Graphs, Groups and Matrices*

by

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1. Introduction

Studies of the automorphism group of a graph are principally concerned with the following two problems: 1) the existence of a graph whose automorphism group is isomorphic to a given group, and 2) determining the group of a graph. In the case of finite graphs, the first problem (posed by König [13]) was largely settled by Frucht [5] who showed that there exist infinitely many nonisomorphic connected graphs G whose automorphism groups are (abstractly) isomorphic with a given finite group A. The same author further showed in [7] that the result still holds if the class of connected graphs is restricted to those which are regular of degree three. This result was sharpened even more by Sabidussi [18] who proved that the condition of being regular of degree three can be replaced by several other conditions. In the general case of graphs with an arbitrary number of vertices, Sabidussi [21] proved that given any group A, there exists a graph whose automorphism group is isomorphic with A.

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Since the automorphism group of a finite graph can be regarded as a permutation group, one can also consider when there exists a graph whose automorphism group is permutationally isomorphic with a permutation group. This problem is much more difficult than the former, and there are correspondingly fewer results on the subject. We will discuss the relevant work in connection with the second problem mentioned above.

Computing the automorphism group of a graph is, in general, quite difficult. Kagno [12] determined the groups of all graphs with up to six vertices; and, more recently, Hemminger [11] supplemented the listing by giving the groups of directed graphs with up to six vertices. Moreover, the automorphism group of a tree has been determined by Prins [17]. Since direct computation is usually not feasible, there is considerable interest in finding the group of a composite graph in terms of the respective groups of the graphs in the composition. Perhaps the simplest composition from this point of view is the representation of a graph as a sum of its connected components. It is well known that the group of the sum of non-isomorphic graphs is just the direct sum of the groups of the respective graphs in the sum. Moreover, Frucht [5] showed that the group of a sum of n graphs which are isomorphic with a graph G is the wreath-product of the symmetric group of degree n with the group of G. Thus, the automorphism group of a graph can be expressed in terms of the groups of its connected components, which simplifies matters somewhat.

Other compositions of graphs which have corresponding group compositions are the so-called cartesian product, composition (or lexicographic product), and Kronecker product. Harary [8] and Sabidussi [19],[20] treated the groups of the cartesian product and composition of graphs. Mowshowitz [14] made similar observations on the Kronecker product.

Our object here is to exploit the connection between the adjacency matrix of a graph and its automorphism group in order to determine the latter. It is trivial to verify that a permutation σ of the nodes of a graph G is an automorphism of G if and only if the permutation matrix corresponding to σ commutes with the adjacency matrix of G. Using this observation, Chao [2] was able to construct graphs whose automorphism groups contain a given transitive group as a subgroup; Mowshowitz [15] proved that if the eigenvalues of the adjacency matrix of a graph are distinct, then every non-trivial automorphism has order 2, so that the group is abelian which in turn implies (by a result of Chao [1]) that for $p > 2$ no graph with p nodes satisfying this condition has a transitive group. Moreover, Chao [3] showed that under the same condition, the automorphism group of a directed graph is abelian.

In what follows, we will take advantage of the fact that the adjacency matrix of a graph is a (0,1)-matrix and thus can be regarded as a matrix over GF(2). This point of view has important consequences for the construction of automorphism groups, as we propose to demonstrate.

Since the results in Sections 2 and 3 apply to the most general type of graph, we need the notion of a net. A net $N = (V,X,f,s)$ consists of a finite set V (containing the nodes of N), a finite set X (containing the directed lines of N), and two functions f and s both mapping X into V. Two lines $x,y \in X$ are said to be parallel if their first and second nodes coincide, i.e., if $fx = fy$ and $sx = sy$, respectively; a line x is called a loop if $fx = sx$. A digraph is a net with no loops and no parallel lines, i.e., an irreflexive relation; and a graph is a symmetric digraph. The adjacency matrix $A = A(N) = (a_{ij})$ of a net N with nodes v_1, v_2, ..., v_p is defined by

$$a_{ij} = |\{x \in X | fx = v_i \text{ and } sx = v_j\}|$$

The automorphism group $\Gamma = \Gamma(N)$ of a net N is the set of all one-one mappings of V onto itself which preserve the adjacency matrix. For graph theoretic terms not defined here, see [9],[10]. Throughout the following, we shall regard the automorphism group of a net, and the symmetric group S_p as groups of permutation matrices.

2. Nets with non-derogatory adjacency matrix

Let N be a p-node net with adjacency matrix $A = A(N)$. As we observed earlier in connection with graphs, an element $P \in S_p$ is in $\Gamma(N)$ if and only if

$$PA = AP \qquad (1)$$

Thus, taking A as a matrix over a field F, $\Gamma(N)$ is contained in the centralizer of A over F.

Theorem 1. Let N be a net with adjacency matrix $A = A(N)$. If A is non-derogatory (i.e., its minimal and characteristic polynomials are identical) over a field F, then $\Gamma(N)$ is abelian.

Proof. Since A is non-derogatory, the centralizer of A is just the ring of polynomials in A over F (see, for example, [22]). Thus, every $P \in \Gamma(N)$ is a polynomial in A, from which the result follows.

Corollary 1a. (Chao [3]) If the adjacency matrix of a digraph D has all distinct eigenvalues, $\Gamma(D)$ is abelian.

Proof. Since A(D) has distinct eigenvalues, it is non-derogatory over the field of complex numbers.

Corollary 1b. (Mowshowitz [15]) If the adjacency matrix $A = A(G)$ of a graph G has all distinct eigenvalues, then $\Gamma(G)$ is abelian and every non-trivial automorphism has order 2.

Proof. First, we observe that A is a symmetric matrix, so that all of its eigenvalues are real. By hypothesis, A is non-derogatory over the reals. Hence, $\Gamma(G)$ is abelian. Moreover, since each polynomial in A over the reals is a symmetric matrix, it is clear that all non-trivial automorphisms have order 2.

3. Nets with irreducible characteristic polynomial

An important special case of Theorem 1 arises when the characteristic polynomial is irreducible over the integers. The following result is a generalization of some observations of Collatz and Sinogowitz [4].

Theorem 2. Let N be a net with $A = A(N)$, $\Gamma = \Gamma(N)$, and the characteristic polynomial $\phi_A(x)$ of degree p. If $\phi_A(x)$ is irreducible over the integers, then Γ is trivial.

Proof. Without loss of generality, suppose the rows of A are arranged into blocks corresponding to the orbits of Γ. Now, let $\zeta=[z_1,\ldots,z_1\ldots z_k,\ldots,z_k]^T$ be a vector with h_i components equal to z_i where h_i is the size of the i-th orbit of Γ. Consider the result of multiplying ζ by A.

$$A\zeta = [\sum_{j=1}^{k} z_j t_{ij},\ldots,\sum_{j=1}^{k} z_j t_{ij},\ldots,\sum_{j=1}^{k} z_j t_{kj},\ldots,\sum_{j=1}^{k} z_j t_{kj}]^T \qquad (2)$$

where t_{ij} is the number of lines incident from a node in the i-th orbit to nodes in the j-th orbit of Γ. (Clearly, the number of such lines is the same for all nodes in the same orbit.)

Now, let $T = (t_{ij})$ for $1 \le i, j \le k$, $\bar{\zeta} = [z_1,\ldots,z_k]^T$, and consider the equation

$$A\zeta = x\zeta \qquad (3)$$

From (2) and (3) we obtain

$$T\bar{\zeta} = x\bar{\zeta} \qquad (4)$$

Hence, $\bar{\zeta}$ is an eigenvector of T, and consequently ζ is an eigenvector of A, so that det(T-xI), the characteristic polynomial of T, divides $\phi_A(x)$. So, if there exists a $P \in \Gamma$ with $P \neq I$, then $1 \leq \deg[\det(T-xl)] < p$, from which the result follows.

Figure 1 exhibits a smallest digraph and graph with irreducible characteristic polynomials.

x^3-x-1 $\qquad$ $x^6-7x^4-2x^3+8x^2+2x-1$

Figure 1. Digraphs with irreducible polynomial

The converse of Theorem 2 is not true, as evidenced by the fact that the characteristic polynomials of trees with an odd number of nodes, and of regular graphs have linear factors.

4. Construction of the group of a graph with non-derogatory matrix

According to Corollary 1b, every non-trivial automorphism of a graph with non-derogatory adjacency matrix (with respect to a given field) has order 2. For purposes of constructing the group of a graph it is useful to regard the adjacency matrix as one with entries in GF(2). So, let G be a p-node graph with group $\Gamma = \Gamma(G)$

and non-derogatory adjacency matrix $A = A(G)$ over GF(2).

Since any matrix Q satisfying $AQ = QA$ is a polynomial in A, we can write

$$Q = \sum_{i=0}^{p-1} a_i A^i \quad , \quad a_i \in GF(2) \tag{5}$$

Now, if Q is a permutation matrix (and thus an element of Γ) we have

$$I = Q^2 = [\sum_{i=0}^{p-1} a_i A^i]^2 = \sum_{i=0}^{p-1} a_i (A^2)^i \tag{6}.$$

Thus, in order to find the elements of Γ, it suffices to examine all polynomials $f(x)$ such that $f(A^2) = I$; for, if $f(A)$ is a permutation matrix, $f(A) \in \Gamma$.

Let $\mu_{A^2}(x)$ denote the minimal polynomial of A^2, and suppose $\deg \mu_{A^2}(x) = m$. Then every polynomial $f(x)$ satisfying $f(A^2) = I$ is of the form

$$f(x) = g(x)\, \mu_{A^2}(x) + 1 \tag{7}$$

for some polynomial $g(x)$. Hence, every matrix Q such that $Q^2 = I$ can be expressed in the form

$$Q = \mu_{A^2}(A) \sum_{i=0}^{p-m-1} b_i A^i + I \tag{8}$$

where $b_i \in GF(2)$.

Lemma 3a. Let A = A(G) be the adjacency matrix of a p-node graph G. Suppose A is non-derogatory over GF(2). Then

$$[\mu_{A^2}(x)]^2 = \mu_{A^2}(x^2) = \begin{cases} \phi_A(x) \text{ if p is even} \\ x\,\phi_A(x) \text{ if p is odd} \end{cases}$$

Proof. First, let us regard the coefficients of $\phi_A(x)$, the characteristic polynomial of A, as integers.

$$\phi_A(x) = \sum_{i=0}^{p} (-1)^i a_i x^{p-i}$$

By Theorem 2 of [16], all the odd subscripted coefficients a_i are even. Hence, if p is even

$$\phi_A(x) = \sum_{i \text{ even}} a_i x^{p-i} = \sum_{i \text{ even}} a_i (x^2)^{\frac{p-i}{2}} ;$$

if p is odd

$$\phi_A(x) = x \sum_{i \text{ even}} a_i (x^2)^{\frac{p-i-1}{2}}$$

Once again, regarding $\phi_A(x)$ as a polynomial over GF(2), we see that

$$\phi_A(x) = \begin{cases} [\sum_{i \text{ even}} a_i x^{\frac{p-i}{2}}]^2 \text{ if p is even} \\ x[\sum_{i \text{ even}} a_i x^{\frac{p-i-1}{2}}]^2 \text{ if p is odd} \end{cases}$$

Now, suppose p is even and let $g(x) = \sum_{i \text{ even}} a_i x^{\frac{p-i}{2}}$.

Clearly, $g(A^2) = 0$. If $h(x)$ is a polynomial of degree $< \frac{p}{2}$ and $h(A^2) = 0$, then $h^2(x)$ is such that $h^2(A) = 0$ and $\deg h^2(x) < p$, contradicting the minimality of p. The argument is exactly analogous for p odd.

From Lemma 3a and the foregoing discussion, we obtain the following.

Theorem 3. Let G be a p-node graph with adjacency matrix $A = A(G)$. If A is non-derogatory over GF(2), then every automorphism $P \in \Gamma(G)$ can be expressed in the form

$$P = \mu_{A^2}(A) \sum_{i=0}^{p-m-1} b_i A^i + I \qquad (9)$$

for some choice of $b_i \in GF(2)$, where $m = \deg \mu_{A^2}(x) = \{ \frac{p}{2} \}$. Thus, $\Gamma(G)$ can be constructed in at most $2^{[\frac{p}{2}]}$ steps.

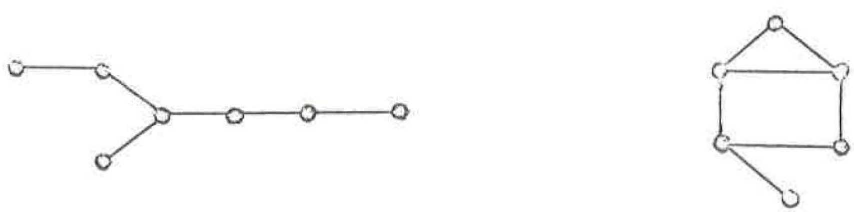
The construction may be facilitated by taking advantage of some additional information. Multiplying both sides of equation (9) on the right by $\mu_{A^2}(A)$ gives

$$P\mu_{A^2}(A) = \mu_{A^2}(A) \qquad (10)$$

So if P is an automorphism of G, it can only interchange nodes of G corresponding to identical rows of $\mu_{A^2}(A)$. Moreover, if u and v are similar nodes, then the rows of $\mu_{A^2}(A)$ corresponding to u and v must constitute a minimal pair of identical rows. This follows from the fact that if $\mu_{A^2}(A)$ has more than two rows identical to the same one, the matrix $\mu_{A^2}(A) \sum_{i=0}^{p-m-1} b_i A^i$ will have a principal submatrix of order > 2 consisting of all ones.

Corollary 3a. Under the hypotheses of Theorem 3, a necessary condition for two nodes of a graph G to be similar is that the corresponding rows of $\mu_{A^2}(A)$ constitute a minimal pair of identical rows. Hence, if the rows of $\mu_{A^2}(A)$ are pairwise distinct, $\Gamma(G)$ is trivial.

Figure 2 exhibits two identity graphs which respectively satisfy and fail to satisfy the condition of Corollary 3a.



Rows of $\mu_{A^2}(A)$ are distinct

$\mu_{A^2}(A)$ has three minimal pairs of identical rows

Figure 2. Identity Graphs

From Theorem 2, it follows that if the characteristic polynomial of the adjacency matrix of a graph is irreducible over GF(2), then its group is trivial. Of course, the polynomial might be irreducible over the integers and not over GF(2) -- for example the digraph of Figure 1 is irreducible over GF(2) but the graph is not (indeed the polynomial of a graph is always reducible over GF(2)). For a digraph with non-derogatory adjacency matrix, over GF(2), it remains to establish a general relationship between the factors of its characteristic polynomial and the cyclic subgroups of its automorphism group.

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