STABILITY RESTRICTIONS ON SECOND ORDER, THREE LEVEL

FINITE DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS

by

J. M. Varah

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ABSTRACT

In this paper we are concerned with second order schemes which are easy to use, and apply readily to nonlinear equations. We examine the stability restrictions for such schemes using linear stability analysis, and illustrate their behaviour on Burgers' equation.

1. Introduction

We take as our standard equation

$$u_t = \nu u_{xx} + f(u,u_x) \tag{1.1}$$

in the region $0 \leq x \leq 1$, $0 \leq t$ with $u(x,0)$, $u(0,t)$, $u(1,t)$ prescribed. We take ν to be a constant and $f(u,u_x)$ a smooth nonlinear function of u and u_x, and assume the equation has a unique solution. Throughout the discussion, we shall treat (1.1) as a scalar equation, but extensions to systems of equations (and variable ν) are easily made. The corresponding linear model equation for stability analysis is

$$u_t = \nu u_{xx} - cu_x \tag{1.2}$$

with constants ν and c.

We assume the usual Δx, Δt mesh in x and t, and denote the finite difference solution $v(j\Delta x, n\Delta t)$ by v_j^n. We shall use the fairly general three-level scheme

$$\frac{\alpha v_j^{n+1} + \beta v_j^n + \gamma u_j^{n-1}}{\Delta t} = \nu D_+D_-(\hat{v}_j) + f(\bar{v}_j, D_0\tilde{v}_j). \tag{1.3}$$

Here α,β,γ are constants, $\hat{v}_j,\bar{v}_j,\tilde{v}_j$ denote (possibly different) linear combinations of v_{j-1},v_j,v_{j+1}, and

$$D_0(v_j) = \frac{v_{j+1} - v_{j-1}}{2(\Delta x)}, \quad D_+D_-(v_j) = \frac{v_{j+1} - 2v_j + v_{j-1}}{(\Delta x)^2}.$$

This scheme includes many of the second-order schemes in current use; it does not, however, include Keller's box scheme [9], or the recently proposed fourth-order schemes for fluid dynamics [2] and [8].

For (1.3) to be second order in both x and t, we need

$$\frac{\alpha u_j^{n+1} + \beta u_j^n + \gamma u_j^{n-1}}{\Delta t} = u_t(j\Delta x, n\Delta t + \theta\Delta t) + O((\Delta t)^2)$$

for some $-1 \leq \theta \leq 1$ and as well,

$$\hat{u}_j, \bar{u}_j, \tilde{u}_j = u(j\Delta x, n\Delta t + \theta\Delta t) + O(\Delta x)^2.$$

Of course, the use of the specific spatial difference operators in (1.3) immediately imposes a restriction on Δx. For the linear equation (1.2) in steady state, $\nu u_{xx} - cu_x = 0$ and boundary conditions $u(0) = 0$, $u(1) = 1$, the solution is

$$u(x) = \frac{e^{(c/\nu)x} - 1}{e^{c/\nu} - 1} .$$

The difference equation is $\nu D_+D_-(v_j) - cD_0(v_j) = 0$, $v_0 = 0$, $v_n = 1$, with solution

$$v_j = \frac{\left(\frac{1+a}{1-a}\right)^j - 1}{\left(\frac{1+a}{1-a}\right)^n - 1}$$

where $\Delta x = 1/n$ and $a = \frac{c(\Delta x)}{2\nu}$. Thus $v_j \cong u(j\Delta x)$ only if $\frac{1+a}{1-a} \cong e^{2a}$ and for this we need (at least) $a < 1$ or

$$\Delta x < \frac{2\nu}{c} . \tag{1.4}$$

This restriction (which comes from accuracy considerations, not stability) is well-known, and has led to other schemes being proposed for equations where ν is small (see for example Hemker [6] and Kellogg and Tsan [10]).

In what follows, we shall examine the stability restrictions on Δx and Δt imposed by specific choices of parameters in the general scheme (1.3). Fully implicit schemes will be discussed in Section 2; then in Section 3 we examine the effect of linearizing the nonlinear part using extrapolation. In Section 4 we describe other linearizations which work well in special cases. Finally in Section 5 we apply these schemes, and illustrate the effect of the stability restrictions, on Burgers' equation.

2. Fully Implicit Schemes

The basic scheme is

$$\frac{(\theta+\frac{1}{2})v_j^{n+1}-2\theta v_j^n+(\theta-\frac{1}{2})v_j^{n-1}}{\Delta t} = \nu D_+D_-(\bar{v}_j) + f(\bar{v}_j, D_0\bar{v}_j) \tag{2.1}$$

with $\bar{v}_j = \theta v_j^{n+1} + (1-\theta)v_j^n$. For any θ, $-1 \le \theta \le 1$, this scheme has second order truncation error in x and t, applied to (1.1) centered at the point $(x_n, t_n + \theta(\Delta t))$. From an accuracy point of view, it makes little difference which θ is chosen; stability considerations however do limit the effective range of θ (see below). Well-known special cases of this scheme are:

(a) Crank-Nicolson ($\theta = \frac{1}{2}$):

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} = \nu D_+D_-\left(\frac{v_j^{n+1}+v_j^n}{2}\right) + f\left(\frac{v_j^{n+1}+v_j^n}{2}, D_0\left(\frac{v_j^{n+1}+v_j^n}{2}\right)\right) \tag{2.2}$$

(b) Gear ($\theta = 1$):

$$\frac{\frac{3}{2}v_j^{n+1}-2v_j^n+\frac{1}{2}v_j^{n-1}}{\Delta t} = \nu D_+D_-(v_j^{n+1}) + f(v_j^{n+1},D_0v_j^{n+1}). \tag{2.3}$$

We refer to the second as Gear because of the connection with the second-order backward differentiation formula popularized by Gear [5].

The basic stability questions are answered by the following theorem.

Theorem 2.1: On the linear problem (1.2), the scheme (2.1) is unconditionally stable for all ν and c, if $\frac{1}{2} \leq \theta \leq 1$.

Proof: The amplification matrix has eigenvalues $\kappa(\xi)$ satisfying

$$(\theta + \tfrac{1}{2} + \theta z)\kappa^2 - (2\theta - (1-\theta)z)\kappa + \theta - \tfrac{1}{2} = 0 \tag{2.4}$$

where $z = 4\nu\lambda \sin^2(\xi/2) + ic\eta \sin \xi$, $\lambda = \Delta t/(\Delta x)^2$, $\eta = \Delta t/\Delta x$, and $|\xi| \leq \pi$. Notice Re $z \geq 0$. Since we want unconditional stability, we need to show $|\kappa(\xi)| \leq 1$ for all $z = w + iy$ in the right half-plane. That $\theta \geq \frac{1}{2}$ is necessary is easily seen by taking $z \to \infty$: one root $|\kappa_2| \to |\frac{1-\theta}{\theta}| \leq 1$ only if $\theta \geq \frac{1}{2}$.

For sufficiency, we use the Schur-Cohn criterion (Henrici [7, page 474]). If the quadratic (2.4) is expressed as $a_2\kappa^2 + a_1\kappa + a_0 = 0$, both roots are inside the unit circle if

$$\delta_1 \equiv |a_0|^2 - |a_2|^2 < 0$$

and
$$|\bar{a}_0 a_1 - a_2 \bar{a}_1| < |\delta_1|.$$

Using $z = w + iy$, we have

$$\delta_1 = -(2\theta + (\theta+2\theta^2)w + \theta^2(w^2+y^2))$$

so that $\delta_1 < 0$ for all $\theta > 0$ if $w \geq 0$. After some manipulation, the second inequality reduces to

$$0 < 4\theta w + (8\theta^2+2\theta-1)w^2 + (4\theta^3+4\theta^2-2\theta)w(w^2+y^2) + (2\theta^3-\theta^2)(w^2+y^2)^2$$

which holds for $w > 0$ if $\theta \geq \frac{1}{2}$. For $w = 0$, we get equality (which means one root $|\kappa_1| = 1$) for all θ when $y = 0$ (this is the consistency condition) and as well for all y, when $w = 0$ and $\theta = \frac{1}{2}$. Since at most one root $|\kappa_1| = 1$, this (von Neumann) condition is sufficient as well as necessary for stability.

Q.E.D.

It is also interesting to note that as $\nu \to 0$, the model equation (1.2) becomes the model hyperbolic equation with no decay in the solution; the only difference scheme of type (2.1) which carries over this property is Crank-Nicolson ($\theta = \frac{1}{2}$). For $\theta > \frac{1}{2}$, all $|\kappa(\xi)| < 1$ even for $w = 0$ so there is attenuation of all Fourier components of the solution.

Although this fully implicit scheme has nice stability properties, discretization of the nonlinear equation (1.1) leads to a nonlinear algebraic system to be solved at each timestep. This necessitates some kind of iterative procedure (such as a Newton iteration) with perhaps several iterations if an approximate Jacobian is used, and thus a large amount of computation per iteration. Thus some form of linearization of the scheme (2.1) is of interest.

3. Linearization by Extrapolation

This was first proposed for the Crank-Nicolson scheme by Lees [12]. The general θ-scheme ($\frac{1}{2} \leq \theta \leq 1$) is

$$\frac{(\theta+\frac{1}{2})v_j^{n+1}-2\theta v_j^n+(\theta-\frac{1}{2})v_j^{n-1}}{\Delta t} = \nu D_+D_-(\bar{v}_j) + f(\hat{v}_j,D_0\hat{v}_j) \tag{3.1}$$

where $\bar{v}_j = \theta v_j^{n+1} + (1-\theta)v_j^n$ and $\hat{v}_j = v_j^n + \theta(v_j^n-v_j^{n-1})$. Again the special cases of interest are:

(a) extrapolated Crank-Nicolson ($\theta=\frac{1}{2}$):

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} = \nu D_+D_-\left(\frac{v_j^{n+1}+v_j^n}{2}\right) + f(\hat{v}_j,D_0\hat{v}_j) \tag{3.2}$$

$$\hat{v}_j = \frac{3}{2}v_j^n - \tfrac{1}{2}v_j^{n-1},$$

(b) extrapolated Gear ($\theta=1$):

$$\frac{\frac{3}{2}v_j^{n+1}-2v_j^n+\frac{1}{2}v_j^{n-1}}{\Delta t} = \nu D_+D_-(v_j^{n+1}) + f(\hat{v}_j,D_0\hat{v}_j) \tag{3.3}$$

$$\hat{v}_j = 2v_j^n - v_j^{n-1}.$$

Again we wish to examine stability for the linear problem $u_t = \nu u_{xx} - cu_x$. Notice that as $c \to 0$, we tend to the same scheme as in the previous section; whereas as $\nu \to 0$, we tend to an explicit scheme, so there is no chance for unconditional stability. The most we can expect is that the scheme be stable if $c\eta < 1$ for example. Again $\eta = \Delta t/\Delta x$, $\lambda = \Delta t/(\Delta x)^2$. However the results are somewhat surprising:

Theorem 3.1: On the hyperbolic problem $u_t = -cu_x$, the extrapolation scheme (3.1) is unstable for all η and all $\theta > 0$.

Proof: The eigenvalues $\kappa(\xi)$ of the amplification matrix satisfy

$$(\theta+\tfrac{1}{2})\kappa^2 - (2\theta + (1+\theta)iy)\kappa + (\theta - \tfrac{1}{2} + \theta iy) = 0$$

with $y = c\eta \sin \xi$. Expanding $\kappa_1(y)$ for small y gives

$$\kappa_1(y) = 1 + iy - \tfrac{1}{2}y^2 + \frac{\theta}{2}iy^3 + (\frac{\theta^2}{2} - \frac{\theta}{4} - \frac{1}{8})y^4 + \ldots$$

and thus

$$|\kappa_1(y)|^2 = 1 + (\theta^2 + \theta/2)y^4 + \ldots > 1$$

so the scheme is unstable for all $\theta > 0$ and all η. Q.E.D.

Thus for small ν, the extrapolation scheme will be unstable.

However the scheme may still be of interest: recall that we needed $\Delta x < 2\nu/c$ to get a reasonable approximation to the steady-state solution of (1.2). Perhaps the stability restriction here may be comparable. So we now ask: how small can we take ν before some $|\kappa(\xi)| > 1$? The $\kappa(\xi)$ satisfy the quadratic

$$(\theta+\tfrac{1}{2})\kappa^2 - 2\theta\kappa + (\theta-\tfrac{1}{2}) = -\kappa\nu\lambda(\theta\kappa+1-\theta)4\sin^2(\xi/2) + c\eta((1+\theta)\kappa - \theta)i\sin\xi.$$

This is somewhat simplified by introducing $d = c\eta$, $g = \nu\lambda$, $w = 4\sin^2(\xi/2)$, and $y = \sin\xi$ (so $y^2 = w - w^2/4$); this gives

$$(\theta+\tfrac{1}{2}+g\theta w)\kappa^2 - (2\theta-(1-\theta)gw+(1+\theta)idy)\kappa + (\theta-\tfrac{1}{2}+id\theta y) = 0. \qquad (3.4)$$

Clearly, for g large and d small (so the diffusion term dominates) all roots are inside the unit circle; and for g small and d large, we have some roots outside the unit circle. Thus we can phrase our stability question as follows: given d, for what values of g, $g \geq g(d)$, are all roots $|\kappa(\xi)| \leq 1$ for $|\xi| \leq \pi$ (or $0 \leq w \leq 4$)?

Theorem 3.2: The roots of (3.4), $|\kappa(\xi)| \leq 1$ for $|\xi| \leq \pi$ if $g \geq \left(\frac{3\theta}{2\theta^2+2\theta-1}\right)d^2$.

Proof: Again applying the Schur-Cohn criterion, we need the following conditions, expressed as polynomials in w:

(i) $\quad -\delta_1(w) = \theta[2 + (g+2\theta g-\theta d^2)w + \theta(g^2+d^2/4)w^2] \geq 0$ for $0 \leq w \leq 4$

(ii) $$\delta_2(w) = 4\theta g + [g^2(8\theta^2+2\theta-1) + 2\theta(2\theta^2-4\theta-1)gd^2 + \theta^2(1+2\theta)d^4]w$$

$$+ \theta[2g^3(2\theta^2+2\theta-1) + (\tfrac{1}{2}+2\theta-\theta^2)gd^2 - 6\theta g^2d^2 + \frac{\theta(1+2\theta)}{2}d^4]w^2$$

$$- \theta^2[(1-2\theta)g^4 - \frac{3}{2}g^2d^2 + (\frac{1+2\theta}{16})d^4]w^3 \geq 0 \text{ for } 0 \leq w \leq 4.$$

The first condition is quite easily dealt with: the quadratic has no positive roots w if all coefficients are non-negative, i.e. if $g \geq (\frac{\theta}{1+2\theta})d^2 \equiv r_1d^2$. This is a rather weak estimate, but sufficient here because of what follows. The second condition appears rather formidable, and it is not enough to merely consider non-negativity of the coefficients. To see the order of magnitude of the relationship between g and d, we can rewrite $\delta_2(w)$ by collecting the g and d terms separately; this gives

$$\delta_2(w) = g[4\theta + (8\theta^2+2\theta-1)(wg) + 2\theta(2\theta^2+2\theta-1)(wg)^2 + \theta^2(2\theta-1)(wg)^3]$$

$$+ \frac{\theta}{2}w(w-4)gd^2(3\theta wg-2\theta^2+4\theta+1) - \frac{\theta^2(1+2\theta)}{16}(w-4)^2wd^4.$$

Completing the square on the terms involving d gives $\delta_2(w) > 0$ when

$$\sqrt{4\theta gw\, p(gw)} - wg(3\theta wg-2\theta^2+4\theta+1) > \theta(1+2\theta)(1-\frac{w}{4})wd^2$$

where $p(gw) = \theta(\theta^2+2)(gw)^3 + 2\theta(\theta^2+\theta+1)(gw)^2 + (\theta^3+6\theta+2)(gw) + 1$. For large d, if $w = 0(1/d^2)$, this can only happen if $wg = 0(1)$ or $g = 0(d^2)$. We emphasize this because it is rather surprising: one might expect $g = O(d)$ would be sufficient.

With this in mind, we now express $\delta_2(w)$ as a polynomial in g:

$$\delta_2(w) = g^4(w^3\theta^2(2\theta-1)) + g^3(2w^2\theta(2\theta^2+2\theta-1)) + g^2((8\theta^2+2\theta-1)w - (4-w)\tfrac{3}{2}w^2\theta^2d^2)$$

$$+g(4\theta-\tfrac{\theta}{2}wd^2(4-w)(1+4\theta-2\theta^2)) - \frac{\theta^2(1+2\theta)}{16}w(w-4)^2d^4. \qquad (3.5)$$

Thus $$w\delta_2(w) \geq \theta^2(2\theta-1)(wg)^4 + 2\theta(2\theta^2+2\theta-1)(wg)^3 + (8\theta^2+2\theta-1-6\theta^2wd^2)(wg)^2$$

$$+ 2\theta(2 - (1+4\theta-2\theta^2)wd^2)(wg) - \theta^2(1+2\theta)w^2d^4.$$

Now set $r = g/d^2$ and $s = wg$; then the right hand side above is

$$\theta^2(2\theta-1)s^4 + 2\theta(2\theta^2+2\theta-1-\tfrac{3\theta}{r})s^3 + (8\theta^2+2\theta-1-\tfrac{2\theta}{r}(1+4\theta-2\theta^2) - \frac{\theta^2(1+2\theta)}{r^2})s^2 + 4\theta s.$$

This has no positive roots (in s) and hence $\delta_2(w) > 0$ if all coefficients are non-negative: for this we need

$$r \geq \frac{3\theta}{2\theta^2+2\theta-1} = r_2(\theta) \quad \text{and} \quad r \geq \frac{(1+2\theta)}{2\sqrt{\theta(\theta^3+6\theta+2)} - (1+4\theta-2\theta^2)} = r_3(\theta).$$

For $\frac{1}{2} \leq \theta \leq 1$, $r_2(\theta) \geq r_1(\theta), r_3(\theta)$, so we finally obtain $|\kappa(\xi)| \leq 1$ if $g \geq r_2(\theta)d^2$. Q.E.D.

We remark that this is only a sufficient condition, and that better bounds for r may be obtained for specific values of θ.

Expressing this result in turns of Δx, Δt, ν, and c, we find that $g \geq rd^2$ means

$$\Delta t \leq \frac{\nu}{rc^2} \,. \tag{3.6}$$

This is a rather unusual stability condition; the more expected result $g \geq rd$ would translate into $\Delta x \leq \frac{\nu}{rc}$, a condition similar to (1.4). However recall that we really needed $g = O(d^2)$ for d large; if we assume

$$d = c\eta = c\,\frac{\Delta t}{\Delta x} \leq 1, \tag{3.7}$$

which is reasonable in any case because of the explicitness of the hyperbolic difference approximation, our stability condition (3.6) becomes

$$\Delta x \leq \frac{\nu}{rc} \,. \tag{3.8}$$

Again we mention that the value of r can be improved from $r = r_2(\theta)$ for fixed values of θ and d. For example $\theta = 1$ gives $r_2(\theta) = 1$ so the stability condition is $\Delta x \leq \nu/c$. However using (3.5) directly with $d = 1$ gives $|\kappa(\xi)| \leq 1$ precisely for $r \geq \frac{1}{2}$, the <u>same</u> condition as (1.4). For $\theta = \frac{1}{2}$, $r_2(\theta) = 3$ but using (3.5) directly with $d = 1$ gives $|\kappa(\xi)| \leq 1$ for $r \geq r_0$, where $\frac{1}{2} < r_0 < 1$.

These results are borne out by numerical experiments: we give in Table 1 results on the linear equation $u_t = \nu u_{xx} - cu_x$ with steady-state solution $u(x) = \dfrac{e^{(c/\nu)x} - 1}{e^{c/\nu} - 1}$. The numbers given refer to errors from this solution, for Crank-Nicolson, Gear, and their extrapolated versions. The fourth row illustrates the different stability limits for $\theta = \frac{1}{2}$ and $\theta = 1$. The errors in the last two cases are large because of the rapid change in the steady-state solution.

TABLE 1

Δx	Δt	ν	c	CN	EXCN	GEAR	EXGEAR
0.1	0.1	1	1	10^{-4}	10^{-4}	10^{-4}	10^{-4}
0.1	0.1	1	10	.034	∞	.034	∞
0.1	0.1	0.1	1	.034	.034	.034	.034
0.1	0.1	10	10	10^{-4}	∞	10^{-4}	10^{-4}
0.05	0.05	.01	1	0.43	∞	0.43	∞
0.05	0.05	.02	1	0.2	0.2	0.2	0.2

We also give results for $\nu = 0$, $c = 1$ using a periodic initial condition $u(x,0)$ - see Table 2.

TABLE 2

Δx	Δt	CN			EXCN		
		t=1	t=5	t=20	t=1	t=5	t=20
0.1	0.09	0.12	0.38		0.38	10^5	
0.05	0.021	0.11	0.09	0.33	0.08	0.29	10^3

4. Other Linearizations

Here we discuss two other alternatives to the fully implicit scheme (2.1). The first is something of a compromise between (2.1) and the extrapolated scheme (3.1), which we call the linearized scheme:

$$\frac{(\theta+\frac{1}{2})v_j^{n+1}-2\theta v_j^n+(\theta-\frac{1}{2})v_j^{n-1}}{\Delta t} = \nu D_+D_-(\bar{v}_j) + f(\hat{v}_j, D_0\bar{v}_j) \qquad (4.1)$$

where again $\bar{v}_j = \theta v_j^{n+1} + (1-\theta)v_j^n$, $\hat{v}_j = v_j^n + \theta(v_j^n - v_j^{n-1})$. The special cases are

(a) linearized Crank-Nicolson ($\theta=\frac{1}{2}$):

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \nu D_+D_-\left(\frac{v_j^{n+1}+v_j^n}{2}\right) + f\left(\hat{v}_j, D_0\left(\frac{v_j^{n+1}-v_j^n}{2}\right)\right) \qquad (4.2)$$

$$\hat{v}_j = \frac{3}{2}v_j^n - \frac{1}{2}v_j^{n-1},$$

(b) linearized Gear ($\theta=1$):

$$\frac{\frac{3}{2}v_j^{n+1} - 2v_j^n + \frac{1}{2}v_j^{n-1}}{\Delta t} = \nu D_+D_-(v_j^{n+1}) + f(\hat{v}_n, D_0 v_j^{n+1}) \qquad (4.3)$$

$$\hat{v}_j = 2v_j^n - v_j^{n-1}.$$

Applied to the linear equation $u_t = \nu u_{xx} - cu_x$, this scheme is identical to the fully implicit scheme (2.1), so this scheme is unconditionally stable for $\theta \geq \frac{1}{2}$. Moreover (4.1) leads to a linear algebraic system for v_j^{n+1} provided $f(u,u_x)$ is linear in u_x. If this is the case, this scheme has all the advantages and none of the disadvantages of the fully implicit scheme (2.1), and we recommend its use. Such is the case with Burgers' equation for example which we discuss in the next section.

The second alternative is the <u>averaging</u> scheme of Lees [11]:

$$\frac{v_j^{n+1} - v_j^{n-1}}{2(\Delta t)} = \nu D_+ D_-\left(\frac{v_j^{n+1}+v_j^n+v_j^{n-1}}{3}\right) + f(v_j^n, D_0 v_j^n). \tag{4.4}$$

Although this fits the general model (1.3), it is not a direct linearization of (2.1) and has no θ-generalization. It is second order, centred at (x_j, t_n), and we refer to it as the <u>averaged Crank-Nicolson</u> method. The linear stability analysis is straightforward: applied to the equation $u_t = \nu u_{xx} - cu_x$, notice first of all that for $\nu \to 0$, the scheme tends to the well-known leap-frog scheme for $u_t = -cu_x$, which is stable for $c\eta \equiv c\frac{\Delta t}{\Delta x} \leq 1$. For $\nu \neq 0$, the eigenvalues $\kappa(\xi)$ of the amplification matrix satisfy the quadratic

$$(1+w)\kappa^2 + (w+iy)\kappa + (w-1) = 0 \tag{4.5}$$

where $w = \frac{8}{3}\nu\lambda\sin^2(\xi/2)$ and $y = 2c\eta \sin \xi$. It is easily seen that the above condition ($c\eta \leq 1$) guarantees stability for all ν.

However, there is a new problem with this scheme, which often appears in schemes which use averaging over two or more time-levels: the parasitic solution (arising from $\kappa_2(\xi)$ rather than $\kappa_1(\xi)$), although stable, can dominate the numerical solution, leading to improper decay for a diffusion-dependent problem and separation of the numerical solution at alternate time-steps. This effect can be most easily seen for the pure diffusion problem $u_t = \nu u_{xx}$; (4.5) reduces to

$$(1+w)\kappa^2 + w\kappa + (w-1) = 0.$$

For $w = 0$, $\kappa_1 = 1$ and $\kappa_2 = -1$ so the parasitic solution is just as large; for small w,

$$\kappa_1 = 1 - \frac{3}{2}w + \frac{9}{8}w^2 + \ldots \cong e^{-z},$$

where $z = \omega^2\nu(\Delta t)$ which gives the paper decay rate (recall $\xi = \omega(\Delta x)$). However,

$$\kappa_2 = -1 + \frac{w}{2} - \frac{w^2}{8} \cong -e^{-z/3} ,$$

and hence this solution dominates, and gives an improper decay rate. Because of the negative sign, the parasitic solution will alternate sign at each time step, leading to separation of the numerical solution into two very different solutions on alternate timesteps. This effect is illustrated in Table 3, where we give the errors in the numerical solution to $u_t = \nu u_{xx}$, $u(x,0) = \sin \pi x$, using CN and this scheme, AVGCN. All errors are absolute errors; in all cases CN gave the right order of magnitude for the (possibly rapidly) decaying solution, but AVGCN did not.

TABLE 3

			error on first step		error at t = 1		comments
Δx	Δt	ν	CN	AVGCN	CN	AVGCN	
.1	.1	1	.01	.076	$.3 \times 10^{-4}$	.002	separation
.05	.05	1	.0034	.018	$.9 \times 10^{-5}$	.0003	separation
.1	.1	5	.003	.53	$.3 \times 10^{-5}$	.012	poor decay
.05	.05	5	.002	.30	10^{-16}	10^{-5}	poor decay

We should add however, that this scheme does not always perform badly; for example as shown in the next section, it performs well on Burgers' equation, especially when the dominant part of the solution is governed by the hyperbolic terms. Nevertheless, it clearly must be used with care.

5. A Numerical Example: Burgers' Equation

We present two examples of Burgers' equation

$$u_t = \nu u_{xx} - uu_x, \quad 0 \leq x \leq 1. \tag{5.1}$$

The first is the exact solution given by Whitham [13, Chapter 4] and also used by Fong [3] of one shock overtaking another for ν small:

$$u(x,t) = 1 - 0.9 \frac{r_1}{r_1+r_2+r_3} - 0.5 \frac{r_2}{r_1+r_2+r_3}$$

where $r_1 = \exp(-(\frac{x-0.5}{20\nu}) - \frac{99t}{400\nu})$, $r_2 = \exp(-(\frac{x-0.5}{4\nu}) - \frac{3t}{16\nu})$, $r_3 = \exp(-(\frac{x-3/8}{2\nu}))$. We are not so much interested in accuracy as in the effects of the stability restrictions on the numerical solution. We present in Table 4 the errors using linearized CN (4.2), which gave the same results as the fully implicit CN of (2.2), extrapolated CN (3.2), and the averaged CN (4.4). Results for other values of θ (notably $\theta = 1$) were very similar.

TABLE 4

			error at $t = 1$			error at $t = 5$		
ν	Δx	Δt	LINCN	EXTCN	AVGCN	LINCN	EXTCN	AVGCN
.1	.1	.1	.0015	.0026	.0014	$.42\text{x}10^{-5}$	$.18\text{x}10^{-5}$	$.67\text{x}10^{-4}$
.01	.1	.1	.63	.76	.76	$.62\text{x}10^{-3}$	∞	∞*
.01	.05	.02	.16	.12	.13	10^{-13}	10^{-13}	10^{-7}

The large errors for $\nu = .01$ are of course due to the difficulty of fitting the shock with only a few points in x. Despite this, the results are surprisingly similar, except that EXTCN blows up when its stability condition (3.6) is violated. Similarly AVGCN blows up because we violate its stability limit $|u \frac{\Delta t}{\Delta x}| \leq 1$ whenever $u > 1$; when Δt was reset to .05, the error was .0028.

The second example of (5.1) is with the initial condition $u(x,0) = \sin \pi x$, discussed in Ames [1, page 87]. For $\nu \to 0$, the solution (see Whitham [13, Chapter 4]) is given by

$$u(x,t) = \sin(\pi \cdot \xi(x,t))$$

where $\xi(x,t)$ is the (unique) solution to

$$\sin \pi\xi = \frac{x - \xi}{t} .$$

This solution develops a discontinuity at $x = 1$ for $t > 1/\pi$: $u(1,t) = 0$, yet for any fixed $t > 1/\pi$, $u(x,t) \to \sin \pi\xi$ as $x \to 1$, where ξ is the root of $\sin \pi\xi = \frac{1 - \xi}{t}$. This root $\xi \to 0$ as $t \to \infty$ so the solution ultimately decays to zero.

The presence of this discontinuity causes an interesting phenomenon in the numerical solution: the solution is good for most of the x-interval ($0 \leq x \leq 1$) but a large spike develops at the last interior x-point($x = 1 - \Delta x$), grows in time, and finally shrinks. This characteristic is common to all the methods; for very small ν there is some additional smaller oscillation in x.

In Table 5 we give the numerical solution at $x = 0.9$ for $\Delta x = \Delta t = 0.1$ for various time-values.

TABLE 5

ν=.01	t=.4	t=.6	t=.8	t=1.0	t=2.0	t=5.0	t=10.0
AVGCN, LINCN, EXTCN	1.0	1.9	2.1	2.2	1.5	0.2	
ν=.001							
AVGCN	1.30	2.37	1.92	2.60	∞	∞	∞
LINCN	0.94	2.06	2.84	3.10	3.34	3.28	400
EXTCN	0.91	2.81	2.79	2.95	3.35	2.93	∞
LIN GEAR	0.78	1.63	2.46	2.92	3.33	3.28	2.59

For $\nu = .01$, all methods behaved very similarly. For $\nu = .001$, the AVGCN blew up because its stability condition $|u \frac{\Delta t}{\Delta x}| < 1$ was eventually violated, and a similar thing happened with EXTCN. As well, however, LINCN grew enormously for $t = 10$, and this cannot be accounted for by linear stability analysis. We believe it is another example of a nonlinear stability for this scheme (for another more extensive example see Fornberg [4]). Notice as well that the instability was not present using the Gear scheme.

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