The Characteristic Polynomial of a Graph*

by

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1. Introduction

The search for isomorphism invariants has led to consideration of various algebraic properties of the adjacency matrix of a graph. In particular, interest has focused on the characteristic polynomial of the adjacency matrix. Of course, the characteristic polynomial does not always distinguish between non-isomorphic graphs. Many examples are known [1], [6]. Of particular interest is the fact that there exist non-isomorphic connected regular graphs with the same polynomial [7], [9]. Let G_1 and G_2 be two such graphs. Consider the graphs

$$H_{i,k-1} = i\, G_1 \cup (k-1-i)\, G_2 \qquad \text{for } 0 \leq i \leq k-1,$$

i.e., $H_{i,k-1}$ is the union of i copies of G_1 and k-1-i copies of G_2. Since G_1 and G_2 are regular, $H_{i,k-1}$ is regular and the complement $\overline{H}_{i,k-1}$ of $H_{i,k-1}$ is regular and connected. Clearly, all the $H_{i,k-1}$ have the same polynomial since their respective adjacency matrices are direct sums of matrices corresponding to G_1 and G_2. Moreover, it is easy to show [2], that if two regular graphs have the same polynomial, then their complements also have the same polynomial. Hence, the $\overline{H}_{i,k-1}$ $(0 \leq i \leq k-1)$ are non-isomorphic connected regular graphs and have the same characteristic polynomial, from which we conclude the following. Given any positive integer k, there exists an integer n such that there are at least k non-isomorphic connected regular graphs with n points all having the same characteristic polynomial.[1]

The present paper is addressed to the problem of determining under what conditions the characteristic polynomial does distinguish between non-isomorphic graphs. In what follows, we will characterize the coefficients of the characteristic polynomial of an arbitrary digraph, and examine the polynomial of a tree in detail.

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[1] The foregoing demonstration is due to A. J. Hoffman (personal communication).

A digraph (or directed graph) D is an irreflexive binary relation on a finite set V = V(D) of elements called the points (or vertices) of D; the collection E = E(D) of ordered pairs of points constitute the lines (or edges) of D. We will write uv for the ordered pair (u,v). By the order of a digraph D, we shall mean the cardinality of V(D). A graph is a symmetric digraph. The adjacency matrix A = A(D) of a digraph D with n points $v_1, v_2, \ldots, v_n$ is defined by its i,j-th entry a_{ij} as follows:

$$a_{ij} = \begin{cases} 1 \text{ if } v_i v_j \text{ is a line of } D \\ 0 \text{ otherwise} \end{cases}$$

for $1 \leq i, j \leq n$. Two digraphs whose adjacency matrices have the same characteristic polynomial will be called cospectral. For graph theoretic terms used without explicit definitions, see [5].

2. Determination of Coefficients

Collatz and Sinogowitz [1] investigated the relationship between the coefficients of the characteristic polynomial of the adjacency matrix of a graph and certain subgraphs. However, no general formula for the coefficients was derived. In this section we will generalize their results and derive such a formula.

Let D be a digraph with n points, and A = A(D) its adjacency matrix. The characteristic polynomial of A is given by $\phi(\lambda) = \det(A-\lambda I)$ which can be expressed

$$\phi(\lambda) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-i} \qquad (1)$$

It is well known [3] that the k-th coefficient $a_k (1 \leq k \leq n)$ is equal to the sum of all principal minors of order k. Since each k order principal submatrix of A is the adjacency matrix of a subdigraph of D containing k points, it is clear that any principal minor of A is the determinant of the adjacency matrix of a subdigraph of D. Thus the coefficients of the characteristic polynomial of A can be expressed in terms of determinants of matrices belonging to subdigraphs of D.

For an arbitrary digraph H of order k, let $f_H(\{i_1, i_2, \ldots, i_r\})$ denote the number of collections of disjoint directed cycles in H of lengths $i_1, i_2, \ldots, i_r$ where $i_j \geq 1$ $(1 \leq j \leq r)$ and $i_1 + i_2 + \ldots + i_r = k$. Using the formula for the determinant of the adjacency matrix of a digraph [5, p. 151], we obtain the following.

Theorem 1. Let D be a digraph of order n. Then for $1 \leq k \leq n$, the k-th coefficient a_k of the characteristic polynomial of A(D) is given by

$$a_k = \sum \left[\prod_{j=1}^{r} (-1)^{i_j+1} \right] \quad f_D(\{f_1, i_2, \ldots, i_r\}) \qquad (2)$$

where the summation is taken over all rank r partitions $\{i_1, i_2, \ldots, i_2\}$ $(1 \leq r \leq k)$ of k; and $a_0 = 1$.

In an undirected graph (symmetric digraph) G each undirected cycle of length greater than 2 contributes two directed cycles of the same length. Of course, an undirected line contributes exactly one directed cycle of length 2, and a loop contributes one directed cycle of length 1. So, if for a given partition $\{i_1, i_2, \ldots, i_r\}$ of k we let

$$g(i_j) = \begin{cases} 1 \text{ if } 1 \leq i_j \leq 2 \\ 2 \text{ if } i_j > 2 \end{cases}$$

and define $\overline{f}_G$ $(\{i_1, \ldots, i_r\})$ as above but for undirected cycles (and lines), (2) becomes

Theorem 2. Let G be a graph of order n. Then for $1 \leq k \leq n$ the k-th coefficient a_k of the characteristic polynomial of A(G) is given by

$$a_k = \sum \left[\prod_{j=1}^{r} (-1)^{i_j+1} g(i_j) \right] \quad \overline{f}_G(\{i_1, \ldots, i_r\}) \qquad (3)$$

where the summation extends over all rank r partitions $\{i_1, i_2, \ldots, i_r\}$ $(1 \leq r \leq k)$ of k; and $a_0 = 1$.

To fix ideas, let us evaluate the coefficients for the graph shown in Figure 1.

[Fig. 1 about here]

From (3) we have

$$a_0 = 1 \qquad a_4 = \overline{f}(\{2,2\}) - 2\overline{f}(\{4\})$$

$$a_1 = \overline{f}(\{1\}) \qquad a_5 = -2\overline{f}(\{2,3\}) + 2\overline{f}(\{5\})$$

$$a_2 = -\overline{f}(\{2\}) \qquad a_6 = -\overline{f}(\{2,2,2\}) + 2\overline{f}(\{2,4\}) + 4\overline{f}(\{3,3\}) - 2\overline{f}(\{6\})$$

$$a_3 = 2\overline{f}(\{3\}) \qquad a_7 = -2\overline{f}(\{2,5\}) + 2\overline{f}(\{2,2,3\}) - 4\overline{f}(\{3,4\}) + 2\overline{f}(\{7\})$$

Counting the relevant subgraphs, we obtain:

$\overline{f}(\{2\}) = 9$	$\overline{f}(\{5\}) = 1$	$\overline{f}(\{2,5\}) = 0$
$\overline{f}(\{3\}) = 3$	$\overline{f}(\{2,2,2\}) = 8$	$\overline{f}(\{2,2,3\}) = 1$
$\overline{f}(\{2,2\}) = 17$	$\overline{f}(\{2,4\}) = 2$	$\overline{f}(\{3,4\}) = 0$
$\overline{f}(\{4\}) = 2$	$\overline{f}(\{3,3\}) = 0$	$\overline{f}(\{7\}) = 0$
$\overline{f}(\{2,3\}) = 5$	$\overline{f}(\{6\}) = 0$	

from which one immediately computes the polynomial given in Figure 1.

From Theorem 1, it is obvious that a digraph D of order n with no cycles and no symmetric lines has characteristic polynomial $\phi_D(\lambda) = \lambda^n$. This fact gives rise to the following

Theorem 3. For any positive integer k there exists an integer n such that there are at least k non-isomorphic weakly connected digraphs with the same characteristic polynomial.

Proof. Let $n = 2k'+1$ where $k' > k$. Consider the collection of digraphs $D_0, D_1, \ldots, D_{k'}$ constructed as follows. D_0 is the directed path of length

$2k'$, i.e., $V(D_0) = \{v_1, v_2, \ldots, v_{2k'+1}\}$, $E(D_0) = \{e_1, e_2, \ldots, e_{2k'}\}$
where $e_i = v_i v_{i+1}$, $1 \leq i \leq 2k'$.
For $1 \leq j \leq k'$ let

$$V(D_j) = V(D_0)$$

and

$$E(D_j) = \{e_1^{(j)}, e_2^{(j)}, \ldots, e_{2k'}^{(j)}\}$$

where

$$e_i^{(j)} = \begin{cases} v_{i+1}\, v_i \text{ for } 1 \leq i \leq j \\ e_i \text{ for } j+1 \leq i \leq 2k' \end{cases}$$

Clearly, the $k'+1$ digraphs $D_0, D_1, \ldots, D_{k'}$ are pairwise non-isomorphic and weakly connected. Moreover, they are acyclic and have no symmetric lines, so that they all have the same characteristic polynomial ($\phi(\lambda) = \lambda^n$), which concludes the proof.

3. Non-Isomorphic Cospectral Trees[2]

Consider an arbitrary tree T. Since the only cycles in T are directed cycles of length 2 corresponding to the lines of T, the summation in (3) need only take into account partitions of the form $\{2^r\} = \{2,2,\ldots,2\}$. Now, writing $h_r(T)$ for $\bar{f}_T(\{2^r\})$ we have the following immediate consequence of Theorem 2.

Corrollary 2.1. Let T be a tree of order n. Then for $1 \leq k \leq n$ the k-th coefficient a_k of $\phi_T(\lambda)$ is given by

$$a_k = \begin{cases} (-1)^r\, h_r(T) \text{ if } k = 2r \text{ for some } r \geq 1 \\ 0 \text{ otherwise} \end{cases} \tag{4}$$

[2]A list of n-point trees, $2 \leq n \leq 10$, together with the coefficients of their characteristic polynomials is given in the Appendix to this paper.

and $a_0 = 1$.

It is evident from (4) that $|a_k|$ is the number of sets consisting of k pairwise non-incident lines of T, which is precisely the number of independent sets of lines of order k in T. Making this observation from another point of view, one sees that $|a_k|$ is the number of matchings of order k in T.

It is of interest to record the foregoing remarks.

Theorem 4.

Let $\phi_T(\lambda) = \sum_{k=0}^{n} (-1)^k a_k \lambda^{n-k}$ be the characteristic polynomial of a tree T with n points. Let $m = \max_{0 \le k \le n} \{k \mid a_k \neq 0\}$. Then

(i) $|a_k|$ = the number of matchings of order k in T

(ii) Any maximal matching in T is of order m, and, thus, the number of such maximal matchings is $|a_m|$.

Counting independent sets of lines in a tree has a useful dual formulation. For a given tree T consider its line graph L(T). L(T) is defined as follows. The points of L(T) correspond to the lines of T; and two points of L(T) are adjacent if and only if the corresponding lines of T are incident. Thus, an independent set of lines of order k in T corresponds to an independent set of points of order k in L(T).

It is known [5] that a graph is the line graph of a tree if and only if it is a connected block graph in which each cut point is on exactly two blocks, and each block is a complete graph. The duality between independent sets of lines and points given by a tree and its line graph affords some leverage in the construction of cospectral trees.

Theorem 5. Let T_1 and T_2 be trees. If T_1 and T_2 are cospectral, then $L(T_1)$ and $L(T_2)$ have the same number of points and lines.

Proof. Since T_1 and T_2 are cospectral, $h_k(T_1) = h_k(T_2)$ for all k. In particular, this holds for k=1, so that $L(T_1)$ and $L(T_2)$ have the same number of points, say n.

$h_2(T_1)$ $[=h_2(T_2)]$ is the number of pairs of non-adjacent points in $L(T_1)$ $[L(T_2)]$. So, the number of lines in $L(T_1)$ is $\binom{n}{2} - h_2(T_1)$ which is equal to $\binom{n}{2} - h_2(T_2)$.

We turn now to the problem of computing the coefficients of the characteristic polynomial of a tree. If T is a tree and v is a point of T, we denote by T-v the tree obtained from T by removing v together with all lines incident to v. If u is a point not in T, we form the tree T+uv by joining the point u to v.

The following is a special case of Theorem 2 of [6].

Lemma 1. Let T be a tree and v a point of T. Then

$$h_k(T+uv) = h_k(T) + h_{k-1}(T-v).$$

Proof. The tree T+uv consists of the lines of T and the additional line uv. So, there are two ways to construct a matching of order k depending on whether or not the line uv is included. In the former case, we need to find a matching of order k-1 in T-v since we cannot choose a line incident to uv. This may be done in $h_{k-1}(T-v)$ ways. In the latter case,all the lines of T are available for choosing a matching of order k. There are $h_k(T)$ ways to do this.

As a simple application of Lemma 1, consider the path P_n on n points. The following is also derived in [6] but in a different form, looking at the polynomial as a function of λ rather than at the coefficients.

Theorem 6. Let P_n be a path on n points.

Then

(i) $h_k(P_{n+1})$ satisfies the recurrence

$$h_k(P_{n+1}) = h_k(P_n) + h_k(P_{n-1})$$

(ii) $h_k(P_{n+1}) = \binom{n-k+1}{k}$.

Proof. Part (i) is an immediate consequence of Lemma 1. We prove part (ii) by induction on k. Clearly, $h_1(P_{n+1}) = \binom{n}{1}$ for any $n > 1$.

So, assume

$$h_k(P_{n+1}) = \binom{n-k+1}{k}.$$

Then

$$h_{k+1}(P_{n+1}) = \sum_{r=2}^{n-2k+1} h_k(P_{n+1-r}) = \sum_{r=2}^{n-2k+1} \binom{n-r+1}{r}$$

$$= \sum_{r=0}^{n-1-2k} \binom{k+r}{k} = \binom{n-(k+1)+1}{k+1},$$

as required.

A more interesting class of trees whose coefficients can be determined rather easily consists of trees homeomorphic to a star. Such a tree is one with a single point of degree > 2, every other point being of degree 1 or 2.

Suppose S is a tree homeomorphic to a star. Let S be of order n+1, and let v be the point in S with degree > 2. Furthermore, let d_i be the number of points in S whose distance from v is $i \geq 1$, and m the maximal distance between v and any other point in S. The tree S is completely characterized by the point v and the parameters $d_1, d_2, \ldots, d_m$, so we shall write $S = S_v(d_1, d_2, \ldots, d_m)$.

Theorem 7. Let $S = S_v(d_1, d_2, \ldots, d_m)$ be a tree homeomorphic to a star. Then $h_k(S)$ satisfies the recurrence

$$h_k(S_v(d_1, \ldots, d_m)) = \sum_{i=2}^{m} \sum_{r=1}^{di} h_{k-1}(S_v(d_1, \ldots, d_{i-1}-1, d_i-r))$$

Proof. Suppose u_1 is a point of S such that the distance $d(u_1, v)$ between u_1 and v is m. Obviously, u_1 is an endpoint of S. Let u_1' be the point adjacent to u_1. The number of matchings of order k containing the line u_1u_1' is clearly

$$h_{k-1}(S_v(d_1,d_2, \ldots, d_{m-1}-1, d_m-1)).$$

Having used u_1u_1', delete it from the tree and choose another point u_2 such that $d(v,u_2) = m$. Let u_2' be the point adjacent to u_2. Now, the number of matchings of order k including u_2u_2' is

$$h_{k-1}(S_v(d_1, d_2, \ldots, d_{m-1}-1, d_m-2)).$$

Continuing in this way until all the d_m points at distance m from v are exhausted, we find the number of matchings of order k contributed by the lines incident to those points to be

$$\sum_{r=1}^{d_m} h_{k-1}(S_v(d_1, d_2, \ldots, d_{m-1}-1, d_m-r)).$$

Repeating this process successively for points at distance m-j from v for $1 \leq j \leq m-2$, we obtain the desired recurrence.

Now we will find the solution of the recurrence given in Theorem 7' using a direct combinatorial argument.

Theorem 8. Let $S = S_v(d_1, \ldots, d_m)$ be as above.

Then

$$h_k(S_v(d_1, \ldots, d_m)) = \sum \binom{d_m}{i_m}\binom{d_{m-1}-i_m}{i_{m-1}}\cdots\binom{d_2-i_3}{i_2}\binom{d_1-i_2}{1}$$

$$+ \sum \binom{d_m}{i_m}\binom{d_{m-1}-i_m}{i_{m-1}}\cdots\binom{d_2-i_3}{i_2} \qquad (5)$$

where the summation for both terms extends over all ordered sequences $(i_2, i_3, \ldots, i_m)$ of non-negative integers satisfying $i_2+i_3+\ldots+i_m = k-1$ and $i_2+i_3+\ldots+i_m = k$ respectively.

Proof. There are two cases to consider depending on whether a line incident to v is chosen or not.

Case 1. A line incident to v is chosen. Then k-1 additional lines must be selected. The selection may be made by choosing i_m from those d_m lines at distance m from v, and i_j from d_j-i_{j+1} $(2 \leq j \leq m-1)$ of the d_j lines at distance j from v, where $i_2+i_3+\ldots+i_m = k-1$. i_{j+1} lines must be excluded from the d_j since i_{j+1} lines were chosen from those at distance j+1 from v. Clearly, the number of ways of making these selections is precisely the first term of (5)

Case 2. No line incident to v is chosen. In this case, the selection is as in case 1 except that k lines must be chosen and d_1 lines are excluded. Again the number of possible selections under the given constraints is exhibited in the second term of (5)

Expressions for the coefficients of other classes of trees may be obtained by examining special types of line graphs. Before considering a case in point we will give a recurrence in terms of the line graph of a tree which parallels the one given in Lemma 1. In what follows, we will write $h_k(L)$ for $h_k(T)$ when $L = L(T)$ for a tree T.

Let T be a tree and $L = L(T)$ its line graph. An endblock K_t (complete of order t) of L is a block which is joined to exactly one other block of L. Let v be the point in common between K_t and the block to which it is joined. Then the graph $L-K_t-v$ is obtained from L by removing only K_t.

Lemma 2. Let $L = L(T)$ be the line graph of a tree T, and let K_t and v be as above. Then $h_k(L)$ satisfies the recurrence

$$h_k(L) = (t-1)\, h_{k-1}(L-K_t-v) + h_k(L-K_t)$$

Proof. There are two cases to consider, depending on whether or not a point of K_t (other than v) is contained in an independent set of order k. Clearly, the first term of the recurrence arises from the former, and the second term from the latter.

Now we consider a line graph $L = L(t_1, \ldots, t_m)$ of the following form. L consists of a block K_m together with m endblocks K_{t_j} $(1 \leq j \leq m)$ joined to K_m.

Call such a graph a line-star.

Theorem 9. Let $L = L(t_1, \ldots, t_m)$ be a line-star as above. Then

(i) $$h_k(L) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le m} (t_{i_1}-1)(t_{i_2}-1)\ldots(t_{i_k}-1)$$

$$+ \sum_{1 \le i_1 < i_2 < \ldots < i_{k-1} \le m} (t_{i_1}-1)(t_{i_2}-1)\ldots(t_{i_{k-1}}-1)(m-k+1)$$

(ii) If $t_1 = t_2 = \ldots = t_m = t$, then

$$h_k(L) = \binom{m}{k}(t-1)^{k-1}(t+k-1).$$

Proof. The two summations in (i) are obtained as follows. Consider an independent set of order k. Either all k points are chosen from the endblocks (first term), or k-1 points are chosen from the endblocks and one point is chosen from k_m (second term).

Part (ii) follows from (i) by substituting t for each $t_j (1 \le j \le m)$.

Corollary 9.1. Suppose $L_1 = (t_1, \ldots, t_m)$ and $L_2 = L(s_1, \ldots, s_n)$ are line-stars corresponding to trees T_1 and T_2, respectively, with $t_j > 1$, $s_j > 1$ and $m \neq n$. Then T_1 and T_2 are not cospectral.

Proof. Taking $n > m$, the result follows immediately from the observation that $h_n(L_1) = 0$, but $h_n(L_2) > 0$.

When $t_1 = t_2 = \ldots = t_m = t$, we will write $L(mt)$ for $L(t_1, t_2, \ldots, t_m)$. As an application of Lemma 2, let us evaluate $h_k(L')$ where L' is the line graph obtained from $L(mt)(t>1)$ by joining an endblock K_s to point v of some K_t. According to the Lemma

$$h_k(L') = (s-1)\, h_{k-1}[L'-K_s-v] + h_k[L'-K_s]$$

Clearly, $L'-K_s-v$ is just $L((t-1), (m-1)t)$, and $L'-K_s$ is $L(mt)$. Hence,

$$h_k(L') = (s-1)\left\{ \binom{m-1}{k}(t-1)^k + \binom{m-1}{k-1}(t-1)^{k-1}(t-2) \right.$$

$$\left. + \binom{m-1}{k-1}(t-1)^{k-1}(m-k+1) + \binom{m-1}{k-2}(t-1)^{k-2}(t-2)(m-k+1) \right\}$$

$$+ \binom{m}{k}(t-1)^{k-1}(t+k-1).$$

We conclude with the following result.

Theorem 10. There exist infinitely many pairs of non-isomorphic cospectral trees.

Proof. Consider the pair of trees T_1 and T_2 shown in Figure 2.

[Fig. 2 about here]

Let u,v, x,y be as in the Figure, and let n be the number of lines in T_1 (and T_2). It is clear that $h_1(T_1) = h_2(T_2) = n$, and $h_k(T_1) = h_k(T_2) = 0$, for $k > 2$. Now $h_2(T_1) = uv$, and $h_2(T_2) = x(y+1) + y$. Equating $h_2(T_1)$ and $h_2(T_1)$, and using the relations $u+v = n-1$, $x+y = n-2$, we obtain

$$y^2 - v^2 + (n-1)v - (n-2)y - (n-2) = 0 \qquad (6)$$

Taking $y = v+1$ in (11) gives

$$v = \frac{3n-5}{3} \qquad (7)$$

So, to obtain a pair of cospectral trees of the desired form, we need only find a value of $n(\geq 7)$ which makes v an integer. Clearly, $n = 7+3k$, $k = 0,1,2,\ldots$ are permissible values, which concludes the proof.

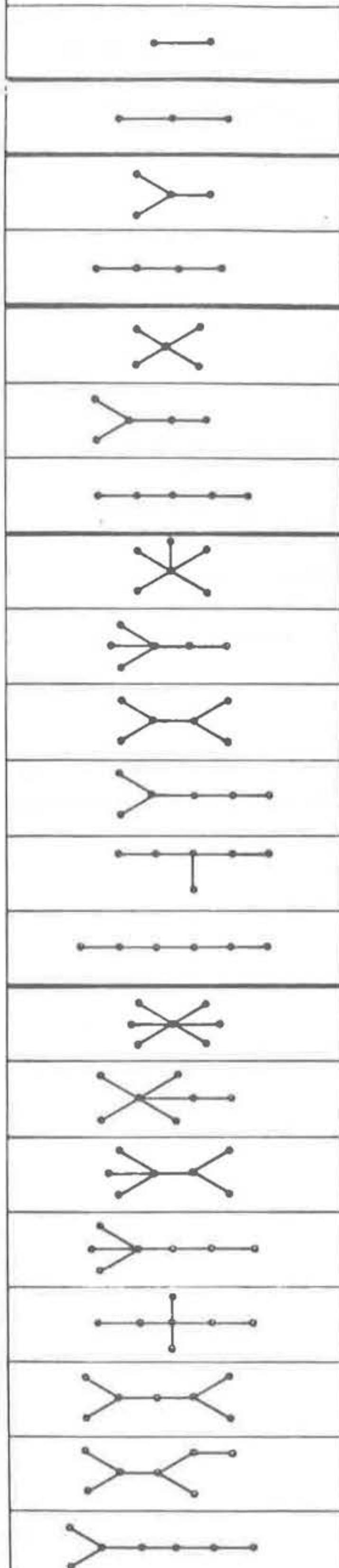
Appendix

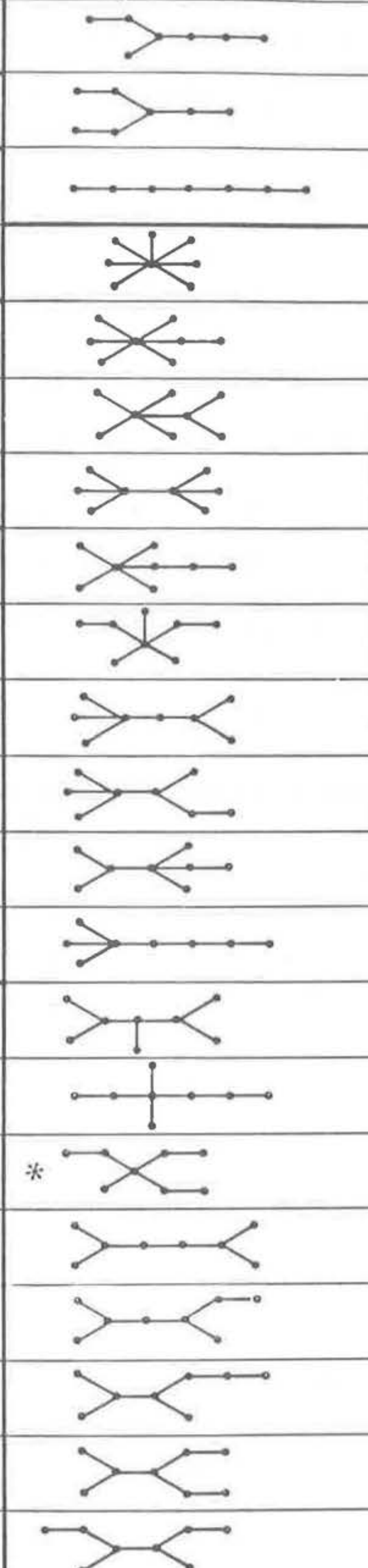
The following is a list of n-point trees, $2 \leq n \leq 10$, together with the coefficients of their characteristic polynomials. The polynomial of a tree T is given by

$$\phi_T(\lambda) = \sum_{i=0}^{n} (-1)^i \, a_i \, \lambda^{n-i}$$

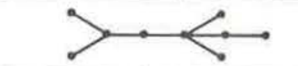
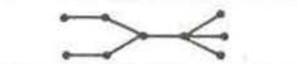
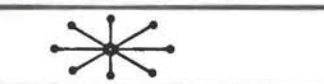
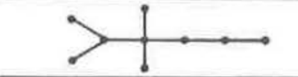
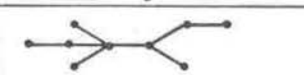
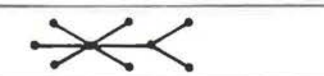
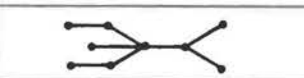
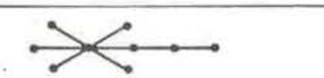
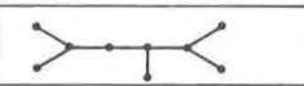
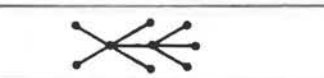
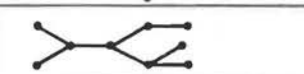
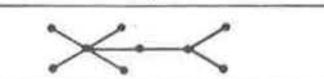
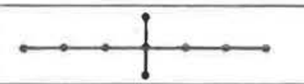
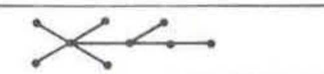
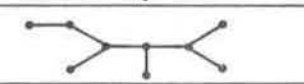
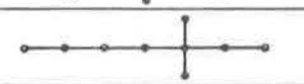
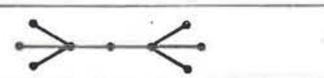
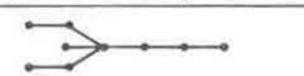
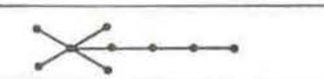
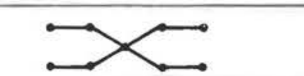
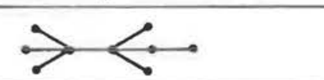
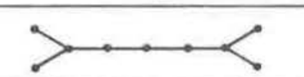
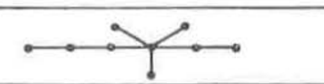
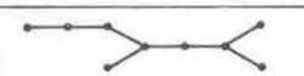
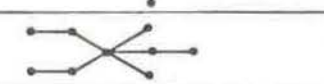
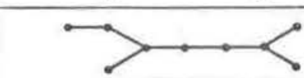
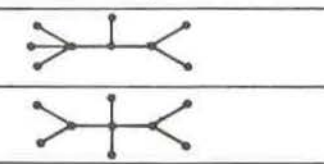
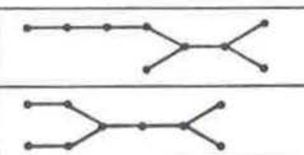
Note that for all trees, $a_0 = 1$ and $a_i = 0$ for odd values of i.

The present list is an expanded and corrected version of an earlier one in Collatz and Sinogowitz [1]; trees preceded by an asterisk are those whose polynomials were given incorrectly in that paper. For a complete catalogue of the characteristic polynomials of graphs on 7 points, see King [8].





TREE	COEFFICIENTS a_2	a_4	a_6	a_8	a_{10}
	-1				
	-2				
	-3	0			
	-3	1			
	-4	0			
	-4	2			
	-4	3			
	-5	0	0		
	-5	3	0		
	-5	4	0		
	-5	5	0		
	-5	5	-1		
	-5	6	-1		
	-6	0	0		
	-6	4	0		
	-6	6	0		
	-6	7	0		
	-6	7	-2		
	-6	8	0		
	-6	8	-2		
	-6	9	-2		
	-6	9	-3		
	-6	9	-4		
	-6	10	-4		
	-7	0	0	0	
	-7	5	0	0	
	-7	8	0	0	
	-7	9	0	0	
	-7	9	0	0	
	-7	9	-3	0	
	-7	11	0	0	
	-7	11	-3	0	
	-7	11	-4	0	
	-7	12	-3	0	
	-7	12	-4	0	
	-7	12	-5	0	
*	-7	12	-7	1	
	-7	13	-4	0	
	-7	13	-5	0	
	-7	13	-6	0	
	-7	13	-7	0	
	-7	13	-7	1	

TREE	COEFFICIENTS a2 a4 a6 a8 a10	TREE	COEFFICIENTS a2 a4 a6 a8 a10
	-7 14 -7 0		-8 17 -6 0
	-7 14 -8 0		-8 17 -7 0
	-7 14 -8 1		-8 17 -8 0
* 	-7 14 -9 1		-8 17 -9 0
	-7 15 -10 1		-8 17 -10 0
	-8 0 0 0		-8 17 -10 0
	-8 6 0 0		-8 17 -11 2
	-8 10 0 0		-8 17 -12 2
	-8 11 0 0		-8 18 -10 0
	-8 11 -4 0		-8 18 -10 0
	-8 12 0 0		-8 18 -12 0
	-8 14 0 0		-8 18 -12 0
	-8 14 -4 0		-8 18 -12 2
	-8 14 -6 0		-8 18 -12 2
	-8 15 0 0		-8 18 -14 3
	-8 15 -4 0		-8 18 -16 5
	-8 15 -6 0		-8 19 -12 0
	-8 15 -7 0		-8 19 -13 0
	-8 15 -10 2		-8 19 -13 2
	-8 16 -6 0		-8 19 -14 2
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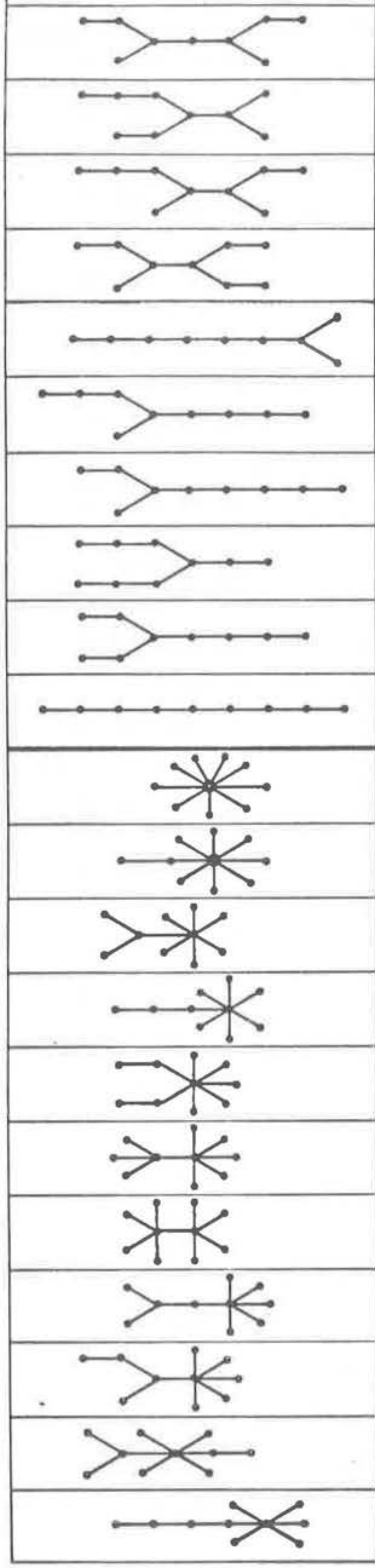

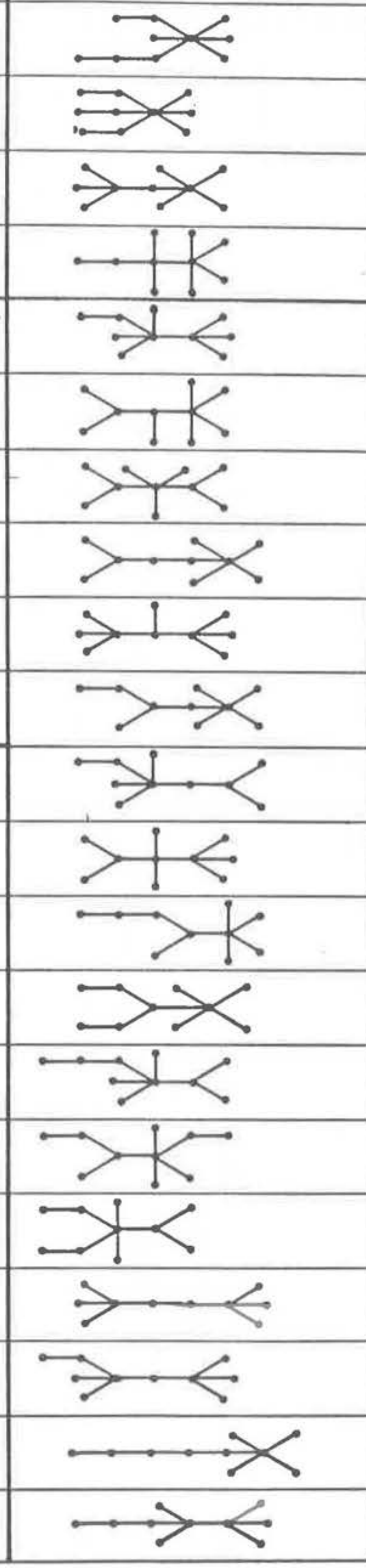

TREE	COEFFICIENTS a2	a4	a6	a8	a10
	-8	19	-14	3	
	-8	19	-15	2	
	-8	19	-15	3	
	-8	19	-16	4	
	-8	20	-16	2	
	-8	20	-17	3	
	-8	20	-17	4	
	-8	20	-18	4	
	-8	20	-18	5	
	-8	21	-20	5	
	-9	0	0	0	0
	-9	7	0	0	0
	-9	12	0	0	0
	-9	13	0	0	0
	-9	13	-5	0	0
	-9	15	0	0	0
	-9	16	0	0	0
	-9	17	0	0	0
	-9	17	-5	0	0
	-9	17	-8	0	0
	-9	18	-5	0	0

TREE	COEFFICIENTS a2	a4	a6	a8	a10
	-9	18	-9	0	0
	-9	18	-13	3	0
	-9	19	0	0	0
	-9	19	-8	0	0
	-9	19	-9	0	0
	-9	20	-8	0	0
	-9	20	-12	0	0
	-9	21	-8	0	0
	-9	21	-9	0	0
	-9	21	-9	0	0
	-9	21	-11	0	0
	-9	21	-12	0	0
	-9	21	-12	0	0
	-9	21	-13	0	0
	-9	21	-14	0	0
	-9	21	-15	3	0
	-9	21	-17	4	0
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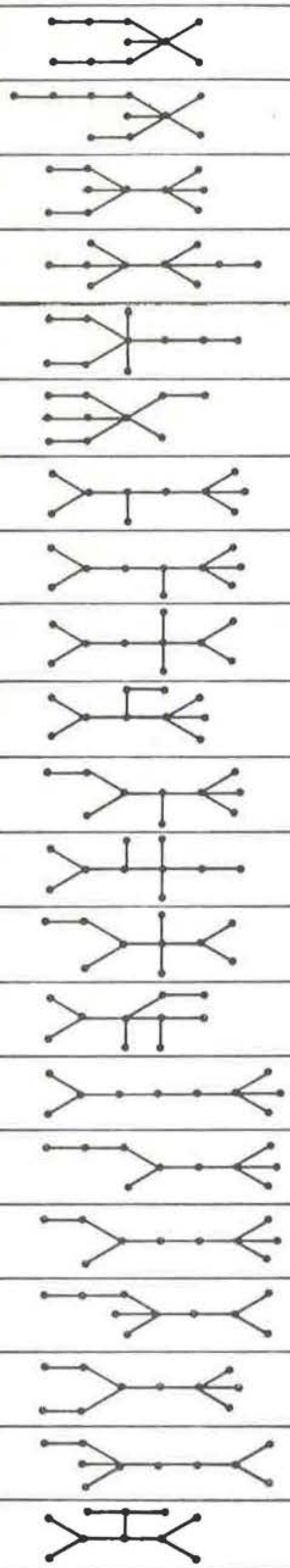
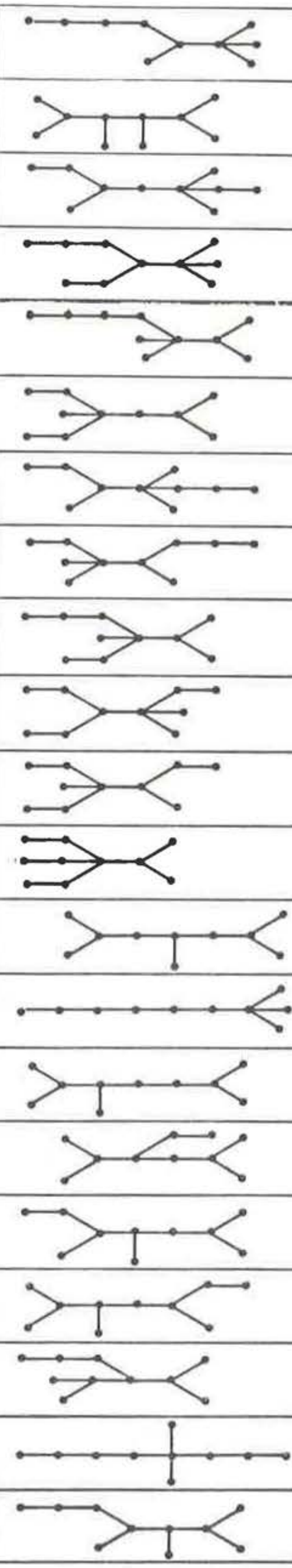
	-9	22	-15	0	0





TREE	COEFFICIENTS					TREE	COEFFICIENTS				
	a_2	a_4	a_6	a_8	a_{10}		a_2	a_4	a_6	a_8	a_{10}
	-9	22	-16	0	0		-9	24	-20	3	0
	-9	22	-16	3	0		-9	24	-20	4	0
	-9	22	-17	3	0		-9	24	-20	5	0
	-9	22	-17	4	0		-9	24	-21	3	0
	-9	22	-19	5	0		-9	24	-21	4	0
	-9	22	-22	9	-1		-9	24	-21	5	0
	-9	23	-14	0	0		-9	24	-22	5	0
	-9	23	-15	0	0		-9	24	-22	6	0
	-9	23	-16	0	0		-9	24	-23	6	0
	-9	23	-17	0	0		-9	24	-23	7	0
	-9	23	-17	3	0		-9	24	-24	9	-1
	-9	23	-18	4	0		-9	24	-25	9	0
	-9	23	-19	4	0		-9	25	-21	0	0
	-9	23	-20	4	0		-9	25	-22	3	0
	-9	24	-17	0	0		-9	25	-22	4	0
	-9	24	-18	0	0		-9	25	-23	4	0
	-9	24	-18	3	0		-9	25	-23	5	0
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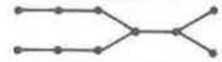
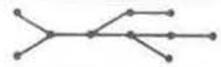
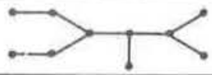
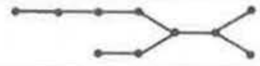
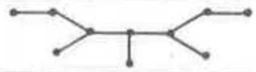
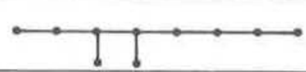
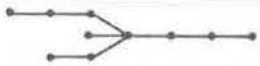
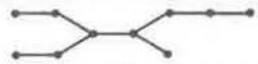
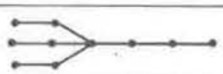
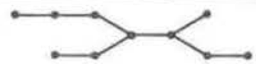
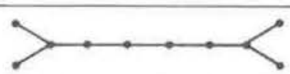
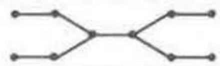
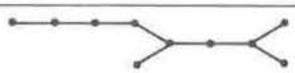
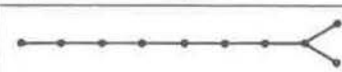
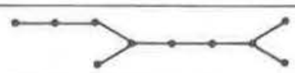
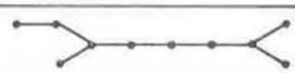
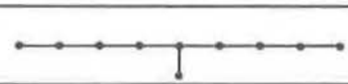
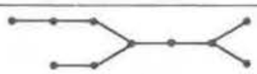
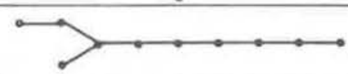
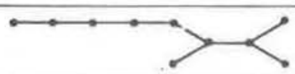
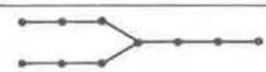
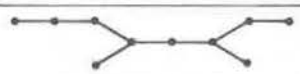
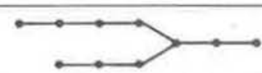
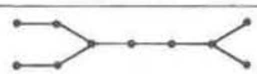
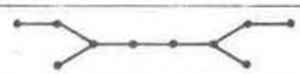
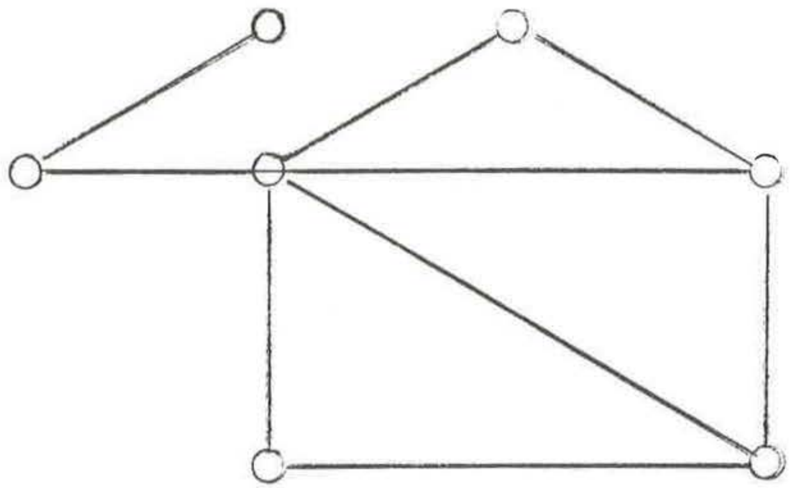
	-9	24	-19	3	0		-9	25	-24	4	0
	-9	24	-19	4	0		-9	25	-24	5	0
	-9	24	-20	0	0		-9	25	-24	6	0

TREE	COEFFICIENTS a_2	a_4	a_6	a_8	a_{10}	TREE	COEFFICIENTS a_2	a_4	a_6	a_8	a_{10}
	-9	25	-24	7	0		-9	26	-28	8	0
	-9	25	-25	7	0		-9	26	-28	9	0
	-9	25	-25	8	0		-9	26	-28	9	0
	-9	25	-25	9	-1		-9	26	-28	10	-1
	-9	25	-26	8	0		-9	26	-28	11	-1
	-9	25	-26	10	-1		-9	26	-29	11	0
	-9	25	-28	12	-1		-9	26	-29	11	-1
	-9	26	-25	4	0		-9	26	-30	13	-1
	-9	26	-26	5	0		-9	27	-30	9	0
	-9	26	-26	6	0		-9	27	-31	11	0
	-9	26	-26	7	0		-9	27	-31	11	-1
	-9	26	-27	7	0		-9	27	-31	12	-1
	-9	26	-27	8	0		-9	27	-32	12	0
	-9	26	-27	8	0		-9	27	-32	13	-1
	-9	26	-27	9	0		-9	27	-32	14	-1
	-9	26	-27	10	-1		-9	28	-35	15	-1



G

$$\phi_G(\lambda) = \lambda^7 - 9\lambda^5 - 6\lambda^4 + 13\lambda^3 + 8\lambda^2 - 4\lambda - 2$$

Figure 1. The Polynomial of a Seven Point Graph

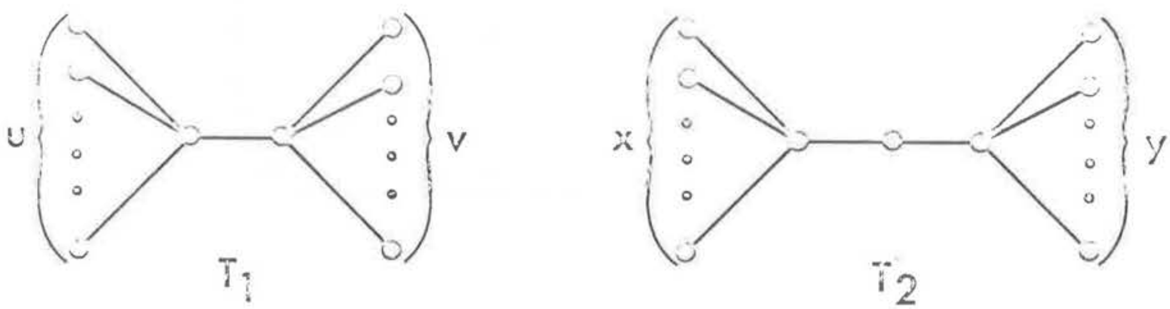


Figure 2. Pairs of Cospectral Trees

References

1. L. Collatz and U. Sinogowitz, Spektren endlicher Graphen. Abh. Math. Sem. Univ. Hamburg 21 (1957), 63-77.

2. H. J. Finck and G. Grohmann, Vollständiges Produkt, chromatische Zahl und charakteristisches Polynom regulärer Graphen (I). Wissen. Z. Tech. Hochscule Ilmenau 11 (1965), 1-3.

3. F. R. Gantmacher, The Theory of Matrices, Volume I (English Translation). Chelsea, New York, 1959.

4. F. Harary, The determinant of the adjacency matrix of a graph. SIAM Review 4 (1962), 202-210.

5. F. Harary, Graph Theory. Addison-Wesley, Reading, Mass., 1969.

6. F. Harary, C. King, A. Mowshowitz and R. C. Read, Cospectral Graphs and digraphs. Bull. London Math. Soc., forthcoming.

7. A. J. Hoffman and D. K. Ray-Chaudhuri, On the line graph of a symmetric balanced incomplete block design. Trans. Amer. Math. Soc. 116 (1965), 238-252.

8. C. King, Characteristic polynomials of 7-node graphs, Scientific Report, University of the West Indies/CC6(AFOSR project 1026-66).

9. S. S. Shrikhande, The uniqueness of the L_2 association scheme, Ann. Math. Statist. 30 (1959), 781-789.