SOLVABLE CASES OF THE TRAVELLING SALESMAN PROBLEM

by

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ABSTRACT

This paper is a chapter in a book on the travelling salesman problem edited by Eugene L. Lawler, Jan Karel Lenstra and Alexander H.G. Rinooy Kan. By a solvable case of the travelling salesman problem is meant a case of the distance matrix for which a polynomial algorithm exists. In this paper several previously known special cases are related and extended. Further, an upper bound is obtained on the cost of an optimal tour for a broad class of matrices.

§1. Introduction

1.1 Survey of Methods and Results

In this chapter the goal is to examine special cases of the distance matrix $\{c_{ij}\}$ for which the travelling salesman's problem can be solved in polynomial time. Since the assignment problem for any distance matrix can be solved in polynomial time, not surprisingly the polynomial solutions of special cases of the travelling salesman problem dealt with here require solving first the assignment problem. The unifying theme for the chapter is the manner in which an optimal tour ψ for the travelling salesman is obtained from an optimal assignment ϕ.

A cyclic permutation $\rho = [p_1,\ldots,p_a]$ takes values defined as follows:

$$\rho(i) = \begin{cases} i, \text{ if } i \neq p_1,\ldots,p_a, \\ p_{j+1}, \text{ if } i=p_j, \text{ and} \\ p_1, \text{ if } i=p_a. \end{cases}$$

Two cyclic permutations are disjoint if they have no cities in common.

For any given assignment ϕ and cyclic permutation ρ define:

$$c\phi(\rho) = c(\phi\rho) - c(\phi).$$

Using this definition the following result is elementary:

Theorem 1. Given any assignments ϕ and ψ there exist disjoint cyclic permutations $\rho_1,\ldots,\rho_k$ for which

$$\psi = \phi\rho_1\ldots\rho_k \text{ and}$$

$$c(\psi) = c(\phi) + c\phi(\rho_1) +\ldots+ c\phi(\rho_k).$$

Proof. Any permutation can be represented as the product of disjoint cyclic permutations, and therefore $\phi^{-1}\psi$ can be so represented. The second result then follows from the fact that $\rho_1,\ldots,\rho_k$ are disjoint.

End of Proof.

An assignment ϕ which is not a tour consists of two or more subtours. Cyclic permutations $\rho_1,\ldots,\rho_k$ can be chosen to connect the subtours into a single tour ψ. The common method of determining an optimal tour ψ used in this chapter is to first determine an optimal assignment ϕ and then to select cyclic permutations for which $\phi\rho_1\ldots\rho_k$ is a tour and $c\phi(\rho_1) +\ldots+ c\phi(\rho_k)$ is as small as possible.

In §2 the technique is first illustrated for upper triangular distance matrices, a solvable case first described in [Lawler, 1971]. In this case a single cyclic permutation ρ suffices and $c\phi(\rho) = 0$. Also in this section the method of selecting ρ is applied to a broad class of matrices C to obtain an upper bound on the cost of an optimal tour. For this class of matrices, as for all matrices, this cost is bounded below by the cost of an optimal assignment for C, but it is also bounded above by the cost of an optimal assignment for the matrix C$'$ obtained from C by dropping one column and one row.

In §3 it is shown that the distance matrix defined in [Fuller, 1972] is upper triangular so that the technique of §2 may be used for its solution. Moreover, because of the special form of the distance matrix an optimal assignment can be obtained by a simple algorithm. An exercise suggests a solvable generalization of the Fuller matrix.

In §4 a class of distance matrices called double sum matrices is defined. For this class of matrices a lower bound on the cost $c(\psi)$ of an optimal tour ψ is obtained by focusing attention on interchanges, that is cyclic permutations of two cities. The interchanges $[p,q]$ considered are those with each of p and q in different subtours of ϕ. The effect of such an interchange on ϕ is to connect into a single subtour the pair of subtours containing p and q. Therefore each such interchange can be regarded as an edge of cost $c\phi([p,q])$ connecting two nodes of a graph with subtours of ϕ as nodes. The cost $c\phi(T)$ of a minimum spanning tree T for the graph is shown to be a lower bound for $c\phi(\rho_1) +\ldots+ c\phi(\rho_k)$ so that $c(\phi) + c\phi(T)$ forms a lower bound for $c(\psi)$.

In §5 a special class of double sum matrices is defined suggested by the distance matrix defined in [Gilmore and Gomory, 1964]. It is shown that the lower bound $c(\phi) + c\phi(T)$ can be achieved for this special class. The interchanges used to define the edges of T can be applied in a particular order to define cyclic permutation $\rho_1,\ldots,\rho_k$ for which

$$c\phi(\rho_1) +\ldots+ c\phi(\rho_k) = c\phi(T).$$

Permutations of the rows and columns of a matrix for an assignment problem can be accommodated by renumberings of the rows and columns since the numberings do not connect a particular row with a particular column. For the travelling salesman problem, on the other hand, the numberings are important because the ith row and the ith column each contain distance information relating to the ith city. A renumbering of the rows or columns alone can however be accommodated. A permutation β of the columns of C after a renumbering of the rows results in $c_{i\beta(j)}$, i=1,...,n, being the distances in column j.

It may be possible to transform a given distance matrix into one of the special forms described in this chapter by permutations of the rows and columns. This does not always mean that the matrix has then a solvable travelling salesman problem. The method of solution for upper triangular matrices can only be applied to matrices which can be brought to this form by a renumbering of the rows, or a renumbering of the columns, but not to those that require a renumbering of the rows and a permutation of the columns.

1.2 Some Omissions

Unfortunately limitations in space have required the elimination of several topics. These include the following.

The motivating applications for the papers [Fuller, 1972] and [Gilmore and Gomory, 1964] are not discussed. The reader is urged to turn to these papers to gain some sense of the interaction between theory and practice.

A time and space analysis of the algorithms described in this chapter has not been made. This is a serious omission in a chapter dealing with practical algorithms but the analysis is not difficult. The appendix to [Fuller, 1972] offers some help for one algorithm.

The special case [Syslo, 1973] which obtains Hamiltonian circuits in directed line graphs from Eulerian circuits in an underlying graph to the line graph is not described. Since Eulerian circuits can be efficiently found, the travelling salesman problem for these special graphs can be efficiently solved.

In the paper [Jenkyns, 1979] bounds for solutions to the travelling salesman problem are obtained using the theory of independence systems. These results are not used or described here.

Exercise 1. The classic paper [Johnson, 1954] describes an optimal solution to a problem with close connections with [Gilmore and Gomory, 1964]. Show that the problem is not a travelling salesman problem.

§2. Upper triangular and graded matrices

2.1 Finding an Optimal Tour for Upper Triangular Matrices

An upper triangular matrix is one in which all distances are zero on and below the main diagonal; that is, $c_{ij} = 0$ for $i \geq j$. The other distances may be positive, negative or zero.

Let C be an upper triangular matrix and let C' be obtained from C by removing the first column and the last row. Any assignment ϕ' for C' can be extended to an assignment ϕ for C by defining $\phi(i)$ to be $\phi'(i)$ for $i \neq n$ and $\phi(n)$ to be 1. Since the arc (n,1) has zero distance the cost of ϕ is the cost of ϕ'. Of course if ϕ' is an optimal assignment for C' it does not follow that ϕ is an optimal assignment for C, although it will be shown later that if ϕ is a tour then it is an optimal tour for C.

Let ϕ' be any optimal assignment for C' and let ϕ not be a tour. An optimal tour will be obtained from ϕ by combining its several subtours into one. This will be accomplished by removing from its subtours arcs (i,j) for which $i \geq j$, what are called backward arcs, and replacing them with other such arcs. Since backward arcs have zero distance the tour obtained will have the same total cost as ϕ' or ϕ.

Note first that each subtour of ϕ must contain at least one backward arc. Hence a backward arc can be selected from each subtour, including (n,1) from the subtour of which it is a member. Let a selection of such arcs be $(p_1,\phi(p_1)),\ldots,(p_k,\phi(p_k))$, where $\phi(p_1) > \phi(p_2) >\ldots> \phi(p_k)$. Here $p_k = n$ and $\phi(p_k) = 1$. Consider now the permutation $\psi = \phi[p_1,\ldots,p_k]$. Each of the arcs $(p_i,\phi(p_i))$ is a backward arc by choice, that is $p_i \geq \phi(p_i)$. Consequently $(p_i,\psi(p_i))$ is a backward arc for $1 \leq i < k$, since $\phi(p_i) > \phi(p_{i+1}) = \psi(p_i)$, and $(p_k,\psi(p_k))$ is a backward arc since $p_k = n \geq \phi(p_1) = \psi(p_k)$. It follows therefore that $c(\psi) = c(\phi)$. Furthermore ψ is a tour. This fact is illustrated in figure 1 for k = 3. To show that ψ is an optimal tour is a little more difficult. To do so it will be shown that any tour for C defines an assignment for C' of the same cost so that necessarily any tour for C with the same cost as an optimal assignment for C' is an optimal tour.

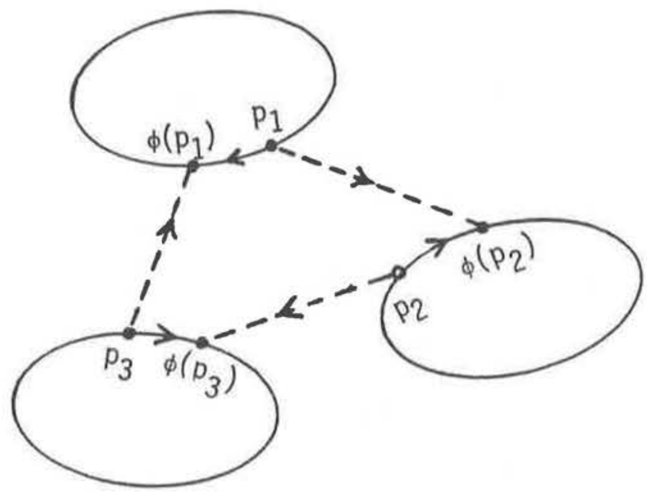

figure 1

Let τ be any tour for C. It defines a path from city n to city 1. Let the cities in the path in the order in which they are encountered be $p_0, p_1, \ldots, p_m, p_{m+1}$, where $p_0 = n$ and $p_{m+1} = 1$. If $m = 0$ then $\tau(n) = 1$ and τ defines an assignment for C' with the same cost as τ. Assume therefore that $m \geq 1$. A cyclical permutation ρ of cities selected from $p_0, \ldots, p_{m+1}$ will be defined with the following property: If ϕ is $\tau\rho$ then $\phi(n) = 1$ and the cost of ϕ is the same as the cost of τ so that ϕ and therefore τ defines an assignment for C' of the same cost as τ. The cost of ϕ is maintained as the same as τ by replacing backward edges of τ by other backward edges.

There must be some i for which $p_m \geq p_{i+1}$ and $p_i > p_{i+1}$. If $p_m \geq p_1$ then 0 is one such i, while if $p_m < p_1$ i is such an index if p_i is the last city in the path for which $p_m < p_i$. Let m_1 be one such i. Then (p_m, p_{m_1+1}) is a backward arc that can replace the backward arc (p_m, p_{m+1}).

The argument of the previous paragraph can now be repeated for the path $p_0, p_1, \ldots, p_{m_1}, p_{m_1+1}$ to define an m_2 for which (p_{m_1}, p_{m_2+1}) is a backward arc that can replace the backward arc (p_{m_2}, p_{m_2+1}). Continued repetitions will define indices $m_1, m_2, \ldots, m_k$ such that $(p_{m_{j-1}}, p_{m_{j+1}})$ is a backward arc that can replace the backward arc $(p_{m_j}, p_{m_{j+1}})$ for $j=1,\ldots,k$, where $m_0 = m$, and such that m_k is 0.

Now define ρ to be $[p_{m_0}, p_{m_1}, \ldots, p_{m_k}]$ and ϕ to be $\tau\rho$. For each j, $0 \le j < k$, $\phi(p_{m_j})$ is $\tau(p_{m_{j+1}})$, that is $p_{m_{j+1}+i}$, while $\phi(p_{m_k})$ is $\tau(p_{m_0})$, that is $\phi(n)$ is 1.

By definition each of the arcs $(p_{m_j}, \tau(p_{m_j}))$, where $0 \le j \le k$, is backward. But so also is each of the arcs $(p_{m_j}, \phi(p_{m_j}))$. Therefore $c(\phi) = c(\tau)$.

This completes the proof of:

Theorem 2. $\psi = \phi[p_1, \ldots, p_k]$ is an optimal tour for C.

This proof cannot be generalized to the case where C is upper triangular after a renumbering of the rows and a permutation of the columns. In this case there is no assurance that every subtour of ϕ uses a backward arc.

2.2 An Upper Bound for Graded Matrices

The method of defining the cyclic permutation $[p_1, \ldots, p_k]$ to break the cycles of ϕ can be used to obtain an upper bound on the cost of an optimal tour for a broad class of distance matrices C.

A matrix C is said to be graded across its rows if $c_{ij} \le c_{ik}$ for $1 \le i \le n$ and $1 \le j \le k \le n$. It is said to be graded up its columns if $c_{ij} \ge c_{kj}$ for $1 \le j \le n$ and $1 \le i \le k \le n$.

Theorem 3. Let C be any distance matrix for which either for some permutation β of its columns C is graded across its rows and its last row is all zero, or for some permutation α of its rows C is graded up its columns and its first column is all zero. Let ϕ' be an optimal assignment to the matrix C' obtained from C by deleting in the first case column $\beta(1)$ and the last row, and in the second case row $\alpha(n)$ and the first column. Then $c(\phi')$ is an upper bound on the cost of a tour of C.

Proof. Consider the first case. Define ϕ as in 2.1 from ϕ' taking account of the permutation β. Because $c_{n\beta(1)} = 0$ it follows that $c(\phi) = c(\phi')$. If ϕ is a tour then $c(\phi) = c(\phi')$ is an upper bound on the cost of an optimal tour. If ϕ is not a tour, select an edge from each subtour of ϕ including the edge $(n,\beta(1))$ from the subtour of which it is a member. As before order the arcs $(p_i,\beta^{-1}\phi(p_i))$ so that $\beta^{-1}\phi(p_1) >\ldots> \beta^{-1}\phi(p_k)$, where $p_k = n$ and $\beta^{-1}\phi(p_k) = 1$. Define $\psi = \beta^{-1}\phi[p_1,\ldots,p_k]$, which as before is a tour.

Further $c(\psi) = c(\phi) +$

$$\sum_{i=1}^{n-1} (c_{p_i\beta^{-1}\phi(p_{i+1})} - c_{p_i\beta^{-1}\phi(p_i)})$$

$$+ c_{n\beta^{-1}\phi(1)} - c_{n\beta^{-1}\phi(n)}.$$

From the conditions on C and the ordering of the p_i's it follows that $c(\psi) \le c(\phi)$. This completes the proof of the first case. The proof of the second case is identical since the matrix of the second case is a transpose of the matrix of the first case.

End of Proof.

Although no important use is made of Theorem 3 in this chapter, it is included here to suggest a possibly interesting use of the cycle breaking technique of this section. An upper bound on the cost of an optimal tour can be useful in branch and bound algorithms for the travelling salesman problem as was described in [Gilmore, 1962] for the more general quadratic assignment problem, or ψ can be used as a quick approximate solution.

Exercise 2 Theorem 3 describes a bound for a matrix that is either graded across the rows and has a zero row, or is graded up its columns and has a zero column. Develop bounds for matrices that are only either graded across their rows or up their columns.

§3. Fuller Matrices

A Fuller matrix described in [Fuller, 1972] can be transformed into an upper triangular matrix. Further, the assignment problem to be solved in applying the technique of the last section has a simple method of solution.

3.1 A Fuller Matrix is Upper Triangular

Let f_i and s_i, $1 \le i < n$, and f_n be non-negative numbers satisfying $0 \le f_n \le f_i \le s_i < 1$. For $1 \le i \le n$ define $c_{in} = 0$ and for $1 \le j \le n-1$ define

$$c_{ij} = \begin{cases} s_j - f_i \text{ if } s_j \ge f_i \\ 1 + s_j - f_i \quad \text{otherwise} \end{cases}$$

The distance matrix so obtained is called here a Fuller matrix. The form of the matrix can be simplified by adding f_i to row i for $1 \le i \le n$, subtracting s_j from column j for $1 \le j \le n-1$, and subtracting f_n, the smallest f, from column n. The new distances then are:

$$c_{in} = f_i - f_n$$

$$c_{ij} = \begin{cases} 0 \quad \text{if } s_j \ge f_i \\ 1 \quad \text{otherwise} \end{cases}$$

Let C be the distance matrix so defined. It can be assumed that the cities have been renumbered so that $f_1 \ge \ldots \ge f_n$. Then C is upper triangular since if $i \geq j$ and $j \leq n-1$ then $s_j \geq f_j \geq f_i$, so that $c_{ij} = 0$, while also $c_{nn} = 0$.

3.2 An Optimal Assignment for C .

To solve the travelling salesman problem for C by the technique described in §2 it is first necessary to find an optimal assignment ϕ' for the matrix C' obtained from C by dropping the first column and the last row. Such an assignment can be found quite simply.

Clearly an optimal assignment for C' will select as many zero distance arcs from columns 2 through n-1 of C as any other assignment, and further will select those arcs from rows with the smallest index i. The latter is necessary to permit the selection of the least cost available arc from the last column, since the distances in the last column are in decreasing order.

Consider the following algorithm for defining an assignment ϕ' for C':

1. For i = 1 to n-1, find a j, $2 \leq j \leq n-1$ if one exists, for which $c_{ij} = 0$ and $\phi'^{1}(j)$ is undefined and set $\phi'(i) = j$, otherwise leave $\phi'(i)$ undefined.
2. Let k be the largest i for which $\phi'(i)$ is undefined and set $\phi'(i) = n$.
3. For each i for which $\phi'(i)$ is undefined select a j for which $\phi'^{-1}(j)$ is undefined and set $\phi'(i) = j$.

Theorem 4. ϕ' is an optimal assignment for C'.

Proof. Consider any other assignment ψ and compare the number of zero cost arcs selected by ϕ' and ψ. The number selected by ψ from the first k rows can be no more than the number selected by ϕ', for $1 \leq k \leq n-1$. For consider a row from which ψ selects but ϕ' does not select a zero cost arc. Necessarily it is a row for which in each column in which the row has a zero cost arc, ϕ' has selected a zero cost arc in a prior row. Consequently since ψ selects a zero cost arc in such a row it must have selected fewer zero cost arcs from prior rows than did ϕ'.

It follows also therefore that if ψ and ϕ' select the same number of zero cost arcs then $\phi'^{-1}(n) \geq \psi^{-1}(n)$. For let $\phi'(k) = \psi(i) = n$, where $k < i$. Necessarily ϕ' selects a zero cost arc for row i while ψ does not. Yet the number of zero cost arcs selected by ϕ' in rows prior to i is not less than the number selected by ψ. Since ϕ' and ψ select the same number of zero cost arcs it follows that in some row following i in which ψ selects a zero cost arc ϕ' does <u>not</u>. But that is impossible from step (2) of the algorithm since $k < i$.

Since ϕ' selects as many zero cost arcs as ψ and $\phi'^{-1}(n) \geq \psi^{-1}(n)$ it follows that $c(\phi') \leq c(\psi)$.

End of Proof.

It is clear from the matrix C that the choice of an optimal tour is not dependent upon the values s_j but only upon how these values compare with the f_i. But it is also clear from the algorithm for determining an optimal tour that the choice is also not dependent upon the values f_i. Indeed the zero and one-distance arcs of C are all that affect the choice.

<u>Exercise 3</u> The application of the algorithm for upper triangular matrices to a Fuller matrix required first the transformation of the matrix as does also the application of the algorithm for an optimal assignment for C'. Describe an algorithm which operates directly on the original f_i and s_j and compare the algorithm with the one described in [Fuller, 1972].

<u>Exercise 4</u> The Generalized Fuller Matrix

The parameters f_i and s_i determining the distances of a Fuller matrix satisfy the condition $0 \leq f_n \leq f_i \leq s_i < 1$ for $1 \leq i \leq n-1$. For a generalized Fuller matrix this condition is relaxed to $0 \leq f_n \leq f_i < 1$ and $0 \leq s_i < 1$ so that the transformed matrix C is no longer upper triangular. Moreover, the matrix does not satisfy the conditions of theorem 3 since although the matrix is graded up its columns its first column is not necessarily all zero. Nevertheless the technique of theorem 3 can be applied. It is only necessary to ensure when executing step (3) of the algorithm for finding an optimal assignment for C' that no cycle is created in which a zero-distance arc does not appear. Then a single zero-distance arc can be chosen from each cycle. Show in this way that an upper bound like that of theorem 3 applies to generalized Fuller matrices. Can an optimal tour be obtained in this way?

§4. Double Sum Distance Matrices

Let $D = (d_{ij})_{i,j=1}^{n}$ be any matrix of elements d_{ij} for which $d_{ij} \geq 0$. A distance matrix C is a double sum matrix based on D if there is a permutation ϕ of the columns of C for which

$$c_{i\phi(j)} = \sum_{k=i}^{n} \sum_{l=1}^{j} d_{kl}.$$

In this section a lower bound on the cost of an optimal tour for a double sum distance matrix is derived. In the next section it is shown that this lower bound can be achieved when the matrix D on which the double sum matrix is based satisfies special constraints. The results of these two sections generalizes those of [Gilmore and Gomory, 1964].

The permutation ϕ of the columns of C used to define C as a double sum matrix plays a special role throughout this section because of the following theorem:

Theorem 5. ϕ is an optimal assignment for C.

Proof. Let ψ be any permutation for which there exist a p and q for which $p < q$ and $\phi^{-1}\psi(p) > \phi^{-1}\psi(q)$. Consider then $c(\psi) - c(\psi[p,q])$, that is

$-c\psi([p,q])$ as defined in §1. Let $p' = \phi^{-1}(p)$ and $q' = \phi^{-1}\psi(q) + 1$. Then

$$
\begin{aligned}
-c\psi([p,q]) &= c_{p\psi(p)} - c_{p\psi(q)} + c_{q\psi(q)} - c_{q\psi(p)} \\
&= \sum_{k=p}^{n} \sum_{l=q'}^{p'} d_{kl} - \sum_{k=q}^{n} \sum_{l=q'}^{p'} d_{kl} \\
&= \sum_{k=p}^{q-1} \sum_{l=q'}^{p'} d_{kl}
\end{aligned}
$$

Figure 2 illustrates the area of D to be summed to obtain this term. Since $d_{kl} \geq 0$ it follows that $c(\psi) \geq c(\psi[p,q])$. Thus any permutation ψ differing from ϕ can be transformed to ϕ by a series of interchanges that do not increase the cost of the assignment.

End of Proof.

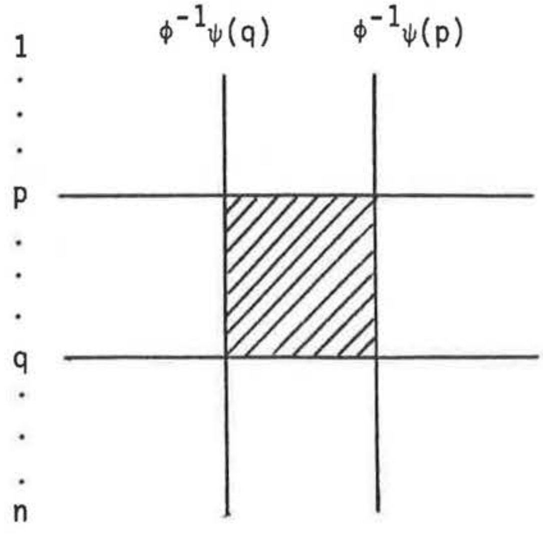

figure 2

Theorem 1 will be used to obtain a lower bound for the cost of a tour ψ. However some definitions are needed first.

A cyclic permutation $\rho = [p_1,\ldots,p_k]$ is said to cover an interchange $[q,q+1]$ if $\min\{p_1,\ldots,p_k\} \le q < \max\{p_1,\ldots,p_k\}$. A permutation $\psi = \phi\cdot\rho_1\cdots\rho_k$ is said to cover an interchange $[q,q+1]$ if one of $\rho_1,\ldots,\rho_k$ covers $[q,q+1]$.

Lemma 1. For any cyclic permutation ρ

$$c\phi(\rho) \ge \Sigma\{d_{qq+1} : \text{where } \rho \text{ covers } [q,q+1]\}$$

Proof. For any ρ

$$c\phi(\rho) = \sum_{q \text{ in } \rho} c_{q\phi(\rho(q))} - c_{q\phi(q)}.$$

Since $c_{q\phi(q)}$ sums only terms d_{ij} for which $i \ge j$ it follows that any term d_{ij} for which $i < j$ that is summed for $c_{q\phi(\rho(q))}$ contributes at least once to the sum $c\phi(\rho)$.

When $\rho = [p_1,p_2]$, where $p_1 < p_2$, then

$$c_{p_1\phi(p_2)} = \sum_{k=p_1}^{n} \sum_{l=1}^{p_2} d_{kl}$$

and therefore the sum includes d_{qq+1} for all q for which $p_1 \le q < p_2$.

Let ρ now be a cyclic permutation in some order of the cities $p_1,\ldots,p_k$, where $k > 2$ and $p_1 < p_2 <\ldots< p_k$. Consider any pair of cities p_{j-1} and p_j from ρ. Since ρ is a cyclic permutation of $p_1,\ldots,p_k$ in some order there is some i, $1 \le i \le j-1$, for which $\rho(p_i) \ge p_j$, for otherwise ρ could be factored into a cyclic permutation of $1,\ldots,j-1$ in some order and a cyclic permutation of $j,\ldots,n$ in some order. But

$$c_{p_i\phi(\rho(p_i))} = \sum_{k=p_i}^{n} \sum_{l=1}^{\rho(p_i)} d_{kl}$$

Since $p_i \le p_{j-1} < p_j \le \rho(p_i)$ this sum includes d_{qq+1} when $p_{j-1} \le q < p_j$.
End of Proof.

From theorem 1 therefore follows:

Lemma 2. For any permutation ψ

$$c(\psi) \geq c(\phi) + \Sigma\{d_{pp+1} : \text{where } \psi \text{ covers } [p,p+1]\}.$$

None of the lemmas established so far make any distinction between an arbitrary permutation and a tour. This deficiency will now be rectified.

If ϕ is itself a tour then it is necessarily an optimal tour for C because of theorem 5. For the remainder of this section it is assumed therefore that ϕ is not a tour but defines two or more subtours of the cities.

Let V be the set of subtours of ϕ. A graph $G^*(\psi) = (V, E^*(\psi))$ is uniquely determined by a permutation ψ as follows. Let as before $\psi = \phi\rho_1 \ldots \rho_k$. Let $E^*(\psi)$ consist of all pairs $\{v_1, v_2\}$, for which at least one of $\rho_1, \ldots, \rho_k$ has members in each of v_1 and v_2. The graph $G^*(\psi)$ permits the statement of a necessary condition for ψ to be a tour:

Lemma 3. If Ψ is a tour then $G^*(\Psi)$ is connected.

Proof. Let G^* be a connected component of $G^*(\psi)$, and let S be the set of all cities of nodes of G^*. Let p be any city of S. For each ρ_i all the cities of ρ_i are either all members of S or all not members. Hence $\rho_1 \ldots \rho_k(p)$ is in S, as is also $\phi\rho_1 \ldots \rho_k(p)$. Therefore $S = \{\psi(p) : p \varepsilon S\}$. It follows that if ψ is a tour S must contain all cities and $G^*(\psi)$ is connected.

End of Proof.

This necessary condition for ψ to be a tour together with lemma 2 will be used to obtain a lower bound on the cost of an optimal tour. To do so another graph $G = (V,E)$ with nodes V is needed. G has multiple edges, with an edge in E corresponding to each interchange $[p,p+1]$ for which p and p+1 are in different subtours of ϕ. For each such interchange there is an edge connecting the two subtours with cost $c\phi([p,p+1]) = d_{pp+1}$. Any permutation $\psi = \phi\rho_1\ldots\rho_k$ defines a subgraph $G(\psi) = (V,E(\psi))$ of G. Each edge of E corresponding to an interchange [p,p+1] covered by ψ is a member of $E(\psi)$ with the same cost d_{pp+1}.

Lemma 4. If $G^*(\psi)$ is connected then so also is $G(\psi)$.

Proof. Consider any edge $\{v_1,v_2\}$ of $G^*(\psi)$. Let ρ_i have a member r in v_1 and s in v_2. It is sufficient to show that there is a path in $G(\psi)$ connecting v_1 and v_2.

Without loss of generality it may be assumed that $r < s$. Consider the interchanges [p,p+1] for which $r \le p$ and $p+1 \le s$. By definition they are all covered by ψ. Consider those that are members of $E(\psi)$, that is for which p and p+1 are in different cycles of V. Consider also the cycles v of V which have a member q for which $r \le q \le s$. The interchanges correspond to edges of $G(\psi)$ connecting v_1 to v_2 via the given cycles.

End of Proof.

The graph G is necessarily connected. There is therefore a spanning tree of G with minimum cost. Let T be such a tree and let c(T) be its cost. A lower bound on the cost of tours for C can now be established.

Theorem 6. Let ψ be any tour. Then $c(\psi) \ge c(\phi) + c(T)$.

Proof. $G(\psi)$ is a subgraph of G and is by lemma 4 connected. Consequently if $c(E(\psi))$ is the sum of the costs of edges in $E(\psi)$ it follows that $c(E(\psi)) \ge c(T)$. But $\Sigma\{d_{pp+1} :$ where ψ covers $[p,p+1]\} \ge c(E(\psi))$. The theorem follows directly therefore from lemma 2.

End of Proof.

Let $\alpha_1,\ldots,\alpha_m$ be all of the interchanges determined by the edges of the minimum spanning tree T of G, where the ordering of the interchanges is arbitrary. The permutation $\alpha_1 \ldots \alpha_m$ determines disjoint cyclic permutation for which $\alpha_1 \ldots \alpha_m = \rho_1 \ldots \rho_k$. Naturally the sum $c\phi(\rho_1) + \ldots + c\phi(\rho_k)$ depends upon the order of application of the interchanges so that if $\psi = \phi\rho_1 \ldots \rho_k$ then $c(\psi)$ depends upon the order. However, the goal of combining the subtours into a single tour is achieved no matter the order of application:

Theorem 7. $\psi = \phi\alpha_1 \ldots \alpha_k$ is a tour if $\alpha_1, \ldots, \alpha_k$ are all the interchanges corresponding to the edges of T taken in any order.

The proof of the theorem is left as an exercise.

Exercise 5 Prove theorem 7.

Exercise 6 Show that a Fuller matrix is not in general double sum.

Exercise 7 Develop upper and lower bounds for the cost of an optimal tour for a double sum matrix using theorems 5 and 3 and provide a formula for the difference between these bounds. Compare this difference with c(T) of theorem 6.

Exercise 8 (Research problem) Are double sum matrices solvable?

§5. A solvable case of double sum matrices

The lower bound on the cost of a tour derived in theorem 6 was obtained under the assumption that C is double sum. Here it will be shown that under an assumption on the matrix D upon which C is based a tour ψ can be constructed with cost equal to the lower bound. The assumption, that each city is one of two types, has been suggested by [Gilmore and Gomory, 1964], and the main result of that paper will be shown to follow from the results of this section. The optimal tour $\psi = \phi\rho_1 \ldots \rho_k$ is constructed by applying the interchanges defined by the spanning tree T in a particular order to create cyclic permutations $\rho_1, \ldots, \rho_k$ for which $c\phi(\rho_1) + \ldots + c\phi(\rho_k) = c(T)$.

5.1 Type 1 and Type 2 Cities and An Order of Application of Interchanges

Consider now the matrix D upon which C is based. A city p, $1 \le p \le n$, may be of one of the following two types although for general D it may be of neither type:

Type 1: For all i and j, $i < p < j$, $d_{ij} = 0$.

Type 2: For all i and j, $i \ge p \ge j$, $d_{ij} = 0$.

The areas of D that must be zero for city p to be type 1 or type 2 are illustrated in figure 3. Note that if p is of type 2 then $C_{i\phi(j)} = 0$ for $i \ge p \ge j$. The assumption on D for the solvable case described in this section is that each city is either of type 1 or of type 2.

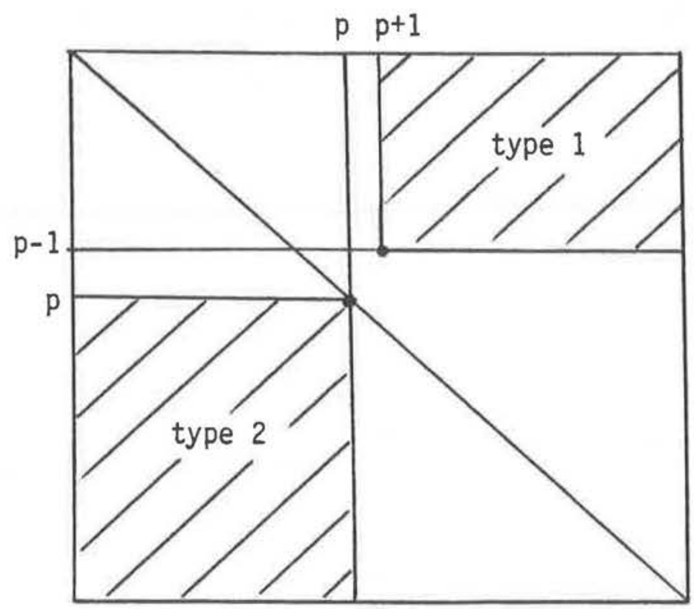

figure 3

Consider now the spanning tree T of §4. Such a tree defines a collection of interchanges $[p,p+1]$ which determine $c(T)$ as the sum of the costs $c\phi([p,p+1])$. By applying the interchanges in a particular order cyclic permutations $\rho_1,\ldots,\rho_k$ can be obtained for which $c\phi(\rho_1) +\ldots+ c\phi(\rho_k) = c(T)$. The possibilities for that order are determined by the following theorem:

Theorem 8. Let ρ be any cyclic permutation.

(a) Let either (i) q be of type 2 and the largest integer in ρ,

or (ii) $q+1$ be of type 1 and the smallest integer in ρ.

Then $c\phi(\rho[q,q+1]) = c\phi(\rho) + c\phi([q,q+1])$

(b) Let either (i) q be of type 1 and the largest integer in ρ,

or (ii) $q+1$ be of type 2 and the smallest integer in ρ.

Then $c\phi([q,q+1]\rho) = c\phi(\rho) + c\phi([q,q+1])$.

Proof. Only cases (ai) and (bi) will be proved. The other cases are left as an exercise.

Consider the case (ai). Note that if ψ is $\phi\rho[q,q+1]$ then

$$\psi(p) = \begin{cases} \phi(q+1) & \text{if } p = q, \\ \phi(\rho(q)) & \text{if } p = q+1, \\ \phi(\rho(p)) & \text{otherwise} \end{cases}$$

Let $\delta_1 = c_{q+1\phi(\rho(q))} - c_{q\phi(\rho(q))}$ and $\delta_2 = c_{q\phi(q+1)} - c_{q+1\phi(q+1)}$

Then $c\phi(\rho[q,q+1]) = c\phi(\rho) + \delta_1 + \delta_2$.

But $\delta_i = 0$ and $\delta_2 = d_{qq+1}$ since q is of type 2 and $q > \rho(q)$. Recall that $c\phi([q,q+1]) = d_{qq+1}$.

Consider lastly the case (bi). Note that if ψ is $\phi[q,q+1]\rho$ then

$$\psi(p) = \begin{cases} \phi(q+1) & \text{if } p = \rho^{-1}(q), \\ \phi(q) & \text{if } p = q+1, \\ \phi(\rho(p)) & \text{otherwise} \end{cases}$$

Let $\delta_1 = c_{\rho^{-1}(q)\phi(q+1)} - c_{\rho^{-1}(q)\phi(q)}$ and $\delta_2 = c_{q+1\phi(q)} - c_{q+1\phi(q+1)}$.
Then $c\phi([q,q+1]\rho) = c\phi(\rho) + \delta_1 + \delta_2$. But

$$\delta_1 = \sum_{i=\rho^{-1}(q)}^{n} d_{iq+1} \quad \text{and} \quad \delta_2 = \sum_{i=q+1}^{n} d_{iq+1}.$$

Therefore

$$\delta_1 + \delta_2 = \sum_{i=\rho^{-1}(q)}^{n} d_{iq+1} = d_{qq+1}$$

since q is of type 1 and $\rho^{-1}(q) < q$.

5.2 An Algorithm for Finding an Optimal Tour

Theorem 8 provides the basis for several algorithms for finding an optimal tour $\psi = \phi\rho_1\ldots\rho_k$ from an optimal assignment ϕ. Here one such algorithm is described.

(1) Let I be the set of interchanges determined by the tree T. Set i to 0.

(2) If I is empty then stop. Otherwise reset i to i+1. Let p be the smallest integer for which an interchange [p,p+1] is in I. Set ρ_i to [p,p+1] and reset I to I-{[p,p+1]}.

(3) Let q be the largest integer in ρ_i. If [q,q+1] is not in I go to (2). Otherwise if q is of type 2 reset ρ_i to $\rho_i\cdot$[q,q+1], and if q is of type 1 reset ρ_i to [q,q+1]$\cdot\rho_i$. Reset I to I-{[q,q+1]} and go to (3).

Step (2) of the algorithm begins the construction of the next cyclic permutation when ρ_i is completed. Step (3) of the algorithm is justified by the cases (ai) and (bi) of theorem 8.

5.3 An Example

The distance matrix C defined in [Gilmore and Gomory, 1964] is double sum and each city is of type 1 or type 2 so that the method of solution described in this section can be applied. To show this is not difficult.

Let f and g be functions defined over the reals satisfying:

$$f(x) + g(x) \geq 0 \tag{1}$$

and define:

$$_{ij} = \begin{cases} \displaystyle\int_{B_i}^{A_j} f(x)dx & \text{if } A_j \geq B_i\text{ ,} \\ \displaystyle\int_{A_j}^{B_i} g(x)dx & \text{otherwise.} \end{cases}$$

The value of c_{ij} can be rewritten as

$$|[-\infty, A_j] \cap [B_i, +\infty]|_f + [A_j, +\infty] \cap [B_i, -\infty]_g \tag{2}$$

where [ ] denotes an interval, $|\quad|_f$ is the integral of f over the enclosed interval, and $|\quad|_g$ the integral of g.

Assume that the cities have been renumbered and a permutation ϕ defined for which

$$B_1 \leq B_2 \leq \ldots \leq B_n \text{ and } A_{\phi(1)} \leq A_{\phi(2)} \leq \ldots \leq A_{\phi(n)}. \tag{3}$$

The distance matrix C can be simplified by subtracting the first column from every column and then subtracting the last row from every row. The resulting new distance from city i to city j is given by

$$(c_{i\phi(j)} - c_{i\phi(1)}) - (c_{n\phi(j)} - c_{n\phi(1)}). \tag{4}$$

Using (2):

$$c_{i\phi(j)} - c_{i\phi(1)} = |[A_{\phi(1)}, A_{\phi(j)}] \cap [B_i, +\infty]|_f + |[-\infty, B_i] \cap [A_{\phi(1)}, A_{\phi(j)}]|_g$$

so that

$$c_{n\phi(j)} - c_{n\phi(1)} = |[A_{\phi(1)}, A_{\phi(j)}] \cap [B_n, +\infty]|_f + [-\infty, B_n] \cap [A_{\phi(1)}, A_{\phi(j)}]|_g$$

Subtracting the second of these from the first yields

$$|[A_{\phi(1)}, A_{\phi(j)}] \cap [B_i, B_n]|_{f+g} \tag{5}$$

as the value of (4).

The distance matrix defined by (5) has been obtained by legitimate column and row operations so that (5) may replace (2) as the definition of c_{ij}. It is clear therefore that although the distance defined by (2) is dependent upon both f and g, a solution to an assignment or travelling salesman problem for that distance is dependent only upon f+g. This observation was drawn in [Gilmore and Gomory, 1964] from the algorithm used to solve the travelling salesman problem. Here the observation is immediate.

Define B_{n+1} to be B_n and let

$$P_k = [B_k, B_{k+1}] \quad \text{for } k=1,\ldots,n.$$

Similarly define $A_{\phi(0)}$ to be $A_{\phi(1)}$ and let

$$Q_l = [A_{\phi(l-1)}, A_{\phi(l)}] \quad \text{for } l=1,\ldots,n.$$

Since

$$[B_i, B_n] = \bigcup_{k=i}^{n} P_k \text{ and } [A_\phi(1), A_\phi(j)] = \bigcup_{l=1}^{j} Q_l$$

it follows that

$$[A_\phi(1), A_{\phi(j)}] \cap [B_i, B_n] = \bigcup_{k=i}^{n} \bigcup_{l=1}^{j} (P_k \cap Q_l).$$

But the P's are all mutually disjoint as are also the Q's. It follows therefore that

$$c_{i\phi(j)} = \sum_{k=i}^{n} \sum_{l=1}^{j} |P_k \cap Q_l|_{f+g}$$

so that C is double sum based on the matrix D with $d_{kl} = |P_k \cap Q_l|_{f+g}$.

Further each city is necessarily of type 1 or of type 2 since for any p either $B_p \le A_{\phi(p)}$ or $B_p > A_{\phi(p)}$. In the first case it follows from (3) that $B_i \le A_{\phi(j)}$ whenever $i \le p \le j$. But $P_{i-1} = [B_{i-1}, B_i]$ and $Q_{j+1} = [A_{\phi(j)}, A_{\phi(j+1)}]$ so that $d_{i-1\ j+1} = 0$; that is $d_{ij} = 0$ for $i \ge p \ge j$ as required for a type 1 city. In the second case it follows from (3) that $B_i > A_{\phi(j)}$ whenever $i \ge p \ge j$. But $P_i = [B_i, B_{i+1}]$ and $Q_j = [A_{\phi(j-1)}, A_{\phi(j)}]$ so that $d_{ij} = 0$ for $i \ge p \ge j$ as required for a type 2 city.

Exercise 8 Show that the greedy algorithm will not in general find an optimal tour for a double sum matrix in which each city is of type 1 or type 2. By the greedy algorithm is meant the algorithm which beginning with one city will find a tour by taking as the next city one of least distance from the present which has not been previously visited.

Exercise 9 Prove cases (aii) and (bii) of theorem 8.

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