)$ is directed and $x = \bigsqcup \varepsilon(W)$. $Comp(X,\varepsilon)$ denotes the set of all computable elements of (X,ε) and is called an effective domain (generated by ε).

For every effectively given domain (X,ε) there is a recursive function $d_\varepsilon: N\to N$ such that for every $j\in N$, $\varepsilon(W_{d_\varepsilon(j)})$ is directed and if $\varepsilon(W_i)$ is directed then $\bigsqcup \varepsilon(W_i) = \bigsqcup \varepsilon(W_{d_\varepsilon(i)})$. This function is due to the following "directing" procedure

- generate W_j as $x_0, x_1, x_2, \ldots\ldots, x_n, \ldots$
- generate an r.e. set $W_{d_\varepsilon(j)} = \{y_0, y_1, y_2, \ldots.\}$ by:

$y_0 = x_0$

$y_{n+1} =$ if $\{\varepsilon(y_0), \ldots, \varepsilon(y_n), \varepsilon(x_n)\}$ has an upper bound in B_X
then $\mu k.\varepsilon(k) = \bigsqcup\{\varepsilon(y_0),\ldots, \varepsilon(y_n), \varepsilon(x_{n+1})\}$
else y_n

This function gives us a numeration $\delta_\varepsilon: N\to Comp(X,\varepsilon)$ defined by $\delta_\varepsilon(i) = \bigsqcup \varepsilon(W_{d_\varepsilon(i)})$. This numeration is called a directed indexing of $Comp(X,\varepsilon)$.

Given effectively given domains (X,ε) and (X',ε'), let $[X\to X']$ be the set of all functions called continuous functions $f:X\to X'$ which preserve the lub of directed subsets, (i.e. if $D\subset X$ is directed then $f(D) = \{f(x)|x\in D\}$ is directed and $f(\bigsqcup D) = \bigsqcup f(D)$.) with the following partial ordering: for $f,g\in[X\to X']$,

$f<g$ iff $f(x)<g(x)$ for all $x\in X$.

It is well-known that $[X\to X']$ is a domain where

$B_{[X\to X]}$ = the set of all possible finite joins of the step functions $[b,b']:X\to X'$ s.t. $b\in B_X$, $b'\in B_X{}'$ and $[b,b'](x):=$ <u>if</u> $b<x$ <u>then</u> b' <u>else</u> $\bot$ where $\bot = \bigsqcup\phi$.

Let $[\varepsilon\to\varepsilon']:N\to B_{[X\to X']}$ be the following numeration:

$[\varepsilon\to\varepsilon'](n):=$ <u>if</u> $\sigma(n)$ has a lub <u>then</u> $\bigsqcup\sigma(n)$ <u>else</u> $\bot$

$\sigma(n) = \{[\varepsilon(i),\varepsilon'(j)]|(i,j)\in Pr(n)\}$

Pr:standard enumeration of finite subsets of NxN.

It is known that if (X,ε) and (X',ε') are effectively given domains then $([X\to X'],[\varepsilon\to\varepsilon'])$ is also an effectively given domain.

$f:X\to X'$ is <u>computable</u> wrt $(\varepsilon,\varepsilon')$ iff $f\in Comp([X\to X'],[\varepsilon\to\varepsilon'])$.

<u>Proposition 3.1 (Kanda [3])</u>

A continuous function f from an effectively given domain (X,ε) to another (X',ε') is computable w.r.t. $(\varepsilon,\varepsilon')$ iff $f\upharpoonright Comp(X,\varepsilon)$ is a morphism from $\delta_\varepsilon:N\to Comp(X,\varepsilon)$ to $\delta_{\varepsilon'}:N\to Comp(X',\varepsilon')$. This equivalence is effective, i.e. from a directed index of f, we can compute a Gödel number of a recursive function which realizes $f\upharpoonright Comp(X,\varepsilon)$ and vice versa.

□

<u>Proposition 3.2</u> (Weihrauch-Schafer [8], Streicher [6]).

Let (X,ε) and (X',ε') be effectively given domains and $f:X\to X'$ be a function. If $f \restriction Comp(X,\varepsilon)$ is a morphism from δ_ε to $\delta_{\varepsilon'}$ then f is continuous.

□

Let us introduce a numeration $(\delta_\varepsilon \to \delta_{\varepsilon'}) : N \to Hom(\delta_\varepsilon, \delta_{\varepsilon'})$ by $(\delta_\varepsilon \to \delta_{\varepsilon'})(i) = \delta_{(\varepsilon \to \varepsilon')}(i) \restriction Comp(X,\varepsilon)$. This is a numeration of $Hom(\delta_\varepsilon, \delta_{\varepsilon'})$ because we have 3.1 and 3.2. We can consider this numeration $(\delta_\varepsilon \to \delta_{\varepsilon'})$ as the directed indexing $\delta_{(\varepsilon \to \varepsilon')}$ of $Comp([X \to X'], [\varepsilon \to \varepsilon'])$ because for each element of $Hom(\delta_\varepsilon, \delta_{\varepsilon'})$ there is a unique computable extension. In other words, to within the identification $Hom(\delta_\varepsilon, \delta_{\varepsilon'}) = Comp([X \to X'], [\varepsilon \to \varepsilon'])$, we have $(\delta_\varepsilon \to \delta_{\varepsilon'}) = \delta_{[\varepsilon \to \varepsilon']}$.

Now as an immediate consequence of 3.1, we have:

Theorem 3.3

$(\delta_\varepsilon \to \delta_{\varepsilon'}) : N \to Hom(\delta_\varepsilon, \delta_{\varepsilon'})$ is acceptable. Thus the category of directed indexings of effective domains is an acceptable subcategory of Num.

□

Let (Y,ε) and (Y',ε') be effectively given domains. Then $Y \times Y'$ is a domain with $B_{Y \times Y'} = B_Y \times B_{Y'}$. Let $\varepsilon \times \varepsilon' : N \to B_{Y \times Y}$ be the following numeration: $\varepsilon \times \varepsilon'(<n,m>) = (\varepsilon(n), \varepsilon'(m))$. Then $(Y \times Y', \varepsilon \times \varepsilon')$ is an effectively given domain. It is obvious that we have $\delta_\varepsilon \times \delta_{\varepsilon'} = \delta_{\varepsilon \times \varepsilon'}$.

A singleton is obviously an effective domain. Therefore in summary, we have:

Theorem 3.4

The category of directed indexings of effective domains is Cartesian closed.

Proof By 2.9, 3.3 and above. □

Due to the theorem 2.10, S-m-n theorem holds for directed indexings.

4. Concluding Remarks

In Kanda [4] and Weihrauch [7], it was shown that "acceptable indexings" of an effective domain are all recursively isomorphic. This result is a special case of our general result 2.5.

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