Alternate Row and Column Elimination for Solving Certain Linear Systems

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1. Introduction

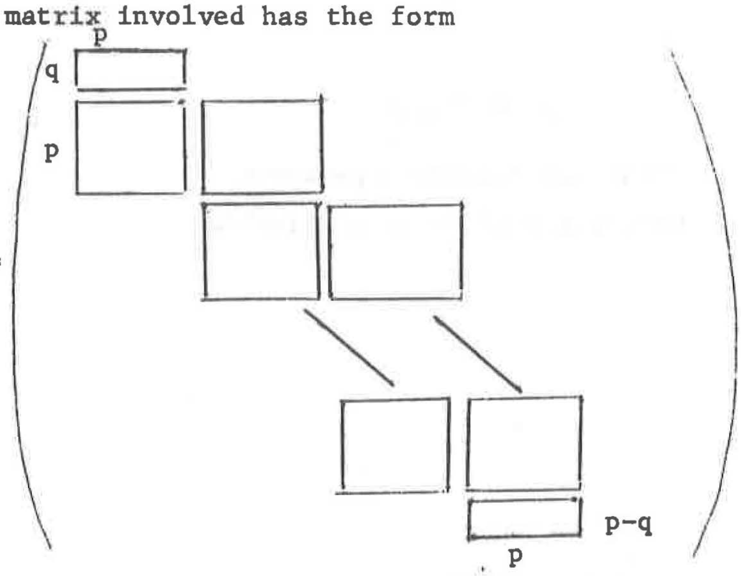
One of the most effective ways of numerically solving a system of two-point boundary value problems is to use a relatively simple, low-order finite difference scheme on a given mesh and then improve the accuracy by extrapolation or deferred correction. If this scheme is also *compact* (that is if it connects as few points as possible consistent with the order of the differential equation) there are certain advantages regarding stability and convergence: see Kreiss [3]. Examples of such schemes for systems of first-order equations are the popular midpoint rule of Keller [1], and certain collocation schemes using simple piecewise polynomials.

Because such schemes involve solving linear systems of the same form time and time again, it is important to have a fast stable method for solving such systems. Of course one can use the usual Gaussian elimination with partial pivoting, but it is of interest to do better. In [6] we compared band and block forms of elimination for such schemes, and other block forms have been discussed by Keller [2], Schechter [5], and White [7]. However the faster methods do limited or no pivoting, so one cannot guarantee stability of the elimination. Here we present a method for such systems which is stable, and faster than the usual Gaussian elimination with pivoting (although not as fast as the possibly unstable block elimination methods).

2. The Method

The matrix involved has the form

A =

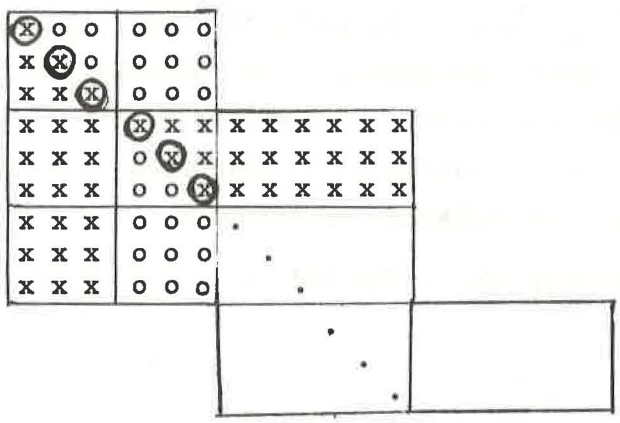


(2.1)

Here all blocks are p x p (p = order of differential equation system) except the first (q x p) and last ((p - q) x p); q denotes the number of boundary conditions at the left hand end (we assume separated boundary conditions).

In solving a linear system with a matrix of the form (2.1), we want to ensure stability of the matrix decomposition. With the usual Gaussian elimination, this is accomplished by row pivoting and row elimination, which introduces extraneous nonzero elements in (2.1). In his thesis ([4]) Lam noted that if one pivoted on the columns for the first q steps of the decomposition, then on rows for the next p - q steps, and alternately thereafter, no extraneous zeros are introduced. But he then proceeded to perform the usual row elimination, and so the multipliers used during the elimination are not bounded a priori, so that the decomposition may still be unstable (although this is less likely than with no pivoting).

However one can in fact guarantee boundedness of the multipliers by eliminating alternately by rows and columns as well. That is, we pivot by columns for the first q steps of the decomposition and use these large pivots to eliminate by columns for these first q steps, so all multipliers used are bounded by 1.0. For the next p - q steps, we pivot by rows and eliminate by rows, again ensuring that all multipliers are bounded by 1.0. This produces a matrix of the form (for $p = 6$, $q = 3$)



Now at the $(p + 1)^{st}$ step, we revert to the column pivoting and column elimination for q rows, then row pivoting and row elimination for p - q rows, and so on through the matrix.

In matrix form this decomposition can be written

$$A = PLBUQ \tag{2.2}$$

where P gives the row permutations, L is lower triangular and contains the column elimination multipliers, U is upper triangular and contains the column elimination multipliers, and Q gives the column permutations. The end result of the decomposition, B, has the following form:

$$B = \left(\begin{array}{cccccccc}
L & O & & & & & & \\
X & R & X & X & & & & \\
X & O & L & O & & & & \\
 & & X & R & X & X & & \\
 & & X & O & L & O & & \\
 & & & & \ddots & \ddots & & \\
 & & & & X & O & L & O \\
 & & & & & & X & R
\end{array}\right) \tag{2.3}$$

where L denotes a lower triangular q x q square block, R an upper triangular square block of order p - q , X an arbitrary (nonzero) block, and 0 a zero block.

Of course for this scheme to be useful, we must be able to solve $B\underline{y} = \underline{z}$ easily. This can be done by a forward and backward block recurrence as follows: partition $\underline{y}^T = (\underline{y}^{(1)}, \underline{y}^{(2)}, \ldots)$ the same as B. Then solve successively for $\underline{y}^{(1)}, y^{(3)}, \underline{y}^{(5)}, \ldots$ by using the odd numbered blocks of B; this submatrix of B is lower triangular, so the process proceeds as forward recurrence. Now solve for the even numbered blocks of $\underline{y}$ starting at the bottom using the even numbered blocks of B; this submatrix is upper triangular, so this process is a backwards recurrence.

Thus using the notation (2.2), the solution of $A\underline{x}=\underline{b}$ proceeds as follows:

$$\begin{aligned} P\underline{x}^{(1)} &= \underline{b} \\ L\underline{x}^{(2)} &= \underline{x}^{(1)} \\ B\underline{x}^{(3)} &= \underline{x}^{(2)} \\ U\underline{x}^{(4)} &= \underline{x}^{(3)} \\ Q\underline{x}^{(5)} &= \underline{x}^{(4)} \end{aligned} \tag{2.4}$$

Of course the first two steps can be done during the decomposition (2.2); then we perform the other steps . No additional storage is required as the elements of U can be stored in the corresponding (zeroed) positions in A. Also, since the multipliers are bounded $(|\ell_{ij}| \le 1, |u_{ij}| \le 1)$ the decomposition is just as stable as the usual Gaussian elimination with row pivoting. Moreover, since the only additional complication is the solving of triangular systems (which is very accurate) the whole process (2.4) is just as accurate as the usual Gaussian elimination.

3. Computation Time

For each p x p block, we have the following operation counts:

a) decomposition: $[\,2p(p-1) + (2p-1)(p-2) + \ldots + (p+q+1)q\,]$ for rows
$[\,2p(p-1) + (2p-1)(p-2) + \ldots + (2p-q+1)(p-q)\,]$ for columns

b) L, U: $[(p-1) + (p-2) + \ldots + q] + [(p-1) + (p-2) + \ldots + (p-q)\,]$

c) $B\underline{z} = \underline{y}$: $[(q+1) + (q+2) + \ldots + 2q] + [2p + (2p-1) + \ldots + (p+q+1)]$.

So the total count, per block,is

$$\frac{5}{6}(p^3-p) + 2pq(p-q) + \frac{p}{2}(3p+1). \tag{3.1}$$

This count should be compared with those given in [6] for block elimination and scaler elimination (with and without pivoting). In particular, for $p=2$, $q=1$ (the case of a single second-order equation) the counts are: 16 for the above method, 23 for scalar elimination with pivoting, 15 for scalar elimination without pivoting, and 16 for block elimination.

The other useful comparison is for p large; we divide by p^3 and keep only the high-order terms. Then as a function of $r=q/p$ ($o<r\leq\frac{1}{2}$) the various counts are

row and column elimination	$5/6 + 2r-2r^2$
scalar elimination (pivoting)	$5/6 + 3r/2$
scalar elimination (no pivoting)	$5/6 + r/2 - r^2/2$
block elimination	$1/3 + 2r - r^2$

Of course the last two methods (which do not guarantee stability) are faster for any r, but it is interesting to compare the stable methods. Our alternate row and column elimination is better for $r > 1/4$, and is 16% faster for $r=1/2$; for $r<1/4$ it is slower, but the worst case occurs when $r\simeq1/8$ and here it is only 3% slower. And for smaller p, the comparison is more biased towards the new method: above, for $p=2$, $q=1$, the new method is 30% faster. Thus it seems reasonable to use the alternate row and column elimination in all cases.

References

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