A NOTE ON LINEAR RESOLUTION STRATEGIES IN CONSEQUENCE-FINDING

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December, 1971

Abstract

The completeness, for consequence-finding, of various linear resolution strategies is studied. Linear resolution with merging and subsumption is complete. A-ordered, linear resolution with merging is, in a certain sense, complete. Linear resolution with merging and C-ordering is incomplete. It is argued that the incompleteness of this latter strategy for consequence-finding recommends it above the other two as a complete strategy for theorem-proving.

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1. Introduction

As initially conceived and applied, the resolution principle of Robinson [1] was a proof-finding procedure, i.e. given a set S of clauses, prove that S is unsatisfiable, when it is. Robinson's method is to generate, from S, a sequence of clauses by resolution. S is unsatisfiable iff the empty clause is so generated. In [2], Lee addresses himself to the problem of characterizing those clauses generated from S when S may be satisfiable. Clearly, any such clause is logically implied by S. Is every clause T, implied by S, so generated? Lee proves a completeness theorem to the effect that some clause T' which subsumes T will be generated from S. Lee calls this generation procedure consequence-finding. If S is viewed as a set of axioms, then consequence-finding is a complete (in Lee's sense) procedure for generating all those theorems whose matrix is a disjunction of literals. As such, it is of some interest to the intellectually difficult problem of discovering new theorems in axiomatic theories. As we shall see later, the theory of consequence-finding may also be of use as a rough comparison of the relative efficiencies of various resolution strategies for proof-finding.

In [3], Lee's completeness results are extended to resolution under various restrictive strategies (I-semantic resolution, positive and negative hyper-resolution, PI-resolution [4]). The present paper considers the effects on completeness of various linear resolution strategies. The basic results are:

1. Linear resolution with the merging, subsumption and tautology conditions of [5] and [6] is complete.

2. Linear resolution with the merging, A-ordering, and tautology conditions of [7] is, in some sense, complete, while that with the C-ordering condition of [7, 8, 9] is incomplete.

We assume the reader to be familiar with the usual terminology and definitions in [1, 4, 5].

2. Results

If S is a set of ground clauses, and T a set of ground literals, write $S_T = \{C \mid C' \in S \text{ and } C = C' - T\}$. If T is the ground clause $\{T_1, T_2, \ldots, T_n\}$, write $\overline{T} = \{\{\overline{T}_1\}, \{\overline{T}_2\}, \ldots, \{\overline{T}_n\}\}$. If S is a set of clauses (ground or general), an m.s.l. (merge, subsumption, linear) deduction of R_n from S is a deduction like that of Figure 1, where

1. $C \in S$
2. No clause in the deduction is a tautology
3. For each j, $C_j \in S$ or $C_j = R_i$ for some $i < j$, in which case

(i) R_i is a merge resolvent.

(ii) The literal of C_j resolved upon is a merge literal of R_i.

(iii) R_{j+1} subsumes R_j.

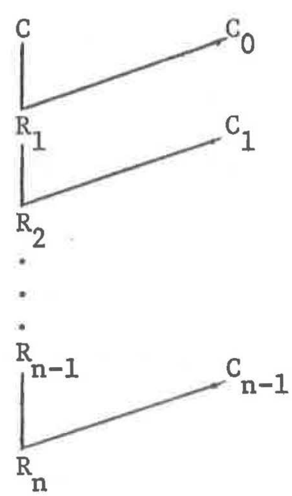

Figure 1.

We omit the proof of the following simple result.

Lemma 1

If $\sum$ is an unsatisfiable set of ground clauses, and T a set of ground literals, then $\sum_T$ is unsatisfiable.

Theorem 1

Suppose that S is a set of ground clauses, and T is a ground clause such that $S\cup\overline{T}$ is unsatisfiable. Then there is an m.s.l. deduction from S of a clause $T' \subseteq T$.

Proof:

By applying Lemma 1 to the unsatisfiable set $S\cup\overline{T}$, we deduce that $S_T\cup\overline{T}$ is unsatisfiable. But no literal of the clauses of $\overline{T}$ has a complement in the clauses of S_T. Hence S_T is unsatisfiable. Let S_T' be a minimally unsatisfiable subset of S_T, so that no clause of S_T' contains the complement of a literal of T. By Theorem 5 of [5] there is an m.s.l. deduction D, say Figure 1, of $R_n = \square$ from S_T'. In D, replace each occurrence of a clause $C_i \varepsilon S_T$ by the clause C_i' of S from which it was derived, and let R_{i+1}', obtained from R_{i+1} in the obvious way, replace R_{i+1}. Let T' be the set of literals of T which have beed added to D to yield this deduction. Then the result is a linear deduction D' of $T' \subseteq T$. Since S_T' contains no clause containing the complement of a literal of T, no clause in D' is a tautology. The merge conditions 3(i) and 3(ii) hold in D' since they hold in D. To see that the subsumption condition 3(iii) is valid in D', notice that if C_j' is a merge resolvent, say $C_j' = R_i'$ for some $i < j$, then R_j' contains all those literals of T which R_i' contains. Since, in D, $R_{j+1} \subseteq R_j$, then, in D', $R_{j+1}' \subseteq R_j'$.

Theorem 1 establishes the ground completeness, in Lee's sense [2], of m.s.l. deduction for consequence-finding. For $S\cup\overline{T}$ is unsatisfiable iff S logically implies T.

We remark here that the ground results in [3] follow easily, using the same method as in the proof of Theorem 1, from the corresponding results in [4]. For example, we sketch an elementary proof of Theorem 4 of [3] which states:

Let T be a ground clause which is false in a ground interpretation of $S \cup \{T\}$, and let A be an ordering of the atoms of $S \cup \{T\}$ such that the atoms of T occur first in A. Then there exists a ground maximal AI-deduction from S of a clause $T' \subseteq T$.

Proof:

Let S_T' be as in the proof of Theorem 1. By Theorem 7 of [4], there exists a ground maximal AI-deduction D of $\square$ from S_T'. In D, replace each occurrence of a clause $C \varepsilon S_T'$ by the clause $C' \varepsilon S$ from which it was derived, and carry out the obvious modifications to the resolvents of D. The resulting deduction is a deduction D' of a clause $T' \subseteq T$. Clearly, by our assumptions about the A-ordering and the fact that T is false in the ground interpretation, D' is a maximal AI-deduction of T'.

We now lift the results of Theorem 1 to the general level.

<u>Theorem 2</u>

Let S be a set of general clauses, and T a clause such that S logically implies T. Then there is an m.s.l. deduction, from S, of a clause E such that E subsumes T.

Proof:

We use the technique, first used in [3], of introducing new distinct constants into the Herbrand Universe. Let $x_1, \ldots, x_n$ be all of the individual variables of T, which we now write $T(x_1, \ldots, x_n)$. Let $b_1, \ldots, b_n$ be new distinct constants not occurring in S or T. Since S logically implies $T(x_1, \ldots, x_n)$, S also logically implies $T(b_1, \ldots, b_n)$. Let $H(b_1, \ldots, b_n)$ be the Herbrand

Universe of $S\cup\{T(b_1, \ldots, b_n)\}$. By Theorem 1 of [3], there exists a finite set of ground instances S' of S over $H(b_1, \ldots, b_n)$ such that S' logically implies $T(b_1, \ldots, b_n)$, i.e. such that $S'\cup\overline{T}\ (b_1, \ldots, b_n)$ is unsatisfiable. By Theorem 1, there exists a ground m.s.l. deduction D', from S', of a clause $E'(b_1, \ldots, b_n) \subseteq T(b_1, \ldots, b_n)$. In D', replace each clause $C' \varepsilon S'$ by that clause $C \varepsilon S$ of which it is a ground instance, and replace each ground resolvent of D' by the general resolvent of its two parent clauses. The result is a general m.s.l. deduction, from S, of a clause E which has, as an instance, $E'(b_1, \ldots, b_n)$. Since S does not contain the symbols $b_1, \ldots, b_n$ there exists a substitution $\sigma(b_1, \ldots, b_n)$ such that

$E\sigma(b_1, \ldots,b_n) \subseteq E'(b_1, \ldots,b_n) \subseteq T(b_1,\ldots,b_n)$, i.e. $E\sigma(x_1,\ldots,x_n) \subseteq T(x_1,\ldots,x_n)$ so that E subsumes T.

Using the same method as in the proof of Theorem 1, we can establish further results relating known properties of linear resolution strategies in theorem-proving to corresponding properties in consequence-finding. We prove our results for the ground case. Lifting proceeds essentially as in the proof of Theorem 2. Let A be an A-ordering of the atoms of S. An <u>m.a.l.</u> (merge, A-ordered, linear) <u>deduction of R_n from S</u> ([7]) is a deduction satisfying the definition of p.2 , with (iii) replaced by

(iii)' The literal resolved upon in producing R_{i+1} is the maximal literal in R_i under A.

<u>Theorem 3</u>

Let S and T be as in the statement of Theorem 1. Let A be an ordering of the atoms of $S\cup\overline{T}$ such that the atoms of T occur first in A. Then there exists an m.a.l. deduction, from S, of a clause $T' \subseteq T$.

Proof:

Let S'_T be as in the proof of Theorem 1. By Theorem 1 of [7] there exists an m.a.l. deduction from S'_T, of $R_n = \square$. Proceed in the obvious way, as in the proof of Theorem 1.

In contrast to this result, one can show that m.c.l. (merge, C-ordered, linear) resolution is incomplete for consequence-finding. An m.c.l. deduction is like that of p.2, except that the literals of C are initially given a fixed ordering, those of C_i are ordered arbitrarily during the course of the deduction, R_{i+1} inherits the ordering of R_i and C_i by concatenating R_i and C_i with merging to the left, and (iii) is replaced by

(iii)'' The literal resolved upon in producing R_{i+1} is the rightmost literal of R_i.

See [7,8,9] for a precise definition. It is easy to see that for $S = \{\{p,x,y\}, \{q,x,\bar{y}\}, \{q,\bar{x},y\}, \{p,\bar{x},\bar{y}\}\}$ there is no m.c.l. deduction of $\{p,q\}$ although S logically implies $\{p,q\}$.

An <u>input deduction, from S, of R_n</u> ([10]) is a linear deduction of R_n, as in Figure 1, such that each $C_i \in S$. A <u>unit deduction</u> of R_n is a resolution deduction of R_n in which each resolvent has at least one parent which is a unit clause. In [10], Chang proves that there is an input deduction, from S, of $\square$ iff there is a unit deduction of $\square$.

<u>Theorem 4</u>

Let S and T be as in the statement of Theorem 1. If there exists a unit deduction, from S, of a clause $T' \subseteq T$, then there is an input deduction, from S, of a clause $T'' \subseteq T$.

Proof:

Since the clauses of $\bar{T}$ are all units, and since $S \cup \bar{T}$ is unsatisfiable, we can use these units, together with the unit deduction of T', to obtain a unit deduction of $\square$. By Theorem 1 of [10], there exists an input deduction, from $S \cup \bar{T}$, of $\square$. In this deduction, delete every resolution operation with parents C and $\{T_i\}$ where $C \in S$ and $T_i \in T$, and replace the corresponding resolvent by C. The resulting deduction is an input deduction, from S, of a clause $T'' \subseteq T$.

The converse of Theorem 4 is false.

3. Remarks

The completeness of m.s.l. deduction for consequence-finding is in some sense a pessimistic result. For this suggests that the m.s.l. strategy for theorem-proving is not as restrictive as it might be since no essential use is made of the fact that a particular clause, namely $\square$, is the one to be generated. On the other hand, deductions involving A-ordering, e.g. maximal AI-deduction or m.a.l. deduction, are complete for consequence-finding only to the extent that all possible A-orderings must be tried in order to generate all possible target theorems, whereas for theorem-proving, any fixed A-ordering will do. This suggests that these strategies do make use of the fact that, for theorem-proving, the target clause is $\square$. Finally, according to this rough measure on the efficiency of strategies for theorem-proving, m.c.l. deduction emerges as the most restrictive of those under consideration since it is incomplete for consequence-finding, and hence makes essential use of the fact that $\square$ is the target theorem. We believe that observations of this kind, relating consequence-finding to theorem-proving, will lead to more precise techniques for comparing efficiencies of resolution strategies.

Acknowledgements

This research was supported in part by National Research Council of Canada grant A7642.

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