# DUALS OF INTUITIONISTIC TABLEAUS

by

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ABSTRACT

We present a dual version of the intuitionistic Beth tableaus with signed formulas introduced in Fitting |9| proving their correctness and completeness with respect to Kripke models.

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## 1. INTRODUCTION.

In introducing the sequent calculi LK for classical first order logic and LJ for intuitionistic first order logic, Gentzen [1] noted the importance of duality and symmetry of the calculus LK. Concerning symmetry Gentzen wrote: " This is made possible by the admission of sequents with several formulae in the succedent....The symmetry thus obtained is more suitable to classical logic. On the other hand the restriction to at most one formula in the succedent will be retained for the intuitionistic calculus LJ." (page 86 of the English translation edited by M.E.Szabo).

The tableaus method in proof theory, first introduced by Beth [2] and Hintikka [3], ultimately derive from the Gentzen sequent calculus, as Smullyan acknowledges (page 15 of [4]) in defining his analytic tableaus. In addition, Gentzen's observations about LK, together with the well known dual properties of first order classical logic, led to the introduction of dual tableaus, see for example the duals of Smullyan tableaus,(elsewhere called positive analytic tableaus) defined in [5].

Turning to intuitionistic first order logic as formalized by Heyting [6], it does not seem that the dual properties of this formal system have been investigated. It should be noticed however that the restriction to only one formula in the succedent of a sequent in the calculus LJ was in many cases unnecessary, see Leblanc and Thomason [7].

After the introduction of Kripke's semantics for intuitionistic logic [8], Fitting [9] introduced intuitionistic tableaus with signed formulas, proving completeness of this method of proof with respect to Kripke models. Then Smullyan [10] showed how the completeness theorem for intuitionistic logic, as well as for $S_4$-Modal logic, can be obtained as a special case of the completeness theorem of classical frameworks.

With such powerful instruments at our disposal it is easy to show that Beth intuitionistic tableaus have duals and to prove their completeness with respect to Kripke models.

## 2. KRIPKE MODELS AND POSITIVE INTUITIONISTIC TABLEAUS.

Before introducing Kripke models we require some definitions.

Let $\mathfrak{S}$ be a countable, non-empty set of parameters: a, b, c,...; and let $\mathfrak{D}$ be a countable set of predicate letters $A_1, A_2, \ldots$ . An atomic formula from $\mathfrak{S}$ and $\mathfrak{D}$ is an expression of the type $A(k_1, k_2, \ldots, k_n)$ where A is an n-place predicate letter in $\mathfrak{D}$ and for each $1 \leq i \leq n$ $k_i$ is an element of $\mathfrak{S}$.

A classical model for $\mathfrak{S}$ and $\mathfrak{D}$ is any complete and consistent set of atomic formulas. Formulas from $\mathfrak{S}$ and $\mathfrak{D}$ are built in the usual way from atomic formulas using the connectives &, v, $\sim$, $\rightarrow$ and the quantifiers $\forall, \exists$ . A signed formula is any expression of the form +B or -B where B is a formula. If S is a set of signed formulas let

$S_+ = \{+B \mid +B \in S\}$ , $S_- = \{-B \mid -B \in S\}$ and

$D(S) = \{\pm X \mid \mp X \in S\}$.

Definition. An intuitionistic Kripke model for $\mathfrak{S}$ and $\mathfrak{D}$, called a model in the sequel, is a triple $(K, R, \Phi)$ where:

K is a non-empty set,

R is a reflexive and transitive relation on K,

$\Phi$ is a mapping from K such that for each p, q in K

(1) $\Phi(p)$ is a classical model for $\mathfrak{S}'$ and $\mathfrak{D}$, $\mathfrak{S}'$ a non-empty subset of $\mathfrak{S}$

(2) For each A in $\mathfrak{D}$ and for each $\underline{a}$ in $\mathfrak{S}$:

$+A(\underline{a}) \in \Phi(p)$ and $pRq \Rightarrow +A(\underline{a}) \in \Phi(q)$;

$-A(\underline{a}) \in \Phi(p)$ and $pRq \Rightarrow$ either $-A(\underline{a})$ or $+A(\underline{a})$ is in $\Phi(q)$.

Definition. Let $\Gamma$ be a set of (signed) sentences. We say that the (signed) sentece $(\pm)B$ is from $\Gamma$ if each parameter and atomic predicate letter used in B appears in some sentence of $\Gamma$.

Definition. Let $(K, R, \Phi)$ be a model. We want to define by induction a sequence of functions on K, $\Phi^{(0)}$ $\Phi^{(1)}, \ldots \Phi^{(n)}, \ldots$ such that for each $n \in N$ and for each p in K $\Phi^{(n)}(p)$ contains only signed sentences from $\Phi(p)$. Let $\Phi^{(0)}(p) = \Phi(p)$ for all p in K. Having defined $\Phi^{(k)}$ such that for each p in K $\Phi^{(k)}(p)$ contains only signed sentences from $\Phi(p)$, then $\Phi^{(k+1)}$ is defined in the following way. For each p in K $\Phi^{(k+1)}(p)$ is the set of signed sentences $\pm B$, B a sentence from $\Phi(p)$, for which one of the following conditions holds:

(1) $\pm B \in \Phi^{(k)}(p)$, respectively;
(2) B is C & D and each of +C and +D is in $\Phi^{(k)}(p)$, respectively, one of -C or -D is in $\Phi^{(k)}(p)$;
(3) B is $C \vee D$ and one of +C or +D is in $\Phi^{(k)}(p)$, respectively, both -C and -D are in $\Phi^{(k)}(p)$;
(4) B is $\sim C$ and for each q in K such that pRq, -C is in $\Phi^{(k)}(q)$, respectively, there is a q in K such that pRq and +C is in $\Phi^{(k)}(q)$;
(5) B is $C \rightarrow D$ and for each q in K such that pRq one of -C or +D is in $\Phi^{(k)}(q)$, respectively, there is a q in K such that pRq and each of +C and -D are in $\Phi^{(k)}(q)$;
(6) B is $\exists x C(x)$ and +C(a) is in $\Phi^{(k)}(p)$ for some sentence C(a) from $\Phi(p)$, respectively, for all sentences C(a) from $\Phi(p)$, -C(a) is in $\Phi^{(k)}(p)$;
(7) B is $\forall x C(x)$ and for all q such that pRq and for all sentences C(a) from $\Phi(q)$ +C(a) is in $\Phi^{(k)}(q)$, respectively, there is a q in K such that pRq and for some sentence C(a) from $\Phi(q)$ -C(a) is in $\Phi^{(k)}(q)$.

Finally let for each p in K $\Phi^{*}(p) = \bigcup_{k \in N} \Phi^{(k)}(p)$.

With this definition, for each p in K $\Phi^{*}(p)$ contains one and only one of +B and -B for each sentence B from $\Phi(p)$.

<u>Definition</u>. A sentence B is valid in a model $(K, R, \Phi)$ if for each p in K such that +B is from $\Phi(p)$, $+B \in \Phi^{*}(p)$. A sentence B is valid if it is valid in all models.

<u>Definition</u>. Let $S = \{+X_1, \ldots, +X_n, -Y_1, \ldots, -Y_m\}$. A model $(K, R, \Phi)$ is a countermodel for the set S if there is a p in K such that $-X_i \in \Phi^{*}(p)$ for each $i \leq n$ and $+Y_j \in \Phi^{*}(p)$ for each $j \leq m$. When this happens we will say that p refuses S.

Finally we recall the following property of Kripke models: if $(K, R, \varphi)$ is a model, for any formula X and elements p, q in K if $+X \in \Phi^{*}(q)$ and pRq then $+X \in \Phi^{*}(q)$.

We are now ready to introduce our dual tableaus. In table A are shown the reduction rules for intuitionistic Beth tableaus as introduced by Fitting [9], while in table B are the positive intuitionistic tableaus. All proof- theoretical definitions given in Fitting apply with minor changes. In particular we recall that a set S of signed sentences is closed if it contains both +X and -X for some sentence X.

We will write $\vdash_I X$, if some positive Beth tableau for +X closes.

$$\frac{S,\ +X \,\&\, Y}{S,\ +X,\ +Y} \qquad \frac{S,\ -X \,\&\, Y}{S,\ -X \mid S,\ -Y}$$

$$\frac{S,\ +X \vee Y}{S,\ +X \mid S,\ +Y} \qquad \frac{S,\ -X \vee Y}{S,\ -X,\ -Y}$$

$$\frac{S,\ +\sim X}{S,\ -X} \qquad \frac{S,\ -\sim X}{S_+,\ +X}$$

$$\frac{S,\ +X \rightarrow Y}{S,\ -X \mid S,\ +Y} \qquad \frac{S,\ -X \rightarrow Y}{S_+,\ +X,\ -Y}$$

$$\frac{S,\ +\exists x X(x)}{S,\ +X(a)}\ * \qquad \frac{S,\ -\exists x X(x)}{S,\ -X(a)}$$

$$\frac{S,\ +\forall x X(x)}{S,\ +X(a)} \qquad \frac{S,\ -\forall x X(x)}{S_+,\ -X(a)}\ *$$

*with the proviso that a does not appear in S or in X.

Table A

1) $\dfrac{S,\ +X\,\&\,Y}{S,+X \mid S,+Y}$ 2) $\dfrac{S,\ -X\,\&\,Y}{S,\ -X,\ -Y}$

3) $\dfrac{S,\ +X \vee Y}{S,\ +X,\ +Y}$ 4) $\dfrac{S,\ -X \vee Y}{S,-X \mid S,-Y}$

5) $\dfrac{S,\ +\sim X}{S_{-},-X}$ 6) $\dfrac{S,\ -\sim X}{S,\ +X}$

7) $\dfrac{S,\ +X \rightarrow Y}{S_{-},\ -X,\ +Y}$ 8) $\dfrac{S,\ -X \rightarrow Y}{S,+X \mid S,-Y}$

9) $\dfrac{S,\ +\exists x X(x)}{S,\ +X(a)}$ 10) $\dfrac{S,\ -\exists x X(x)}{S,\ -X(a)}$ *

11) $\dfrac{S,\ +\forall x X(x)}{S_{-},\ +X(a)}$ * 12) $\dfrac{S,\ -\forall x X(x)}{S,\ -X(a)}$

*with the proviso that a does not appear in S or in X

Table B

## 3. CORRECTNESS OF POSITIVE INTUITIONISTIC BETH TABLEAUS.

The correctness of Beth intuitionistic tableaus is a straightforward consequence of the following theorem:

Theorem (Fitting [9]). If $C_1, C_2, \ldots, C_n$ is a tableau and $C_i$ is realizable, so is $C_{i+1}$.

In the case of positive Beth tableaus the dual concept of realizability will be the notion of countermodel defined earlier.

Theorem 1. Suppose the set of signed sentences above the line in any of the rules of table B admit a countermodel, then so does the set below the line (or at least one of the sets below the line in the cases of rules B1, B4, B8).

In order to prove the theorem we need the following lemma, whose tedious proof we omit because it parallels exactly the proof of lemma 2.2 of Fitting [9].

Lemma 1. If $\{S,+X(b)\}$,( $\{S,-X(b)\}$ ), admits a countermodel and a is a parameter which does not occur in S or in X(b), then $\{S,+X(a)\}$, $\{S,-X(a)\}$ resp.), admits a countermodel.

Proof of the theorem.

There are twelve cases to consider according to which rule we apply.

B1. Suppose there is a $p \in K$ in a model $(K, R, \Phi)$ which refuses $\{S,+X \& Y\}$. Then $D(S) \subseteq \Phi^*(p)$ and $-X \& Y \in \Phi^*(p)$ , which implies that either $-X$ or $-Y$ is in $\Phi^*(p)$. Therefore either p refuses $\{S,+X\}$ or p refuses $\{S,-X\}$.

B2. If there is a $p \in K$ in a model $(K, R, \Phi)$ which refuses $\{S,-X \& Y\}$, then $D(S) \subseteq \Phi^*(p)$, $+X \& Y \in \Phi^*(p)$ and therefore $+X$ and $+Y$ are both in $\Phi^*(p)$. Hence p refuses $\{S,-X,-Y\}$.

B3 and B4 are proved analogously.

B5. Suppose there is a $p \in K$ in a model $(K,R,\Phi)$ which refuses $\{S,+\sim X\}$. Then $-\sim X$ is in $\Phi^*(p)$, which implies that there exists a $q \in K$ such that pRq and $+X \in \Phi^*(q)$. Moreover each signed sentence $-B$ in S is such that $+B$ is in $\Phi^*(p)$ and hence in $\Phi^*(q)$. Therefore q refuses $\{S_-,-X\}$.

B6. Suppose there is a $p \in K$ in a model $(K,R,\Phi)$ which refuses $\{S,-\sim X\}$. Then $+\sim X$, and hence $-X$, is in $\Phi^*(p)$ and $D(S) \subseteq \Phi^*(p)$. Therefore p refuses $\{S,+X\}$.

B7. If some $p \in K$ in a model $(K,R,\Phi)$ refuses $\{S,+X \rightarrow Y\}$, then $-X \rightarrow Y$ is in $\Phi^*(p)$ which implies that there exists a $q \in K$ such that pRq and both $+X$ and $-Y$ are in $\Phi^*(q)$. Moreover each $-B$ in S is such that $+B$ is in $\Phi^*(p)$ and a fortiori in $\Phi^*(q)$. Hence q refuses $\{S_-,-X,+Y\}$.

B8. If some $p \in K$ in a model $(K,R,\Phi)$ refuses $\{S,-X \to Y\}$, then $+X \to Y$ is in $\Phi^*(p)$. Hence either $-X$ or $+Y$ is in $\Phi^*(p)$ which also contains $D(S)$. Hence p refuses either $\{S,+X\}$ or $\{S,-Y\}$.

B9. If some $p \in K$ in a model $(K,R,\Phi)$ refuses $\{S, +\exists xX(x)\}$, then $D(S) \subseteq \Phi^*(p)$ and $-\exists xX(x) \in \Phi^*(p)$. If a is used in $\Phi(p)$, then $-X(a)$ is in $\Phi^*(p)$. Hence p refuses $\{S,+X(a)\}$. Otherwise let c be a parameter used in $\Phi(p)$. Then p refuses $\{S,+X(c)\}$and a does not appear in S or in X(c); hence the lemma applies.

B10. Let us suppose there is a model $(K,R,\Phi)$ and a $p \in K$ which refuses $\{S, -\exists xX(x)\}$. Then $D(S) \subseteq \Phi^*(p)$ and $+\exists xX(x)$ is in $\Phi^*(p)$. Therefore, for some c, $+X(c) \in \Phi^*(p)$ and p refuses $\{S,-X(c)\}$. If $a = c$ we are done; otherwise, since a does not occur in S or X(c), we invoke the lemma.

B11. Suppose there is a $p \in K$ in a model $(K,R,\Phi)$ which refuses $\{S, +\forall xX(x)\}$. Then $-\forall xX(x)$ is in $\Phi^*(p)$. Hence there exists a $q \in K$ such that pRq and a parameter c from $\Phi(q)$ such that $-X(c) \in \Phi^*(q)$. Moreover each $-B$ in S is such that $+B \in \Phi^*(p)$; hence $+B$ is in $\Phi^*(q)$ so that q refuses $\{S_-,+X(c)\}$. If $a = c$ we are done; otherwise we invoke the lemma again since a does not occur in S or X(c).

B12. If some p in a model $(K,R,\Phi)$ refuses $\{S, -\forall xX(x)\}$, we have that $D(S) \subseteq \Phi^*(p)$, $+\forall xX(x) \in \Phi^*(p)$. Hence $+X(c)$ is in $\Phi^*(p)$. If a is one of these c we are done; otherwise this implies that a does not appear in S, X(c), while p refuses $\{S,-X(c)\}$. Therefore the lemma applies.

Corollary 1. If $\vdash_I X$, then X is valid.

Proof. By hypothesis there is a closed tableau fo +X of the form $C_1 = \{+X\}, C_2, \ldots, C_n$, where $C_n$ is a configuration all of whose elements are closed sets. Suppose that X is not valid. Then there exists a model $(K,R,\Phi)$ with a $p \in K$ which refuses $\{+X\}$. But then by the theorem each $C_i$ and, in particular, $C_n$ must contain at least one set of signed sentences which admits a countermodel. This gives a contradiction since each set of $C_n$ is closed.

## 4. COMPLETENESS OF POSITIVE INTUITIONISTIC BETH TABLEAUS.

The completeness proof closely parallels the one given by Fitting [9] and derived by Kripke [8]. First we will introduce the notion of a dual Hintikka collection. Then we will show that any dual Hintikka collection admits a countermodel. Finally we will show that if no positive tableau for +X closes, than +X can be extended to a dual Hintikka collection, from which completeness of positive tableaus with respect to Kripke models follows immediately.

If S is a set of signed sentences, let $\mathcal{P}_S$ be the set of all parameters occurring in the formulas of S.

Definition. A dual Hintikka collection is a collection $\mathcal{C}$ of sets of signed sentences such that for any S in $\mathcal{C}$, no positive tableau for S closes and the following holds:

1) if $+X \& Y \in S$ then $+X \in S$ or $+Y \in S$
2) if $-X \& Y \in S$ then $-X \in S$ and $-Y \in S$
3) if $+X \quad Y \in S$ then $+X \in S$ and $+Y \in S$
4) if $-X \quad Y \in S$ then $-X \in S$ or $-Y \in S$
5) if $-\sim X \in S$ then $+X \in S$
6) if $+\sim X \in S$ then there is a T in $\mathcal{C}$ such that $\mathcal{P}_S \subseteq \mathcal{P}_T$, $S_- \subseteq T$ and $-X \in T$
7) if $-X \to Y \in S$ then $+X \in S$ or $-Y \in S$
8) if $+X \to Y \in S$ then there is a T in $\mathcal{C}$ such that $\mathcal{P}_S \subseteq \mathcal{P}_T$, $S_- \subseteq T$ and $-X \in T$ and $+Y \in T$
9) if $+\exists x X(x) \in S$ then for all a in $\mathcal{P}_S$ $+X(a) \in S$

10) if $-\exists x X(x) \in S$ then for some $a$ in $\mathcal{C}_S$ $-X(a) \in S$

11) if $-\forall x X(x) \in S$ then for all $a$ in $\mathcal{C}_S$ $-X(a) \in S$

12) if $+\forall x X(x) \in S$ then there is a $T$ in $\mathcal{G}$ such that $\mathcal{C}_S \subseteq \mathcal{C}_T$, $S_- \subseteq T$, and an $a \in \mathcal{C}_T$ such that $+X(a) \in T$.

Given any dual Hintikka collection $\mathcal{G} = \{S_p\}_{p \in I}$ we can define a model $(K, R, \Phi)$ as follows. $K = I$; $R$ is a reflexive and transitive relation on $I$ such that $pRq$ iff $S_{p,-} \subseteq S_{q,-}$ and $\mathcal{C}_{S_p} \subseteq \mathcal{C}_{S_q}$ ; and for each $p$ in $K$ $\Phi(p)$ is a classical model from $\mathcal{C}_{S_p}$ defined by: $+A(a) \in \Phi(p)$ iff $-A(a)$ is in $S_p$ for each atomic sentence $A(a)$.

<u>Theorem 2.</u> For any dual Hintikka collection $\mathcal{G}$ the corresponding model defined above is a counter-model for any set $S_p$ in $\mathcal{G}$. More precisely for any $p \in I$: if $\pm X \in S_p$, then $\mp X \in \Phi^*(p)$.

<u>Proof.</u> Since $\Phi^*(p) = \bigcup_{n \in N} \Phi^{(n)}(p)$, it is enough to show that for each $p \in I$ and for each integer $n$ $S_p \cap \Phi^{(n)}(p)$ is empty.

The assertion is true by definition when $n = 0$. Suppose the assertion true for each integer $k \leq n$. We now prove it is true for $k = n + 1$ by assuming the contrary and deriving a contradiction. Suppose there is a $Z$ which is both in $S_p$ and in $\Phi^{(n+1)}(p)$. We have twelve cases to examine according to the form of $Z$.

(1) $Z$ is $+X \& Y$. Then either $+X$ or $+Y$ is in $S_p$ while both of them are in $\Phi^{(n)}(p)$. This contradicts the induction hypothesis.

(2) Z is $-X \& Y$. Then both $-X$ and $-Y$ are in $S_p$ while at least one of them must be in $\Phi^{(n)}(p)$, which gives a contradiction.

(3) and (4) The cases $Z = \pm X \vee Y$ are treated similarly.

(5) Z is $-{\sim}X$. Since Z is in $\Phi^{(n+1)}(p)$, there exists a q such that pRq and $+X \varepsilon \Phi^{(n)}(q)$. But $-{\sim}X$ is also in $S_q$, because $S_{p,-} \subseteq S_{q,-}$; hence $+X$ is also in $S_q$, a contradiction.

(6) Z is $+{\sim}X$. Since Z is in $S_p$, there exists a q such that $S_{p,-} \subseteq S_{q,-}$ and $-X$ is in $S_q$. But by hypothesis, $+{\sim}X$ is also in $\Phi^{(n+1)}(p)$ and pRq; hence $-X$ is in $\Phi^{(n)}(q)$, which is impossible.

(7) Z is $-X \to Y$. Since Z is in $\Phi^{(n+1)}(p)$, there is a q such that pRq and $+X$ and $-Y$ are in $\Phi^{(n)}(q)$. But because $S_{p,-} \subseteq S_{q,-}$, $-X \to Y$ is also in $S_q$; that is, either $+X$ or $-Y$ is in $S_q$, a contradiction.

(8) Z is $+X \to Y$. Since Z is in $S_p$, there is a q such that $S_{p,-} \subseteq S_{q,-}$ and both $-X$ and $+Y$ are in $S_q$. But pRq and Z in $\Phi^{(n+1)}(p)$ implies Z is also in $\Phi^{(n+1)}(q)$ which means that either $-X$ or $+Y$ must be in $\Phi^{(n)}(q)$, a contradiction.

(9) Z is $+\exists x X(x)$. Since Z is in $S_p$, $+X(a)$ is in $S_p$ for all a in $\mathcal{B}_S$. But, because Z is also in $\Phi^{(n+1)}(p)$ there must be some $a \varepsilon \mathcal{B}_{S_p}$ such that $+X(a)$ is in $\Phi^{(n)}(p)$, a contradiction.

(10) Z is $-\exists x X(x)$. Since Z is in $S_p$, there must be

an a in $\mathcal{S}_{S_p}$ such that $-X(a)$ is in $S_p$. But, because $-\exists xX(x)$ is also in $\Phi^{(n+1)}(p)$, then $-X(a)$ is in $\Phi^{(n)}(p)$ which is impossible.

(11) Z is $-\forall xX(x)$. Since Z is in $\Phi^{(n+1)}(p)$, there is a q such that pRq, and an a in $\mathcal{S}_{S_q}$ such that $-X(a)$ is in $\Phi^{(n)}(p)$. But Z in $S_p$ implies Z is also in $S_q$ because $S_{p,-} \subseteq S_{q,-}$ ; hence $-X(a)$ is in $S_q$ which is impossible.

(12) Z is $+\forall xX(x)$. Since Z is in $S_p$, there is a q such that $S_p \subseteq S_q$, $S_{p,-} \subseteq S_{q,-}$ and $+X(a)$ is in $S_q$. On the other hand, in view of the relation pRq, the hypothesis that $+\forall xX(x)$ is in $\Phi^{(n+1)}(p)$ implies that $+X(a)$ is also in $\Phi^{(n)}(q)$, a contradiction.

Definition. A dual Hintikka element with respect to a set $\mathcal{S}$ of parameters is a set of signed sentences S such that no tableau closes for S, $\mathcal{S}_S \subseteq \mathcal{S}$, and S satisfies conditions 1,2,3,4,5,7,9,10,11 in the definition of a dual Hintikka collection.

Theorem 3. Let S be a set of signed sentences such that no tableau for S closes, and $\mathcal{S}$ a countable set of parameters such that $\mathcal{S}_S \subseteq \mathcal{S}$. Then S can be extended to a dual Hintikka element with respect to $\mathcal{S}$.

Proof. Let $Z_1, Z_2, \ldots, Z_n, \ldots$ be an enumeration of all subformulas from S using only parameters from $\mathcal{S}$. We want to define a sequence of signed sets $S_{i,j}$ for each $i \in N$ and $0 \leq j \leq i$, such that no closed tableau exists for any of them. Let $S_{0,0} = S$.

Having defined $S_{n-1,n-1}$, let $S_{n,0} = S_{n-1,n-1}$. Now, in order to define $S_{n,k+1}$ given $S_{n,k}$ for $0 \leq k < n$, let us consider $Z_k$. If neither of $+Z_k$ or $-Z_k$ is in $S_{n,k}$, let $S_{n,k+1} = S_{n,k}$. Otherwise, since $S_{n,k}$ is consistent by hypothesis, at most one of $+Z_k$, $-Z_k$ is in $S_{n,k}$. Again we have to consider all possible cases.

If $+Z_k$ is in $S_{n,k}$ and has the form $+\sim X$, $+X \to Y$, $+\forall xX(x)$, then $S_{n,k+1} = S_{n,k}$. Otherwise: if $Z_k$ is $X \& Y$ and $-Z_k$ is in $S_{n,k}$, let $S_{n,k+1} = \{S_{n,k}, -X, -Y\}$. Since no tableau closes for $S_{n,k}$ by the induction hypothesis, no tableau can close for $S_{n,k+1}$. If $+Z_k$ is in $S_{n,k}$ let $S_{n,k+1}$ be $\{S_{n,k}, +X\}$ or $\{S_{n,k}, +Y\}$ according to which of the two configurations does not close.

The other propositional cases are treated similarly.

$Z_k$ is $\exists xX(x)$ and $+\exists xX(x)$ is in $S_{n,k}$. Then $S_{n,k+1}$ is obtained by adding $+X(a)$ to $S_{n,k}$ for each parameter a used in $S_k$. It is clear that no tableau closes for $S_{n,k+1}$ if no tableau closes for $S_{n,k}$. If $-\exists xX(x)$ is in $S_{n,k}$, then since by construction $S_{n,k}$ contains only a finite number of parameters, take from S the first unused parameter $a_i$ and let $S_{n,k+1} = \{S_{n,k}, -X(a_i)\}$. The case $Z_k$ is $-\forall xX(x)$ is treated similarly to the case $+\exists xX(x)$.

It is evident by construction that $T = \bigcup_{n \in N} S_{n,n}$ is a dual Hintikka element with respect to $\mathcal{S}$.

Theorem 4. (Completeness) If X is valid, then $\vdash_I X$.

Proof. We will show the contrapositive, that is if no tableau for +X closes then {+X} admits a countermodel and therefore X cannot be valid.

Let $\{\eta_m\}_{m\varepsilon N}$ be a collection of countable sets of parameters such that $\eta_i \cap \eta_j$ is empty for each i,j $\varepsilon$ N, $\eta_1$ contains all parameters in X and let $\mathcal{S}_n = \bigcup_{m=1}^{n} \eta_m$ for each n $\varepsilon$ N.

Let us call a p-sentence any sentence of the type $+\sim Y$, $+Y \rightarrow Z$, $+\forall xY(x)$. We will now show how to construct a dual Hintikka collection starting with +X. Since no tableau for +X closes, extend {+X} to a dual Hintikka element with respect to $\mathcal{S}_1$. Let $S_1$ be the set of signed sentences so obtained. Now enumerate all p-formulas of $S_1$ and take the first one. We then extend one of the sets $\{S_{1,-},-X\}$, $\{S_{1,-},-X,+Y\}$, and $\{S_{1,-},+A(a)\}$ to a dual Hintikka element with respect to $\mathcal{S}_2$, note that a $\varepsilon\mathcal{S}_2$, according as the first p-formula of $S_1$ has the form $+\sim X$, $+X \rightarrow Y$ or $+\forall xX(x)$ respectively.

Call $S_2$ the result of such an extension and consider the p-formula in question "used". Clearly no tableau can close for $S_2$ if no tableau for $S_1$ closes. In general at step n we will have the collection $\{S_1,S_2,\ldots,S_{2^n}\}$ where each $S_i$ is a dual Hintikka element with respect to $\mathcal{S}_i$. Take the first unused p-formulas appearing in each of these sets and repeat the process above according to the form of the p-formula, obtaining in this way a sequence $\{S_1,S_2,\ldots,S_{2^{n+1}}\}$.

It is clear that the collection $\mathcal{G}$ constructed according to this procedure is a dual Hintikka collection. Indeed each element $S_i$ of $\mathcal{G}$, being a dual Hintikka element with respect to $\mathcal{B}_i$, must satisfy conditions 1,2,3,4,5,7,9,10,11. Moreover if $+\sim X$ is in $S_i$, for some i, then by construction there must be an $S_k$ in $\mathcal{G}$ such that $S_{i,-} \subseteq S_k$, $\mathcal{B}_{S_i} \subseteq \mathcal{B}_{S_k}$ and $-X \varepsilon S_k$ so that condition 6 is satisfied. If $+X \to Y$ is in $S_i$ there must be an $S_k$ in $\mathcal{G}$ such that $S_{i,-} \subseteq S_k$, $\mathcal{B}_{S_i} \subseteq \mathcal{B}_{S_k}$ and both -X and +Y are in $S_k$, which takes care of condition 8. Finally condition 12 too is satisfied because if $+\forall x X(x) \varepsilon S_i$, for some i, then by construction there must be an $S_k$ in $\mathcal{G}$ such that $S_{i,-} \subseteq S_k$, $\mathcal{B}_{S_i} \subseteq \mathcal{B}_{S_k}$, and an a in $\mathcal{B}_{S_k}$ such that $+X(a) \varepsilon S_k$.

Since $\mathcal{G}$ is a dual Hintikka collection, we can apply Theorem 2 and construct the corresponding model. In such a model $\Phi^*(p_1) \cap S_1$ is empty but $+X \varepsilon S_1$; hence $-X \varepsilon \Phi^*(p_1)$ and X is not valid.

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## B I B L I O G R A P H Y

[1] G.GENTZEN - "Untersuchungen über das logische Schliessen." Mathematische Zeitschrift, 39, 1935. English translation in "The collected papers of Gerard Gentzen", ed.by M.E.Szabo, North Holland, 1969.

[2] E.W.BETH - "The foundations of mathematics." North Holland, 1959.

[3] K.J.J.HINTIKKA - "Form and content in quantification theory." Acta Philosophica Fennica, 8, 1955

[4] R.M.SMULLYAN - "First order logic." Springer-Verlag, 1968.

[5] H.LEBLANC, D.P.SNYDER - "Duals of Smullyan trees." Notre Dame Journal of Formal Logic, vol. XIII, 3, 1972.

[6] A.HEYTING - "Die formalen Regeln des intuitionistichen Logik." S.B.Berlin, Phys.-math. Kl., 1930.

[7] H.LEBLANC, R.H.THOMASON - "The demarcation line between intuitionistic logic and classical logic." Zeitsch.f.math.Logik und Grundl.d. Math.,12, 1966.

[8] S.KRIPKE - "Semantical analysis of intuitionistic logic I." in Formal Systems and recursive functions, North Holland, 1965.

[9] M.C.FITTING - "Intuitionistic logic, Model Theory and Forcing." North Holland, 1969.

[10] R.M.SMULLYAN - "A generalization of intuitionistic and modal logic." in Truth, Syntax and Modality, ed.by H.Leblanc, North Holland, 1973.