ON PSEUDO-SIMILAR VERTICES IN TREES

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Abstract

Two dissimilar vertices u and v in a graph G are said to be pseudo-similar if $G\backslash u \cong G\backslash v$. A characterization theorem is presented for trees (later extended to forests and block-graphs) with strictly pseudo-similar (i.e. pseudo-similar but dissimilar) vertices. It follows from this characterization that it is not possible to have three or more mutually strictly pseudo-similar vertices in trees. Furthermore, pseudo-similarity combined with an extension of pseudo-similarity to include the removal of first neighbourhoods of vertices is sufficient to imply similarity in trees. Neither of these results holds if we replace trees by arbitrary graphs.

1. Introduction

Two vertices u and v in a graph G[†] are <u>similar</u>, denoted $u \sim_G v$ (or simply $u \sim v$ when G is clear from the context), if there exists an automorphism of G mapping u onto v. We are concerned, in this paper, with the notion of similarity and a related notion called pseudo-similarity among vertices in arbitrary trees.

An obvious consequence of the definition of similarity is that $u \sim_G v$ implies $G\backslash\{u\} \cong G\backslash\{v\}$[††], which we abbreviate as $G\backslash u \cong G\backslash v$. According to Harary and Palmer [3], an incorrect proof of the celebrated Reconstruction Conjecture was based on the supposed truth of the converse, namely that $G\backslash u \cong G\backslash v$ implies $u \sim_G v$. While this converse holds in certain interesting situations (e.g. in regular graphs) it is not true in general and counter-examples exist even among trees, the smallest of which is illustrated in Figure 1. Two vertices u and v satisfying $G\backslash u \cong G\backslash v$ are said to be <u>pseudo-similar</u> in G. If, in addition, $u \not\sim_G v$, they are said to be <u>strictly pseudo-similar</u>.

The notion of pseudo-similarity has received considerable attention for both graphs and trees [1, 3, 6]. Early work of Harary and Palmer [3] focused on pseudo-similarity in connected block-graphs. Trees form the most interesting class of connected block-graphs and, in section 5, we show that there is no loss of generality in restricting the study of similarity and pseudo-similarity in block-graphs to the special case of trees. Harary

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† We denote by V(G) (respectively E(G)) the vertex (respectively edge) set of G.

†† If $S \subseteq V(G)$, then $G\backslash S$ denotes the subgraph of G induced on $V(G)\backslash S$.

and Palmer's main result is an interesting characterization of strictly pseudo-similar cutpoints in connected block graphs (equivalently, strictly pseudo-similar vertices in trees). This characterization will be discussed in more detail in section 3.

Harary and Palmer present a general construction for graphs and trees with strictly pseudo-similar vertices. In related work, Krishnamoorthy and Parthasarathy [6] construct a family of graphs with arbitrarily large sets of mutually strictly pseudo-similar vertices. An obvious open question raised by this work is whether such a construction exists for trees. We show, as a direct corollary of a new characterization of pseudo-similar vertices in trees, that a tree can not have a set of more than two mutually strictly pseudo-similar vertices.

In [1], the notion of pseudo-similarity and the results of Harary and Palmer [3] and Krishnamoorthy and Parthasarathy [6] are extended to k-pseudo-similarity and full k-pseudo-similarity. For a graph G=(V,E), let Γ_v^k denote the set of vertices of distance less than or equal to k from vertex v in G (by definition $\Gamma_v^0=\{v\}$; Γ_v^1 is abbreviated Γ_v). Two vertices u and v are said to be <u>k-pseudo-similar</u> if $G\backslash\Gamma_u^k \cong G\backslash\Gamma_v^k$. Vertices u and v are full <u>k-pseudo-similar</u> if they are i-pseudo-similar for all $i \le k$. In general, even full k-pseudo-similarity, for all k, does not imply similarity. In fact, there exist families of graphs with arbitrarily many pairwise strictly full k-pseudo-similar vertices, for all k[1].

In this paper, we restrict the study of k-pseudo-similarity to vertices of arbitrary trees. We show that, in contrast to the more general setting, full 1-pseudo-similarity is sufficient to imply similarity in trees. Similar results for edge pseudo-similarity are discussed in section 7.

Section 2 introduces notation and definitions specific to this paper and presents some preliminary lemmas concerning the subtree structure of trees. Section 3 develops our characterization of pseudo-similarity in trees. This characterization is extended to arbitrary forests in section 4. The tree characterization is further exploited in section 5 to prove that full 1-pseudo-similarity is equivalent to similarity in trees. Sections 6 and 7 present further extensions (to block graphs) and related results on edge pseudo-similarity.

2. Trees and Branches

We will find it convenient to refer to rooted trees without always specifying the root. Our convention is that, unless otherwise specified, whenever some, possibly sub or superscripted, upper case letter (e.g. X_i) denotes a rooted tree, then the corresponding lower case letter, with identical sub or superscripting, (e.g. x_i) denotes the root of that tree. When it becomes necessary to root an otherwise unrooted tree T at some vertex, say r, we will denote the resulting rooted tree (T,r).

If two rooted trees X and Y are isomorphic (that is, the isomorphism preserves the root) then we denote this by $X\overset{*}{\cong}Y$.

If $X_1,\cdots X_k$ are distinct rooted trees then we denote by $<X_1,\cdots,X_k>$ the the (unrooted) tree with vertex set $\bigcup_{i=1}^{k}V(X_i)$ and edge set $\{(x_j,x_{j+1})|1\le j<k\}\cup\bigcup_{i=1}^{k}E(X_i)$. (Note that $<X_1,\ldots,X_k>$ is indistinguishable from $<X_k,\ldots,X_1>$.) Graphically, if we represent the rooted tree X_i as in figure 2 (a), then figure 2 (b) denotes the tree $<X_1,\ldots,X_k>$.

The motivation for introducing this "chaining" of trees should be clear from the following:

Proposition 2.1. If T is any tree with two specified vertices u and v, then there exist s≥1 distinct rooted trees $X_1,\cdots,X_s$, such that $T=<X_1,\cdots,X_s>$, $u=x_1$ and $v=x_s$.

Every rooted tree X has a set $\{X_1,\cdots,X_k\}$ of disjoint rooted subtrees (we will call them primary subtrees) with the property that $V(X)=\{x\}\cup\bigcup_{i=1}^{k}V(X_k)$ and $E(X)=\{(x,x_i)|1\le i\le k\}\cup\bigcup_{i=1}^{k}E(X_k)$.

Obviously, two isomorphic rooted trees have isomorphic sets of primary subtrees. With similar motivation, we find it useful to generalize the notion of a primary subtree to that of a branch.

The rooted tree B is a branch of the rooted tree T (equivalently T has a B-branch) if either:

i) $B \overset{*}{\cong} T$; or

ii) B is a branch of a primary subtree of T.

(In the second case B is said to be a proper branch of T.)

If T is an unrooted tree then the rooted tree B is said to be a branch of T if, for some rooting (T,r) of T, B is a branch of (T,r).

The following proposition is a straightforward consequence of the definition of a branch.

Proposition 2.2. If B is a branch of the rooted tree T, then there exist $r \geq 1$ rooted trees $Y_1,\ldots,Y_r$, such that

i) $B \overset{*}{\cong} Y_r$; and

ii) $T = (\langle Y_1,\ldots,Y_r\rangle, y_1)$.

As proposition 2.2 points out, a branch is not just a rooted subtree. In particular, if B is a branch of T then T with B removed is either empty or connected. This fact is exploited in the following:

Lemma 2.3. If B is any branch of $\langle X,Y\rangle$, then either

i) B is a branch of X;

ii) B is a branch of Y;

iii) X is a proper branch of B; or

iv) Y is a proper branch of B.

Proof. If both x and y belong to B, but neither X nor Y is a proper branch of B, then $\langle X,Y\rangle$ is disconnected by the removal of B, contrary to our assumptions. □

Corollary 2.4. If $|X| \leq |Y|$ then x belongs to a unique branch of size $|X|$ in $\langle X,Y\rangle$, namely X.

Lemma 2.5. If B is any branch of $(\langle X_1,\ldots,X_s\rangle,x_1)$, then either:

i) $B = (\langle X_i,\ldots,X_s\rangle,x_i)$, where $1\leq i\leq s$; or

ii) B is a proper branch of X_i, for some i, $1\leq i\leq s$.

Proof. When $s\leq 2$ the result follows directly from the definitions. When $s>2$ the result follows by straightforward induction on s. □

Corollary 2.6. If $|X_i| \leq |X_s|$, $1\leq i<s$, then $(\langle X_1,\ldots,X_s\rangle,x_1)$ has a unique branch of size $|X_s|$, namely X_s.

Lemma 2.7. If $(\langle X_1,X_2\rangle,x_1) \overset{*}{\cong} (\langle Y_1,Y_2\rangle,y_1)$ and $|X_2| \geq |Y_1|$, then $X_1 \overset{*}{\cong} Y_1$ and $X_2 \overset{*}{\cong} Y_2$.

Proof. Let α be any isomorphism taking $(\langle X_1,X_2\rangle,x_1)$ onto $(\langle Y_1,Y_2\rangle,y_1)$. Then $\alpha(X_2)$ forms a proper branch of $(\langle Y_1,Y_2\rangle,y_1)$. By lemma 2.5, this branch is either a proper branch of Y_1, contradicting the fact that $|X_2| \geq |Y_1|$, or is a branch of Y_2. Since $\alpha(x_2)$ is adjacent to y_1, it follows that $\alpha(X_1) = Y_1$ and $\alpha(X_2) = Y_2$. □

Corollary 2.8. If $(\langle X_i,\ldots,X_{s-1}\rangle, x_i) \overset{*}{\cong} (\langle X_{j+1},\ldots,X_s\rangle, x_{j+1})$, where $i \leq j < s$ then $X_k \overset{*}{\cong} X_{j+k}$, $i \leq k < s-j$, and $X_s \overset{*}{\cong} (\langle X_{s-j},\ldots,X_{s-1}\rangle, x_{s-j})$.

Proof. Straightforward induction on s-j. □

Corollary 2.9. If $(\langle X_1,\ldots,X_{s-1}\rangle, x_{s-1}) \overset{*}{\cong} (\langle X_2,\ldots,X_s\rangle, x_2)$ then $X_i \overset{*}{\cong} X_{s-i+1}$, $1 \leq i \leq s$.

Proof. Straightforward induction on s. □

Corollary 2.10. If $(\langle X_1,\ldots,X_s\rangle, x_s) \overset{*}{\cong} (\langle X_1,\ldots,X_s\rangle, x_1)$ then $X_i \overset{*}{\cong} X_{s-i+1}$, $1 \leq i \leq s$.

Proof. Straightforward induction on s. □

We denote by Br{X;T;v} the number of (not necessarily disjoint) X-branches containing vertex v in the (possibly rooted) tree T. Br{X;T} denotes the number of (not necessarily disjoint) X-branches in T. Obviously, if $T_1 \cong T_2$ then $Br\{X;T_1\} = Br\{X;T_2\}$. The following lemma allows us to relate the branch structure of certain trees to the branch structure of their subtrees.

Lemma 2.11. If X, Y and Z are rooted trees satisfying $|X| \leq |Z| \leq |Y|$ then

$$Br\{Z;\langle X,Y\rangle\} = Br\{Z;\langle X,Y\rangle;x\} + Br\{Z;Y\}.$$

Proof. Immediate from lemma 2.3. □

Lemma 2.12. If B and (T,r) are rooted trees satisfying $|B| < |T|$, then

$$Br\{B;(T,r)\} = Br\{B;T\} - Br\{B;T;r\}.$$

Proof. It follows from proposition 2.2 that the only branch of (T,r) containing r is (T,r) itself. Hence, if $|B| < |T|$ any B-branch of T containing r is not a branch of (T,r). □

3. A Characterization of Strict Pseudo-Similarity in Trees

We are now prepared to develop a new characterization of trees with strictly pseudo-similar vertices. Before doing so let us recall the characterization presented by Harary and Palmer [3] expressed in our notation.

Theorem A. [3] If T is any tree with strictly pseudo-similar vertices u and v then either

i) there exist rooted trees Y_k^j, $0 \leq j \leq 2$, $1 \leq k \leq t$, for some $t > 2$, where $Y_k^j \overset{*}{\cong} Y_k^0$, $1 \leq j \leq 2$, such that $T = \langle Y_1^0,\ldots,Y_t^0,Y_1^1,\ldots,Y_t^1,Y_1^2,\ldots,Y_{t-1}^2\rangle$, $u = y_t^0$ and $v = y_t^1$; or

ii) there exists a vertex w in the component T' of T\u containing v such that w and v are strictly pseudo-similar in T'.

Theorem A provides a quite explicit characterization of minimal trees with strictly pseudo-similar vertices. An obvious question is whether a similar characterization holds for all trees with strictly pseudo-similar vertices.

Recalling proposition 2.1, it is easy to confirm the following:

Proposition 3.1. If T is any tree with distinct pseudo-similar vertices u and v, then there exist $s \geq 2$ distinct rooted trees $X_1,\ldots,X_s$, such that $T = \langle X_1,\ldots,X_s\rangle$, $u = x_1$, $v = x_s$, $\langle X_1,\ldots,X_{s-1}\rangle \cong \langle X_2,\ldots,X_s\rangle$ and $X_1\backslash x_1 \cong X_s\backslash x_s$.

For the remainder of this section let $T = \langle X_1,\ldots,X_s\rangle$, $P = \langle X_1,\ldots,X_{s-1}\rangle$. $Q = \langle X_2,\ldots,X_s\rangle$ and $R = \langle X_2,\ldots,X_{s-1}\rangle$. Note that $T = \langle X_1,(Q,x_2)\rangle = \langle (P,x_{s-1}),X_s\rangle$, $P = \langle X_1,(R,x_2)\rangle$, and $Q = \langle (R,x_{s-1}),X_s\rangle$. Obviously, $|P| = |Q|$ if and only if $|X_1| = |X_s|$.

Lemma 3.2. $P \cong Q$ and $X_1 \overset{*}{\cong} X_s$ if and only if $x_1 \sim_T x_s$.

Proof. If $x_1 \sim_T x_s$ then it follows from corollary 2.10 that $P \cong Q$ and $X_1 \overset{*}{\cong} X_s$.

Conversely, suppose $P \cong Q$ and $X_1 \overset{*}{\cong} X_s$. If $s \leq 3$, then the fact that $x_1 \sim_T x_s$ is immediate. For $s > 3$, we proceed by induction on $|T|$, assuming that the hypothesis is true for all trees smaller than T. Let α be any isomorphism taking P onto Q. $\alpha(X_i)$ (respectively, $\alpha(x_i)$) denotes the image of X_i (respectively, x_i) under α. (It is assumed that $\alpha(X_i)$ is rooted at $\alpha(x_i)$). Suppose that $\alpha(x_1) \neq x_s$; otherwise there is nothing to prove. There are two cases:

i) $\alpha(x_1) \in X_s \backslash x_s$. Since $\alpha(X_1) \not\subseteq X_s \backslash x_s$ (because $|X_1| = |X_s|$), it follows from lemma 2.3 that (R, x_{s-1}) is a proper branch of $\alpha(X_1)$ and $(\alpha(R), \alpha(x_2))$ is a proper branch of X_s. Hence, by proposition 2.2, there exist $r \geq 1$ rooted trees $Y_1, \ldots, Y_r$, such that $X_s = (\langle Y_1, \ldots, Y_r, (\alpha(R), \alpha(x_2))\rangle, y_1)$ and $\alpha(X_1) = (\langle (R, x_{s-1}), Y_1, \ldots, Y_r\rangle, y_r)$ (see figure 3(a)). Since $X_1 \overset{*}{\cong} X_s$, it follows by corollary 2.9, that $(\alpha(R), \alpha(x_2)) \overset{*}{\cong} (R, x_{s-1})$, and hence $x_2 \sim_R x_{s-1}$ and $x_1 \sim_T x_s$.

ii) $\alpha(x_1) \in X_i$, where $i < s$. By lemma 2.3, we know that $\alpha(X_1)$ is a branch of (R, x_{s-1}) and X_s is a branch of $(\alpha(R), \alpha(x_2))$. Hence, by proposition 2.2, there exist $r \geq 2$ rooted trees $Y_1, \ldots, Y_r$, such that $(R, x_{s-1}) = (\langle Y_1, \ldots, Y_{r-1}\rangle, y_{r-1})$, $(\alpha(R), \alpha(x_2)) = (\langle Y_2, \ldots, Y_r\rangle, y_2)$, $\alpha(X_1) \overset{*}{\cong} Y_1$, and $X_s \overset{*}{\cong} Y_r$ (see figure 3(b)). It follows, by our induction hypothesis, that y_1 and y_r are similar in $\langle Y_1, \ldots, Y_r\rangle$. Hence, $(R, x_{s-1}) \overset{*}{\cong} (\alpha(R), \alpha(x_2))$ or $x_2 \sim_R x_{s-1}$, from which it follows that $x_1 \sim_T x_s$. □

A direct consequence of lemma 3.2 is the following:

Corollary 3.3. [3, Theorem 4] If T is any tree with pseudo-similar leaves u and v, then $u \sim_T v$.

<u>Lemma 3.4</u>. If $P \cong Q$ and $x_1 \not\sim_T x_s$ then $|X_j| \leq |X_1|$, $1 < j < s$.

<u>Proof</u>. Suppose to the contrary that $|X_j| \leq |X_1|$, $1 < j < p < s$ and $|X_p| > |X_1|$. Let $Y = \langle X_1, \ldots, X_{p-1} \rangle$. Since $T = \langle X_1, (Q, x_2) \rangle$ it follows, by lemma 2.11, that $Br\{(Y, x_{p-1}); T\} = Br\{(Y, x_{p-1}); T; x_1\} + Br\{(Y, x_{p-1}); (Q, x_2)\}$. Similarly, since $T = \langle (P, x_{s-1}), X_s \rangle$ it follows, by lemma 2.11, that $Br\{(Y, x_{p-1}); T\}$ $= Br\{(Y, x_{p-1}); T; x_s\} + Br\{(Y, x_{p-1}); (P, x_{s-1})\}$. But clearly $Br\{(Y, x_{p-1}); T; x_1\} > 0$ and hence $Br\{(Y, x_{p-1}); T; x_s\} > 0$ or $Br\{(Y, x_{p-1}); (P, x_{s-1})\} > Br\{(Y, x_{p-1}); (Q, x_2)\}$. We consider the two cases separately:

i) $Br\{(Y, x_{p-1}); (P, x_{s-1})\} > Br\{(Y, x_{p-1}); (Q, x_2)\}$. Since $Br\{(Y, x_{p-1}); P\}$ $= Br\{(Y, x_{p-1}); Q\}$, it follows, by lemma 2.12, that $Br\{(Y, x_{p-1}); Q; x_2\} > Br\{(Y, x_{p-1}); P; x_{s-1}\} \geq 0$. Consider any (Y, x_{p-1})-branch B in Q containing the vertex x_2. B must also contain the vertex x_p, since otherwise B is a branch of $\langle X_2, \ldots, X_{p-1} \rangle$, contradicting the fact that $|Y| > |\langle X_2, \ldots, X_{p-1} \rangle|$. Similarly B must contain all of X_j, $2 \leq j \leq p$, since otherwise $|B| > |Q \backslash V(X_j)|$, by lemma 2.3

$$\begin{aligned} |B| &> |Q \backslash V(X_j)| \\ &\geq |P| - |X_p| \\ &\geq |Y|. \end{aligned}$$

Hence $|B| \geq |X_2| + \ldots + |X_p|$, contradicting our assumption that $|X_p| > |X_1|$.

ii) $Br\{(Y, x_{p-1}); T; x_s\} > 0$. Consider any (Y, x_{p-1})-branch B in T containing the vertex x_s. X_s must be a branch of B; otherwise, by Lemma 2.3, B contains $\langle X_1, \ldots, X_{s-1} \rangle$, contradicting the fact that

$|Y| < |X_1| + \ldots + |X_{s-1}|$. But, since $|X_j| \leq |X_1|$, $1 < j < p$, it follows, by corollary 2.6, that B contains a unique branch of size $|X_1|$, namely X_1 itself. Thus $X_1 \overset{*}{\cong} X_s$ and, by lemma 3.2, $x_1 \sim_T x_s$, contradicting our assumptions.

Since both cases lead to contradictions, it follows that $|X_j| \leq |X_1|$, $1 < j < s$. $\square$

Lemma 3.5. If $P \cong Q$ and $x_1 \not\sim_T x_s$, then for some integers $i \geq 1$, $t > 1$, and $1 \leq h \leq t$, there exist rooted trees Y_k^j, $0 \leq j \leq i+1$, $1 \leq k \leq t$, where $Y_k^j \overset{*}{\cong} Y_k^0$, $0 \leq j \leq i+1$, $1 \leq k \leq t$, such that $T = \langle Y_1^0,\ldots,Y_t^0,Y_1^1,\ldots,Y_t^1,\ldots,Y_1^{i+1},\ldots,Y_{h-1}^{i+1}\rangle$, $x_1 = y_t^0$ and $x_s = y_h^i$.

Proof. Let α be any isomorphism taking P onto Q. We consider four cases:

i) $\alpha(x_1) \in X_j \backslash x_j$, where $2 \leq j < s$. By lemma 2.3, $\alpha(X_1)$ is either a branch of X_j (impossible, by lemma 3.4), or $\alpha(X_1)$ contains X_s as a proper branch (which contradicts the size of X_1).

ii) $\alpha(x_1) = x_j$, where $2 \leq j < s$. Since $|X_1| = |X_s|$, we must have $\alpha(X_1) = (\langle X_2,\ldots,X_j\rangle,x_j)$ and $(\langle\alpha(X_2),\ldots,\alpha(X_{s-1})\rangle,\alpha(x_2))$ $= (\langle X_{j+1},\ldots,X_s\rangle,x_{j+1})$. It follows, by corollary 2.8, that $X_k \overset{*}{\cong} X_{k+j-1}$, $2 \leq k < s-j+1$, and $X_s \overset{*}{\cong} (\langle X_{s-j+1},\ldots,X_{s-1}\rangle,x_{s-j+1})$. Thus the lemma holds with $t = j-1$, $i = \lceil (s-2)/t \rceil$, $h = s-1-(i-1)t$, and $Y_p^q \overset{*}{\cong} X_{p+1}$, $1 \leq p \leq t$, $0 \leq q \leq i+1$.

iii) $\alpha(x_1) = x_s$. Since $X_1 \overset{*}{\not\cong} X_s$, by lemma 3.2, it follows that $\alpha(X_1)$ contains $\langle X_2,\ldots,X_{s-1}\rangle$ and thus there exists a rooted tree Z, such that $\alpha(X_1) = (\langle X_2,\ldots,X_{s-1},Z\rangle,z)$ and $X_s = (\langle Z,\alpha(X_2),\ldots,\alpha(X_{s-1})\rangle,z)$. Thus the lemma holds with $t = s-1$, $i = 1$, $h = t$, and $Y_p^q \overset{*}{\cong} X_{p+1}$, $1 \leq p < t$, $0 \leq q \leq 2$, and $Y_t^q \overset{*}{\cong} Z$, $0 \leq q \leq 1$.

iv) $\alpha(x_1) \in X_s \backslash x_s$. Since $|X_1| = |X_s|$, it follows, by lemma 2.3, that $\alpha(X_1)$ contains $<X_2,\dots,X_{s-1}>$ and thus there exist $r \geq 1$ rooted trees $Z_1,\dots,Z_r$ such that $\alpha(X_1) = (<X_2,\dots,X_{s-1},Z_1,\dots,Z_r>,z_r)$ and $X_s = (<Z_1,\dots,Z_r,\alpha(X_2),\dots,\alpha(X_{s-1})>,z_1)$. Thus the lemma holds with $t = s+r-2$, $i = 1$, $h = s-1$, and

$$Y_p^q \overset{\star}{\cong} \begin{cases} X_{p+1} & 1 \leq p \leq s-2,\ 0 \leq q \leq 2 \\ \\ Z_{p-s+2} & s-1 \leq p \leq s+r-2,\ 0 \leq q \leq 1. \end{cases}$$

In each case it follows that if $t = 1$ then $X_1 \overset{\star}{\cong} X_s$ and hence $x_1 \sim_T x_s$, by lemma 3.2, contradicting our hypotheses. □

An immediate consequence of lemma 3.5 is that lemma 3.4 can be strengthened as follows:

Corollary 3.6. If $P \cong Q$ and $x_1 \not\sim_T x_s$, then $|X_1| = |X_s| > |X_j|$, $1 < j < s$.

Lemma 3.5 is extended to a characterization of trees with strictly pseudo-similar vertices in the following theorem.

Theorem 3.7. If T is any tree with strictly pseudo-similar vertices u and v, then for some integers $i \geq 1$, $t > 1$, and $1 \leq h \leq t$, there exist rooted trees Y_k^j, $0 \leq j \leq i+1$, $1 \leq k \leq t$, where $Y_k^j \overset{\star}{\cong} Y_k^0$, $0 \leq j \leq i+1$, $1 \leq k \leq t$, such that

$$T = <Y_1^0,\dots,Y_t^0,Y_1^1,\dots,Y_t^1,\dots,Y_1^{i+1},\dots,Y_{h-1}^{i+1}>,\ u=y_t^0,\ v = y_h^i, \text{ and}$$

$$<Y_1^0,\dots,Y_t^0>\backslash y_t^0 \cong <Y_h^i,\dots,Y_t^i,Y_1^{i+1},\dots,Y_{h-1}^{i+1}>\backslash y_h^i.$$

Proof. Immediate from proposition 3.1 and lemma 3.5. □

Another direct consequence of lemma 3.5 (more specifically, corollary 3.6) is that any set of mutually strictly pseudo-similar vertices of a tree, unlike a general graph, has cardinality at most two. Specifically,

<u>Theorem 3.8</u>. Let T be any tree with vertices u, v, and w. If u and v are strictly pseudo-similar in T and u and w are strictly pseudo-similar in T, then v = w.

<u>Proof</u>. By proposition 3.1 and corollary 3.6, it follows that there exist $s \geq 2$ distinct rooted trees $X_1,\ldots,X_s$ such that $T = \langle X_1,\ldots,X_s \rangle$, $u = x_1$, $v = x_s$, and $|X_1| = |X_s| > |X_j|$, $1 < j < s$. Similarly, there exist $r \geq 2$ distinct rooted trees $Y_1,\ldots,Y_r$ such that $T = \langle Y_1,\ldots,Y_r \rangle$, $u = y_1$, $w = y_r$, and $|Y_1| = |Y_r| > |Y_j|$, $1 < j < r$. By lemma 2.7, $X_1 \stackrel{*}{\cong} Y_1$ and hence $|X_s| = |Y_r|$. But, by corollary 2.6, (T, u) has a unique branch of size $|X_s|$ namely X_s. Hence $X_s = Y_r$ and, in particular, v = w. $\square$

4. Strict Pseudo-Similarity in Forests

Suppose F is any forest with strictly pseudo-similar vertices u and v. If u and v belong to the same component T of F then it should be clear that u and v are strictly pseudo-similar in T and hence the characterization of the previous section (theorem 3.7) holds. What if u and v belong to distinct components T_1 and T_2 in F? This turns out to be possible only when $T_1 \cong T_2$, say $\alpha(T_1) = T_2$, and $\alpha(u)$ and v are strictly pseudo-similar in T_2.

Lemma 4.1. If T_1 and T_2 are distinct trees with $u \in V(T_1)$ and $v \in V(T_2)$, then u and v are (pseudo-) similar in $T_1 \cup T_2$ if and only if u and v are (pseudo-) similar in the tree $T_1 \cup T_2 \cup \{(u,v)\}$.

Proof. The result for pseudo similarity is obvious since $(T_1 \cup T_2)\backslash u \equiv (T_1 \cup T_2 \cup \{(u,v)\})\backslash u$ and $(T_1 \cup T_2)\backslash v \equiv (T_1 \cup T_2 \cup \{(u,v)\})\backslash v$. If u and v are similar in $T_1 \cup T_2$, then $(T_1,u) \overset{*}{\cong} (T_2,v)$. If α is any isomorphism taking (T_1,u) onto (T_2,v) then the automorphism δ given by

$$\delta(x) = \begin{cases} \alpha(x) & \text{if } x \in V(T_1) \\ \\ \alpha^{-1}(x) & \text{if } x \in V(T_2) \end{cases}$$

interchanges u and v in $T_1 \cup T_2$ (and hence also in $T_1 \cup T_2 \cup \{(u,v)\}$). If u and v are similar in $T_1 \cup T_2 \cup \{(u,v)\}$ then there is an automorphism exchanging them (see, for example, corollary 1 of [3]), and hence $(T_1,u) \overset{*}{\cong} (T_2,v)$ and $u \sim_{T_1 \cup T_2} v$. □

Theorem 4.2. If F is any forest with strictly pseudo-similar vertices u and v, where u and v belong to components T_1 and T_2 respectively, then there exists a vertex $w \in V(T_2)$ such that $(T_1,u) \stackrel{*}{\cong} (T_2,w)$ and v and w are strictly pseudo-similar in T_2.

Proof. If $T_1 = T_2$ the result is obvious. Otherwise, we know, by lemma 4.1, that u and v are strictly pseudo-similar in $T_1 \cup T_2 \cup \{(u,v)\}$. It follows, by theorem 3.7, that for some $t > 1$, there exist rooted trees Y_k^j, $0 \leq j \leq 1$, $1 \leq k \leq t$, where $Y_k^0 \stackrel{*}{\cong} Y_k^1$, $1 \leq k \leq t$, $u = y_t^0$, $v = y_1^1$, and $\langle Y_1^0,\ldots,Y_t^0\rangle \backslash y_t^0 \cong \langle Y_1^1,\ldots,Y_t^1\rangle \backslash y_1^1$. But $T_1 = \langle Y_1^0,\ldots,Y_t^0\rangle$ and $T_2 = \langle Y_1^1,\ldots,Y_t^1\rangle$. Choosing $w = y_t^1$ the result follows directly. □

5. Full k-Pseudo-Similarity in Trees

In section 3, we presented a new characterization of trees with pseudo-similar vertices. It is natural to ask if this characterization can be extended to the notion of full k-pseudo-similarity (cf. section 1). Surprisingly perhaps, this characterization is very simple since for $k = 1$ (and hence for all $k \geq 1$) full k-pseudo-similarity is equivalent to similarity in trees.

Theorem 5.1. If $T\backslash u \cong T\backslash v$ and $T\backslash\Gamma_u \cong T\backslash\Gamma_v$, then $u \sim_T v$.

Proof. Suppose not. Let T be any smallest counter example. Since $T\backslash u \cong T\backslash v$ it follows by theorem 3.7 that there exist integers $i \geq 1$, $t > 1$, and $1 \leq h \leq t$, and rooted trees Y_k^j, $0 \leq j \leq i+1$, $1 \leq k \leq t$, where $Y_k^j \overset{*}{\cong} Y_k^0$, $0 \leq j \leq i+1$, $1 \leq k \leq t$, such that $T = \langle Y_1^0,\ldots,Y_t^0,Y_1^1,\ldots,Y_t^1,\ldots,Y_1^{i+1},\ldots,Y_{h-1}^{i+1}\rangle$, $u = y_t^0$ and $v = y_h^i$. We consider two cases:

i) u and v are adjacent in T; that is, $i = h = 1$. In this case, $T = \langle Y_1^0,\ldots,Y_t^0,Y_1^1,\ldots,Y_t^1\rangle$. Let $Y^i = \langle Y_1^i,\ldots,Y_t^i\rangle$, $0 \leq i \leq 1$. Since $T\backslash u \cong T\backslash v$, it follows that $Y^0\backslash y_t^0 \cong Y^1\backslash y_1^1$. Furthermore, since $T\backslash\Gamma_u \cong T\backslash\Gamma_v$, we know that $Y^0\backslash\Gamma_{y_t^0} \cong Y^1\backslash\Gamma_{y_1^1}$. Hence $Y^0\backslash y_t^0 \cong Y^0\backslash y_1^0$ and $Y^0\backslash\Gamma_{y_t^0} \cong Y^0\backslash\Gamma_{y_1^0}$ and so, by our minimality assumption, $y_t^0 \sim_{Y^0} y_1^0$. Hence $(Y^0,y_t^0) \overset{*}{\cong} (Y^1,y_1^1)$, and $u \sim_T v$, contradicting our assumption.

ii) u and v are not adjacent in T. Let $Q = \langle Y_1^1,\ldots,Y_t^1,\ldots,Y_1^{i+1},\ldots,Y_{h-1}^{i+1}\rangle$ and $P = \langle Y_1^0,\ldots,Y_t^0,\ldots,Y_1^i,\ldots,Y_{h-1}^i\rangle$. Since $T\backslash\Gamma_u \cong T\backslash\Gamma_v$, it follows that $Q\backslash y_1^1 \cong P\backslash y_{h-1}^i$, and hence $P\backslash y_1^0 \cong P\backslash y_{h-1}^i$. Since $|Y_1^0| = |Y_1^1|$, it follows, by corollary 3.6, that $y_1^0 \sim_P y_{h-1}^i$. Thus by corollary 2.10,

$Y_j^0 \overset{*}{\cong} Y_{h-j}^0$, $1 \leq j < h$, and $Y_j^0 \overset{*}{\cong} Y_{t+h-j}^0$, $h \leq j \leq t$. Hence for $1 \leq k \leq t$, $Y_j^k \overset{*}{\cong} Y_{h-j}^{i+1-k}$, $1 \leq j < h$, and $Y_j^k \overset{*}{\cong} Y_{t+h-j}^{i-k}$, $h \leq j \leq t$, from which it follows immediately that $u \sim_T v$, contradicting our assumption. $\square$

6. Pseudo-Similarity in Block Graphs

As we mentioned in the introduction, Harary and Palmer [3] developed their characterization of trees with pseudo-similar vertices in the apparently more general context of connected block graphs. However, as the results of this section demonstrate, with respect to questions of similarity and pseudo-similarity, connected block graphs and their special case, trees, are essentially equivalent. A block-graph is perhaps most easily defined as a graph each of whose blocks (i.e. maximal biconnected components) is complete [2]. Harary and Palmer [3] exploit a tree description, called the block-cutpoint-tree [5], of a block graph G. This tree has as its vertex set the union of the blocks of G and the cutpoints of G. A block-vertex b is joined to a cutpoint-vertex v in the block-cutpoint-tree exactly when v belongs to the block b in G.

We call a subgraph of a graph G a pseudo-block if

i) it is a block of G; or

ii) it is a non-cutpoint vertex of G.

Proposition 6.1. Provided G has two or more vertices, every vertex of G belongs to at least two pseudo-blocks of G. (Thus every vertex becomes what we might call a pseudo-cutpoint).

We define the pseudo-block-cutpoint-tree T(G) (see figure 4) of a connected block-graph G as follows:

i) if G is an isolated vertex then $T(G) \cong P_3$ (the path on three vertices); and

ii) if G contains two or more vertices then $T(G)$ has as its vertex set the union of the vertices of G and the pseudo-blocks of G. A pseudo-block-vertex b is joined to a vertex-vertex v in $T(G)$ exactly when v belongs to b in G. If the block graph G has connected components $G_1,\ldots,G_t$ then we denote by $T(G)$ the graph $T(G_1) \cup \ldots \cup T(G_t)$.

Unlike the block-cutpoint tree the pseudo-block-cutpoint tree provides a unique description of G (that is, G can be uniquely recovered from $T(G)$). Furthermore, recalling that every vertex v in G has its unique image, call it $T(v)$, in $T(G)$, this correspondence is preserved by vertex removal. These observations are formalized in the following lemma whose proof is straightforward.

Lemma 6.2. Let G and H be arbitrary block graphs. Then

i) $T(G) \cong T(H)$ if and only if $G \cong H$; and

ii) $T(G\backslash v) \cong T(G)\backslash T(v)$ if v is a cutpoint of G and $T(G\backslash v) \cup K_1 \cong T(G)\backslash T(v)$ if v is not a cutpoint of G.

The following theorem follows directly from the definition of pseudo-block-cutpoint-trees and lemma 6.2.

Theorem 6.3. Let G be any block graph with vertices u and v. Then

i) $u \sim_G v$ if and only if $T(u) \sim_{T(G)} T(v)$; and

ii) u and v are pseudo-similar in G, if and only if $T(u)$ and $T(v)$ are pseudo-similar in $T(G)$.

It follows from theorem 6.3 that u and v are strictly pseudo-similar in G if and only if $T(u)$ and $T(v)$ are strictly-pseudo-similar in the pseudo-block-cutpoint-tree $T(G)$. Hence the characterization of theorem 3.7 extends directly to connected block graphs (and, by the arguments of section 4, to arbitrary block graphs).

7. Edge Pseudo-Similarity

We have to this point been discussing the similarity (or pseudo-similarity) of pairs of vertices in a tree. These notions have natural analogues for edges as well, as do questions regarding the relationship between similarity and pseudo-similarity [4]. Fortunately, it is not necessary to rederive all of our vertex-based results in order to establish the corresponding results for edge similarity.

Two edges x and y in a graph G are similar, denoted $x \sim_G y$ (or simply $x \sim y$ when G is clear from the context), if there exists an automorphism of G taking x onto y. To be consistent with our earlier notion we let $G \backslash x$ denote the graph with the edge x (but not its endpoints) removed. Two edges x and y satisfying $G \backslash x \cong G \backslash y$ are said to be pseudo-similar in G. As before edges which are pseudo-similar but not similar are said to be strictly pseudo-similar.

Questions concerning edge similarity and pseudo-similarity are easily reduced to questions of vertex similarity and pseudo-similarity by means of the subdivision graph associated with a given tree. If G = (V,E) is any graph then the subdivision graph of G, denoted S(G), is the bipartite graph $(V \cup E, E')$ where $v \in V$ is joined to $e \in E$ to form an element $(v,e) \in E'$ exactly when v is an endpoint of e in G.

Proposition 7.1. T is a tree if and only if S(T) is a tree.

Of particular importance for questions concerning edge similarity and pseudo-similarity in trees is the following lemma whose proof follows in a straightforward way from the above definitions.

Lemma 7.2.

(a) If T_1 and T_2 are trees then $S(T_1) \cong S(T_2)$ if and only if $T_1 \cong T_2$.

(b) If T is any tree and $e \in E(T)$ (and hence $e \in V(S(T))$), then $S(T\backslash e) \cong S(T)\backslash e$.

The following theorem follows directly from the above definitions and lemma 7.2.

Theorem 7.3. If T is any tree with edges x and y, then

i) $x \sim_T y$ if and only if $x \sim_{S(T)} y$; and

ii) x and y are edge-pseudo-similar in T if and only if x and y are vertex-pseudo-similar in S(T).

It follows from theorem 7.3 that edges x and y are strictly pseudo-similar in T if and only if they are strictly pseudo-similar (as vertices) in S(T). Hence the characterization of theorem 3.7 leads directly to a characterization of trees with strictly pseudo-similar edges.

Theorem 7.4. If T is any tree with strictly pseudo-similar edges x and y then for some integers $i \geq 1$, $t > 1$ and $1 \leq h \leq t$, there exist rooted trees Y_k^j, $0 \leq j \leq i+1$, $1 \leq k \leq t$, where $Y_k^j \overset{*}{\cong} Y_k^0$, $0 \leq j \leq i+1$, $1 \leq k \leq t$, such that $T = \langle Y_1^0,\ldots,Y_t^0,Y_1^1,\ldots,Y_t^1,\ldots,Y_1^{i+1},\ldots,Y_{h-1}^{i+1}\rangle$. $x = (y_t^0,y_1^1)$, $y = (y_{h-1}^i,y_h^i)$, and $\langle Y_1^0,\ldots,Y_t^0\rangle \cong \langle Y_h^i,\ldots,Y_t^i,Y_1^{i+1},\ldots,Y_{h-1}^{i+1}\rangle$.

Proof. This characterization is a direct consequence of the characterization of S(T) given by theorem 7.3. □

If x is any edge of the graph G then Γ_x denotes the set of edges (including x) that are incident on at least one endpoint of x (Γ_x^k could be defined analogously; cf. section 1.) Two edges x and y are full-1-pseudo-similar if $G \backslash x \cong G \backslash y$ and $G \backslash \Gamma_x \cong G \backslash \Gamma_y$. As with vertices, full-1-pseudo-similarity implies similarity of edges.

<u>Theorem 7.5</u>. If T is any tree with edges x and y, then $T \backslash x \cong T \backslash y$ and $T \backslash \Gamma_x \cong T \backslash \Gamma_y$ implies $x \sim_T y$.

<u>Proof</u>. The argument parallels the proof of theorem 5.1, with theorem 3.7 replaced by theorem 7.4. It is sufficient to look only at case ii of the proof. □

8. Conclusions

We have presented a new characterization of trees with strictly pseudo-similar vertices. This characterization leads directly to related characterizations of forests and block-graphs with strictly pseudo-similar vertices and of trees with strictly pseudo-similar edges.

In addition, we have been able to conclude from our characterization that, unlike the situation for general graphs, in trees it is not possible to have three or more mutually strictly pseudo-similar vertices, nor is it possible to have strictly-full-1-pseudo-similar vertices.

Acknowledgement

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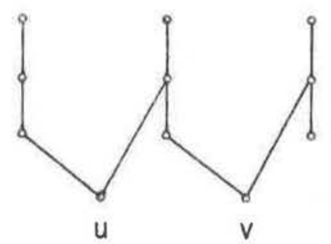

Figure 1. Tree with strictly pseudo similar vertices u and v.

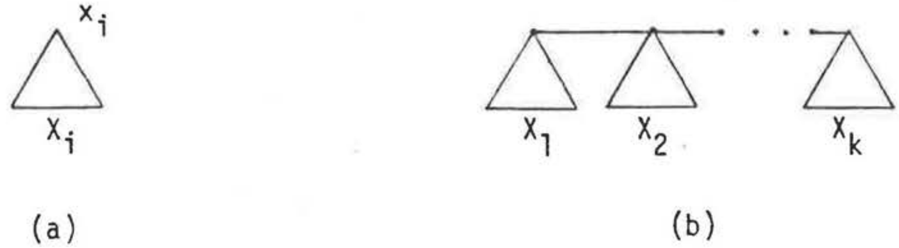

Figure 2.

Q:

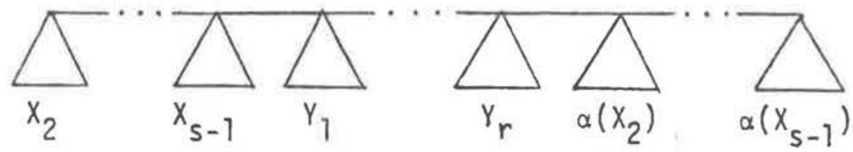

Figure 3(a).

Q:

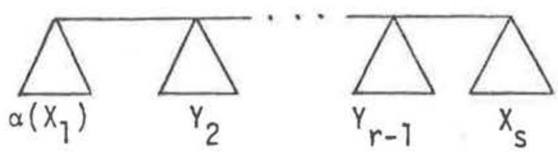

Figure 3(b).

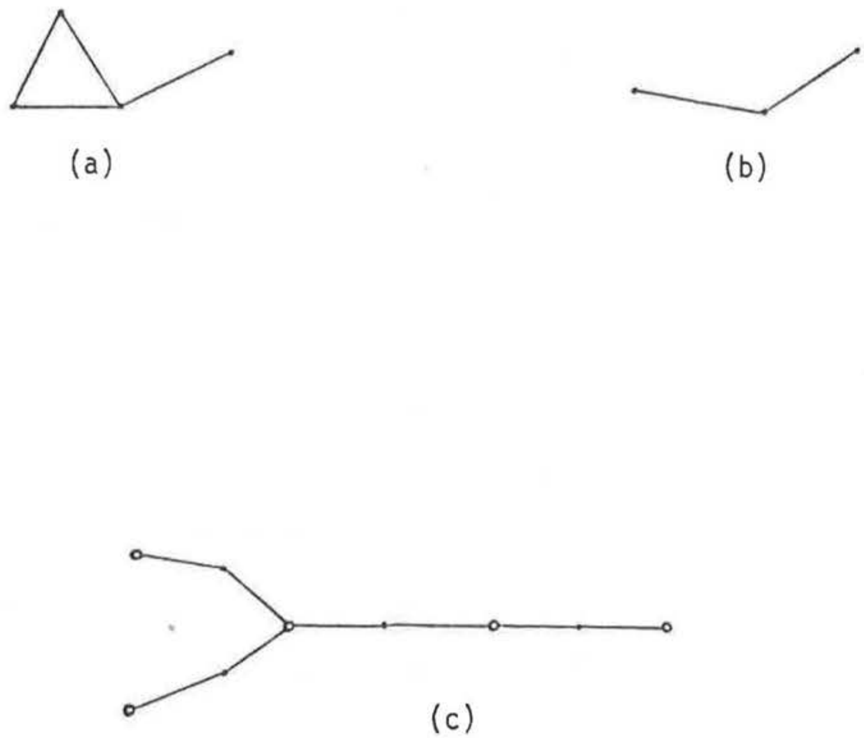

Figure 4. Block-graph (a) with its block-cutpoint tree (b) and pseudo-block-cutpoint-tree (c)

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