# Performance Measures for Constrained Systems *

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**Abstract**

We present a geometric theory of the performance of robot manipulators, applicable to systems with constraints, which may be non-holonomic. The performance is quantified by a geometrical object, the induced metric tensor, from which scalars may be constructed by invariant tensor operations to give performance measures. The measures thus defined depend on the metric structure of configuration and work space, which should be chosen appropriately for the problem at hand. The generality of this approach allows us to specify a system of joint connected rigid bodies with a large class of metrics. We describe how the induced metric can be computed for such a system of joint connected rigid bodies and describe a MATLAB program that allows the automatic computation of the performance measures for such systems. We illustrate these ideas with some computations of measures for the SARCOS dextrous arm [16], and the Platonic Beast, a multi-legged walking machine [12].

## 1 Introduction

What is the best way to hammer a nail? To carry a heavy object? One of the goals of robotics is to understand and automate the manipulation of the physical world by computer. For a given task (a desired change in the state of the objects) one can ask questions such as: "What configuration of the manipulator is optimal?", "What is the best manipulator for this

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task?", and "What class of tasks can be performed by a particular robot?". The answers to these questions are critical to the design, selection, and programming of robots.

A geometrical theory of performance is presented, which assigns a numerical value ("performance measure") to an elementary interaction between the manipulator and the object, within the context of a task. The questions mentioned above can then be translated into optimization and feasibility problems. This theory is a step towards a complexity theory of robots and robot tasks.

In previous work, [18], we constructed such a theory for unconstrained systems. This paper generalizes the theory to systems with constraints. This provides the necessary foundation for the development of general purpose software for constructing performance measures for very broad class of joint connected multibody systems.

Performance measures are already widely used for design and posture optimization for robot arms. Several local measures have been proposed in the past, [19, 3, 5, 11, 15, 4, 6, 2, 8, 9, 1, 13], which are reviewed in [18]. In this paper we will deal with the construction of local performance measures for various tasks, for generic robotic devices.

Our work has the following novel features:

- Our measures can be used for complex manipulators, including constrained multibody systems.
- A unification of measures proposed before in one theoretical framework, based on differential geometry.
- A "toolkit" to make your own performance measure suited for the specific manipulator and task at hand.
- Implemented software that can compute performance measures for systems of joint connected rigid bodies directly from a specification of the system. The software (in MATLAB) can be obtained from the authors.

As a simple illustration of these ideas, consider a redundant robot arm with seven revolute joints. If the robot is grasping a fixed object, the arm forms a closed kinematic chain. However, due to the redundancy, there are infinitely many postures to do this. If we have a local performance measure, i.e. a number $\mu$ associated with every posture (in the context of a specific task), we can resolve this redundancy by finding the posture with optimal $\mu$.

From a design perspective we can find the optimal design for a set of tasks, by selecting the design with the largest average value of a relevant local performance measure. If this involves a continuous infinity of tasks that the manipulator is expected to perform, we obtain such a design measure by the integration of a local measure over some finite region in space.

Finally, for a given task it is often possible to determine if the manipulator can perform it by examining a performance measure.

A complete task will consist of a sequence of elementary tasks. A task planner could utilize local performance measures by examining the feasibility of these subtasks in advance.

All the measures referred to above arise naturally in the framework presented here and several new measures can be derived. The measures derived in our formalism are invariant under general coordinate transformations in configuration space, and therefore correspond to physical properties of the manipulator, and are not just mathematical constructs.

Our measures are generated by the definition of a metric tensor on a configuration space $\hat{\mathcal{C}}$, defined below. From this metric we derive an "induced" metric tensor on the work space, which measures distance by the manipulator movement needed to generate it.

The paper is organized as follows. Section 2 describes the theory of performance for constrained systems. In section 3 we define the induced metric for constrained systems.In section 4 we derive an expicit formula for the induced metric. In section 5 we describe how this formalism can be used to compute performance measures for systems of joint connected rigid bodies. An implementation of the theory is described in section 6, and we give some applications for the SARCOS arm and the Platonic Beast walking robot in section 7. Conclusions are presented in section 8. Some technical points regarding rigid body dynamics are delegated to the appendix 9.

# 2 A Theory of Performance of Constrained Systems

We shall illustrate the ideas throughout the paper with a simple example of a constrained system, the six bar closed loop linkage, depicted in fig. 1.

> **Example:** The system consists of six links, four actuated joints $(\theta_L^1, \theta_L^2, \theta_R^1, \theta_R^2)$ and two passive joints $(\theta_L^3, \theta_R^3)$. The end-effector is attached to the center of the middle link. The distance between joints $L^1$ and $R^1$ is $d$ and the length of the remaining five links is 1.
>
> One interpretation of the system is that of an object (the middle link) with configuration $(x, y, \alpha)$ manipulated by two planar fingers with joint angles $(\theta_L^1, \theta_L^2)$ and $(\theta_R^1, \theta_R^2)$ with contacts at $\mathbf{p}_L$ and $\mathbf{p}_R$. A dual interpretation of the system is as a standing posture of a walking machine. The joints $\theta_L^1$ and $\theta_R^1$ will be passive in this case, representing the feet of the machine and the other four joints will be actuated. $\square$

We model the interaction between a generalized manipulator (any collection of robotic manipulating devices) and a generalized object (the collection of material bodies that are manipulated) as follows. The set of all possible manipulator configurations is described by the *configuration space*, which we denote by $\mathcal{C}$, and its dimension is $n$. A point in $\mathcal{C}$ represents a configuration of the manipulator, and different points in $\mathcal{C}$ represent different configurations. Similarly, all object configurations are described by the *work space*, which we

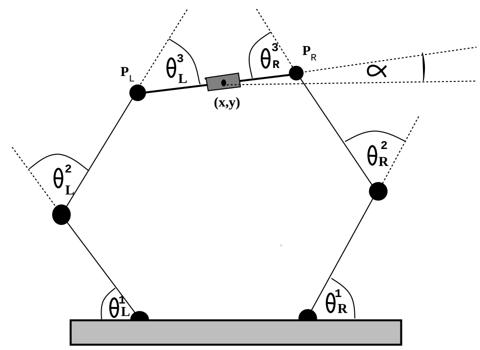


Figure 1: Closed loop manipulator.

denote by $\mathcal{W}$. We denote its dimension by $m$. A task can now be seen as a motion from one point in $\mathcal{W}$ (the present configuration of the object) to another point. This motion is to be achieved by the manipulator, and we describe the coupling between the two by a mapping

$$\kappa : \mathcal{C} \longrightarrow \mathcal{W}$$

which associates a point in $\mathcal{W}$ with every point in $\mathcal{C}$. For typical robot arms with a manipulating device attached to the end of the arm, this mapping is the forward kinematics of the system. However, the connection between the object and the manipulator can be more complex, such as a contact point between a part of the manipulator and a part of the object.

Many systems are most easily described in terms of a more general system upon which constraints are placed (e.g.,[14]). For example, a walking machine could be most easily described by the configuration of all its joints and the position and orientation of its main body. If the machine is walking, we have the constraint that some subset of its feet should be on the ground.

The configurations of the constrained systems we are interested in have a natural description in the form of a set of $n$ coordinates

$$\hat{\boldsymbol{q}} = \begin{pmatrix} \hat{q}^1 \\ \vdots \\ \hat{q}^n \end{pmatrix}$$

upon which some constraint equations are imposed. The coordinates $\hat{\boldsymbol{q}}$ can be thought of as representing configurations that may violate the constraints, which usually has a physical interpretation as the opening up of some closed kinematic loop. Following [7], we call the unconstrained coordinates *descriptors*, i.e. a set of coordinates that are sufficient to describe the system, but may be more than necessary.

The unconstrained coordinates $\hat{\boldsymbol{q}}$ are called the *manipulator* descriptors. The space of all descriptors including the ones that violate the constraints is called the manipulator descriptor space, denoted by $\hat{\mathcal{C}}$, of dimension $\hat{n}$. The points in $\mathcal{C} \subset \hat{\mathcal{C}}$ are labeled by those coordinates $\boldsymbol{q}$ that satisfy the $l$ constraints. The dimension of $\mathcal{C}$ is taken to be $n$.

The extended work space $\hat{\mathcal{W}}$ is defined by the kinematic map

$$\hat{\kappa} : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{W}}.$$

We take its dimension to be $\hat{m}$. After the constraints on $\hat{\mathcal{C}}$ are taken into account, it is reduced to the proper work space $\mathcal{W}$ of dimension $m$. Coordinates on $\hat{\mathcal{W}}$ are denoted by $\hat{\boldsymbol{x}}$ and on $\mathcal{W}$ by $\boldsymbol{x}$.

**Example:** The descriptor space $\hat{\mathcal{C}}$ of the device is labeled by the four joint angles $(\theta_L^1, \theta_L^2, \theta_R^1, \theta_R^2)$ which must satisfy one constraint on $\hat{\mathcal{C}}$.

The coordinates of the passive joints $\mathbf{p}_L$ and $\mathbf{p}_R$ are given by

$$\begin{aligned} p_L^x &= -d/2 - c_L^1 - c_L^{12}, \\ p_L^y &= s_L^1 + s_L^{12}, \\ p_R^x &= d/2 + c_R^1 + c_R^{12}, \\ p_R^y &= s_R^1 + s_R^{12}, \end{aligned}$$

where we have written

$$\begin{aligned} c_L^1 &= \cos\theta_L^1, \\ c_R^1 &= \cos\theta_R^1, \\ c_L^{12} &= \cos(\theta_L^1 + \theta_L^2), \\ c_R^{12} &= \cos(\theta_R^1 + \theta_R^2), \end{aligned}$$

and similarly for the sine functions.

There is one constraint (so $l = 1$), namely that the loop closes,

$$\|\boldsymbol{p}_R - \boldsymbol{p}_L\| = 1. \tag{1}$$

We shall now consider only the position of the center of the middle link, and we define the map $\hat{\kappa}$ by

$$\boldsymbol{x} = (\boldsymbol{p}_L + \boldsymbol{p}_R)/2. \tag{2}$$

□

Note that $\mathcal{W}$ and $\hat{\mathcal{W}}$ are sections of the $(x, y)$ plane, but $\mathcal{W} \subset \hat{\mathcal{W}}$. For example, if $d = 2$, the point $(-2, 0)^T$ is in $\hat{\mathcal{W}}$ (both arms lying flat), but not in $\mathcal{W}$ due to the constraint.

We note that descriptor spaces are not unique. For example, we could have chosen as descriptors the position and orientation of each link, and imposed additional constraints for the six revolute joints.

Tensor indices in $\mathcal{T}\hat{\mathcal{C}}$ (the tangent space to $\hat{\mathcal{C}}$) are denoted by lowercase hatted Latin letters, tensor indices in $\mathcal{T}\hat{\mathcal{W}}$ by lowercase hatted Greek letters. Indices in $\mathcal{T}\mathcal{C}$ and $\mathcal{T}\mathcal{W}$ are the same, but without hat.

To quantify displacements in the manifolds, we need to define a distance. As the manifolds are non-Euclidean in general, we define a vector norm on the tangent bundles $\mathcal{T}\hat{\mathcal{C}}$ and $\mathcal{T}\hat{\mathcal{W}}$. This corresponds intuitively to defining the length of infinitely small line segments everywhere in the manifolds. These infinitesimal line segments correspond, in the limit, to vectors in the tangent space associated with the manifold at the location of the infinitesimal line segment. Distance on $\hat{\mathcal{C}}$ represents the amount of "work" the manipulator has to do for the corresponding motion. It can be defined in many ways, which depends on the application at hand. Distance on $\hat{\mathcal{W}}$ represents how much is achieved with the corresponding movement. The distances on the constrained manifolds follow locally by projection.

The performance can now be described by an "induced" distance on $\mathcal{W}$, that describes how much the manipulator has to move in order to achieve a given infinitesimal displacement in $\mathcal{W}$. Taking the example of a robot arm with an end-effector attached to the tip, the distance on $\mathcal{C}$ measures movements of the arm, the distance on $\mathcal{W}$ measures movements of the end-effector and the induced distance on $\mathcal{W}$ measures the minimal movement of the arm needed to achieve this motion in $\mathcal{W}$.

The induced norm on $\mathcal{W}$ is defined as follows. Let the manipulator be in configuration $\boldsymbol{q}$, with the end-effector at $\boldsymbol{x}$. See figure 2. We define the induced distance $\|\boldsymbol{dx}\|_{\text{induced}}$ from $\boldsymbol{x}$ to $\boldsymbol{x} + \boldsymbol{dx}$ as the length of the shortest path in $\mathcal{C}$ (with respect to its metric) that generates a motion in $\mathcal{W}$ from $\boldsymbol{x}$ to $\boldsymbol{x} + \boldsymbol{dx}$. It is not difficult to prove that the induced norm on $\mathcal{T}\mathcal{W}$, as defined above, is a vector norm if the norm on $\mathcal{T}\hat{\mathcal{C}}$ is [17]

We now make the additional assumption that the norm on $\mathcal{T}\hat{\mathcal{C}}$ is a *quadratic* norm, i.e. it can be given by $\|\widehat{\boldsymbol{dq}}\| = \sqrt{\hat{\boldsymbol{dq}}^T \hat{\boldsymbol{h}} \hat{\boldsymbol{q}}}$, where $\hat{\boldsymbol{h}}$ is a symmetric, nondegenerate, positive definite matrix: the metric tensor of the manifold. A manifold with such a quadratic norm is a Riemannian manifold. (Relaxing the condition of positive definiteness on the metric tensor leads to a pseudo-Riemannian manifold.) A non-trivial result is that if the norm on $\hat{\mathcal{C}}$ is quadratic, then the induced norm is also quadratic. This will be proved by explicit construction below.

**Example:** We shall take the $\hat{\mathcal{C}}$-space metric for our example to be given by the line element

$$ds^2 = (d\theta_L^1)^2 + (d\theta_L^2)^2 + (d\theta_R^1)^2 + (d\theta_R^2)^2,$$

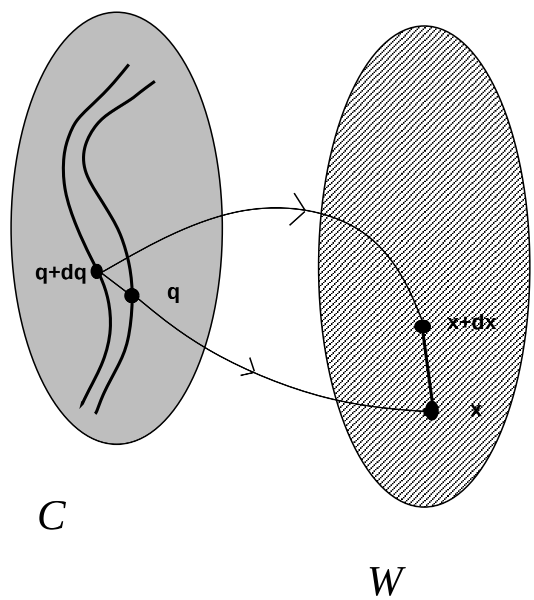


Figure 2: Construction of induced distance

i.e. the kinematic metric. This metric defines the distance between configurations in terms of the actuated joint movements. So the metric tensor $\hat{\boldsymbol{h}}$ is just the $4 \times 4$ unit matrix. The induced norm on $\mathcal{W}$ will now enable us to quantify the mobility of the end-effector, as will be illustrated below. □

# 3 The Induced Metric for Constrained Systems

We now indicate how the induced metric can be computed.

Suppose the system is in some configuration $\widehat{\boldsymbol{q}}$, which is physically realizable (satisfies the constraints) and consider an infinitesimal motion $\widehat{\boldsymbol{dq}}$. The constraints on the motion of the system are written using the Einstein summation convention as

$$F(\widehat{\boldsymbol{q}})^{\Lambda}_{\hat{i}} \widehat{dq}^{\hat{i}} = 0, \tag{3}$$

where the $l \times \hat{n}$ constraint matrix (there are $l$ constraints, labeled by $\Lambda$) $\boldsymbol{F}$ depends on the configuration of the system. We shall usually omit its argument $\hat{\boldsymbol{q}}$. If there exists a vector function $f^{\Lambda}(\hat{\boldsymbol{q}})$, such that

$$F(\hat{\boldsymbol{q}})^{\Lambda}_{\hat{i}} = \frac{\partial f^{\Lambda}(\hat{\boldsymbol{q}})}{\partial \hat{\boldsymbol{q}}^{\hat{i}}},$$

the constraints are integrable and the system is holonomic.

**Example:** The matrix $\boldsymbol{F}$ is obtained by differentiating eq. 1 with respect to the four joint angles. From now on, we shall simplify the algebra by taking all joint angles to be $60^o$, and $d = 1$. In this case, we have

$$\boldsymbol{F} = -\sqrt{3}(1, 1/2, 1, 1/2),$$

which means that joint motions must satisfy

$$d\theta_L^1 + \frac{1}{2}d\theta_L^2 + d\theta_R^1 + \frac{1}{2}d\theta_R^2 = 0.$$

$\square$

A basis for $\mathcal{TC}$ is formed by the $n$ null-space vectors $\boldsymbol{W}_i$ of $\boldsymbol{F}$, which satisfy

$$\boldsymbol{F}\boldsymbol{W}_i = \boldsymbol{0},$$

$i = 1, \ldots, n$. A physically realizable motion $\widehat{\boldsymbol{dq}}$ of the system must satisfy the constraint given in eq. 3, and such a motion can then be written as

$$\widehat{dq}^{\hat{i}} = dq^i W_i^{\hat{i}}. \tag{4}$$

The length $ds^2$ of an infinitesimal displacement $\boldsymbol{dq}$ in $\mathcal{TC}$ is

$$ds^2 = \boldsymbol{dq}^T \boldsymbol{h} \boldsymbol{dq},$$

with the $\mathcal{C}$ space metric $\boldsymbol{h}$ given by

$$\boldsymbol{h} = \boldsymbol{W}^T \hat{\boldsymbol{h}} \boldsymbol{W}, \tag{5}$$

where the $\hat{n} \times n$ matrix $\boldsymbol{W}$ has the null space vectors $\boldsymbol{W}_i$ as its columns.

**Example:** We take

$$\boldsymbol{W} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & -2 & -1/2 \end{pmatrix},$$

which gives

$$\boldsymbol{h} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5/2 \end{pmatrix}.$$

$\square$

The workspace $\hat{\mathcal{W}}$ is defined locally by the mapping $\mathcal{T}\hat{\mathcal{C}} \rightarrow \mathcal{T}\hat{\mathcal{W}}$, which we write as

$$\widehat{\boldsymbol{dx}} = \hat{\boldsymbol{J}}\widehat{\boldsymbol{dq}}. \tag{6}$$

The dimension of $\mathcal{T}\hat{\mathcal{W}}$ is $\hat{m}$, so the Jacobian $\hat{\boldsymbol{J}}$ (which may depend on $\hat{\boldsymbol{q}}$) is a $\hat{m} \times \hat{n}$ matrix.

Eq. 6 places constraints on motions in $\hat{\mathcal{W}}$, and locally defines $\mathcal{W} \subset \hat{\mathcal{W}}$ and $\mathcal{T}\mathcal{W} \subset \mathcal{T}\hat{\mathcal{W}}$. The constraints arise through the constraints from eq. 3 on $\widehat{\boldsymbol{dq}}$, and through the nature of the Jacobian $\hat{\boldsymbol{J}}$, which may place additional constraints on $\widehat{\boldsymbol{dx}}$. This may be caused by a singular posture, for example.

For physically realizable motions $\widehat{\boldsymbol{dq}}$, we have

$$\widehat{\boldsymbol{dx}} = \boldsymbol{J}\boldsymbol{dq},$$

with $\boldsymbol{dq}$ defined by eq. 4, and

$$\boldsymbol{J} = \hat{\boldsymbol{J}}\boldsymbol{W}. \tag{7}$$

The Jacobian $\boldsymbol{J}$ is a $\hat{m} \times n$ matrix.

**Example:** By differentiating 2 we find

$$\hat{\boldsymbol{J}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & \sqrt{3}/2 & -\sqrt{3} & -\sqrt{3}/2 \\ 0 & -1/2 & 0 & -1/2 \end{pmatrix},$$

and

$$\boldsymbol{J} = \begin{pmatrix} 0 & 0 & 5\sqrt{3}/4 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

□

Physically realizable motions in $\mathcal{W}$ can be written as

$$\widehat{dx}^{\hat{\mu}} = dx^{\mu} U_{\mu}^{\hat{\mu}}, \tag{8}$$

with the $m$ $\hat{m}$-vectors $\boldsymbol{U}_{\mu}$ (to be determined below) forming a basis for contravariant vectors in $\mathcal{W}$, analogous to the null space vectors of the constraints in eq. 4. We take the dimension of $\mathcal{T}\mathcal{W}$ to be $m$ (which is also the rank of the Jacobian $\boldsymbol{J}$), so that $\mu = 1, \ldots, m$.

We now recall the definition for the induced length of an infinitesimal displacement $\widehat{\boldsymbol{dx}}$ in $\mathcal{T}\mathcal{W}$, as given in section 2

- The induced length of $\widehat{\boldsymbol{dx}} \in \mathcal{T}\mathcal{W}$ is the length of the shortest $\widehat{\boldsymbol{dq}} \in \mathcal{T}\mathcal{C}$ that generates $\widehat{\boldsymbol{dx}}$ through eq. 6.

Or,

$$\|\widehat{\boldsymbol{dx}}\|^2 = \min_{\widehat{\boldsymbol{dq}}}(\boldsymbol{dq}^T \boldsymbol{h} \boldsymbol{dq}), \tag{9}$$

subject to $\widehat{\boldsymbol{dx}} = \boldsymbol{J}\boldsymbol{dq}$ and $\widehat{\boldsymbol{dx}} \in \mathcal{TW}$.

As shown explicitly below, the induced metric can be written as

$$\|\widehat{\boldsymbol{dx}}\|^2 = \boldsymbol{dx}^T \boldsymbol{g} \boldsymbol{dx}.$$

We call $\boldsymbol{g}$ the *induced metric tensor* on $\mathcal{W}$. Sometimes we also define a metric tensor $\hat{\boldsymbol{g}}$ on $\hat{\mathcal{W}}$, related to $\boldsymbol{g}$ by

$$\boldsymbol{g} = \boldsymbol{U}^T \hat{\boldsymbol{g}} \boldsymbol{U},$$

which is analogous to eq. 5.

# 4 Explicit Construction of the Induced Metric

The induced metric tensor $\boldsymbol{g}$ will now be constructed explicitly.

Compute the singular value decomposition [5] of $\boldsymbol{J}$ as

$$\boldsymbol{J} = \boldsymbol{X}\boldsymbol{\Sigma}\boldsymbol{V}^T,$$

with the $\hat{m} \times \hat{m}$ matrix $\boldsymbol{X}$ and the $n \times n$ matrix $\boldsymbol{V}$ unitary, and the $\hat{m} \times n$ matrix $\boldsymbol{\Sigma}$ of the form

$$\boldsymbol{\Sigma} = \begin{pmatrix} \tilde{\boldsymbol{\Sigma}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where the $m \times m$ matrix $\tilde{\boldsymbol{\Sigma}}$ is of the form

$$\tilde{\boldsymbol{\Sigma}} = \begin{pmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sigma_m \end{pmatrix},$$

with the singular values $\sigma_1, \ldots, \sigma_m$ of $\boldsymbol{J}$ in non-decreasing order, i.e. $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$. We note that the rank of the Jacobian, $m$, can be smaller than the dimension of $\mathcal{TC}$. The latter is determined only by the constraints $\boldsymbol{F}$; the former is also determined by the kinematics.

The matrix $\boldsymbol{X}$ contains the desired basis for contravariant vectors in $\mathcal{TW}$ (see eq. 8) in the form

$$U_\mu^{\hat{\mu}} = X_\mu^{\hat{\mu}},$$

with $\hat{\mu} = 1, \ldots, \hat{m}$, and $\mu = 1, \ldots, m$. $\boldsymbol{U}$ is thus a submatrix of $\boldsymbol{X}$.

Let us define

$$\widetilde{\boldsymbol{dq}} = \boldsymbol{V}^T \boldsymbol{dq},$$

and

$$\widetilde{\boldsymbol{dq}}' = \begin{pmatrix} \widetilde{\boldsymbol{dq}}_1 \\ \vdots \\ \widetilde{\boldsymbol{dq}}_m \end{pmatrix},$$

and

$$\widetilde{\boldsymbol{dq}}^* = \begin{pmatrix} \widetilde{\boldsymbol{dq}}_{m+1} \\ \vdots \\ \widetilde{\boldsymbol{dq}}_n \end{pmatrix},$$

so that

$$\widetilde{\boldsymbol{dq}} = \begin{pmatrix} \widetilde{\boldsymbol{dq}}' \\ \\ \widetilde{\boldsymbol{dq}}^* \end{pmatrix}.$$

The forward kinematics as given in eq. 6 now becomes

$$\boldsymbol{dx} = \tilde{\boldsymbol{\Sigma}} \widetilde{\boldsymbol{dq}}', \tag{10}$$

which does not depend on $\widetilde{\boldsymbol{dq}}^*$. The solution to eq. 10 is

$$\widetilde{\boldsymbol{dq}}' = \tilde{\boldsymbol{\Sigma}}^{-1} \boldsymbol{dx}, \tag{11}$$

where $\widetilde{\boldsymbol{dq}}^*$ should be chosen to minimize

$$ds^2 = \widetilde{\boldsymbol{dq}}^T \tilde{\boldsymbol{h}} \widetilde{\boldsymbol{dq}},$$

with

$$\tilde{\boldsymbol{h}} = \boldsymbol{V}^T \boldsymbol{h} \boldsymbol{V}.$$

We write the $n \times n$ matrix $\tilde{\boldsymbol{h}}$ as

$$\tilde{\boldsymbol{h}} = \begin{pmatrix} \tilde{\boldsymbol{h}}_1 & \tilde{\boldsymbol{h}}_2^T \\ \tilde{\boldsymbol{h}}_2 & \tilde{\boldsymbol{h}}_3 \end{pmatrix},$$

with $\tilde{\boldsymbol{h}}_1$ an $m \times m$ matrix, $\tilde{\boldsymbol{h}}_2$ $(n-m) \times m$, and $\tilde{\boldsymbol{h}}_3$ $(n-m) \times (n-m)$. Then $ds^2$ takes the form

$$ds^2 = \widetilde{\boldsymbol{dq}}'^T \tilde{\boldsymbol{h}}_1 \widetilde{\boldsymbol{dq}}' + 2 \widetilde{\boldsymbol{dq}}^{*T} \tilde{\boldsymbol{h}}_2 \widetilde{\boldsymbol{dq}}' + \widetilde{\boldsymbol{dq}}^{*T} \tilde{\boldsymbol{h}}_3 \widetilde{\boldsymbol{dq}}^*.$$

Imposing

$$\frac{\partial ds^2}{\partial \widetilde{\boldsymbol{dq}}^*} = \boldsymbol{0}$$

yields

$$\widetilde{\boldsymbol{dq}}^* = -\tilde{\boldsymbol{h}}_3{}^{-1}\tilde{\boldsymbol{h}}_2\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{dx}.$$

It follows that the induced length of $\widehat{\boldsymbol{dx}}$ is given by

$$ds^2 = \boldsymbol{dx}^T\boldsymbol{g}\boldsymbol{dx},$$

with the induced metric tensor given by

$$\boldsymbol{g} = \tilde{\boldsymbol{\Sigma}}^{-1}(\tilde{\boldsymbol{h}}_1 - \tilde{\boldsymbol{h}}_2^T\tilde{\boldsymbol{h}}_3^{-1}\tilde{\boldsymbol{h}}_2)\tilde{\boldsymbol{\Sigma}}^{-1}. \tag{12}$$

Note that, if $m = n$, the term with $\tilde{\boldsymbol{h}}_3$ disappears from eq.12, i.e.

$$\boldsymbol{g} = \tilde{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{h}}_1\tilde{\boldsymbol{\Sigma}}^{-1}.$$

**Example:** For our system we find

$$\boldsymbol{X} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} 5\sqrt{3}/4 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix},$$

$$\boldsymbol{V} = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \end{pmatrix},$$

$$\tilde{\boldsymbol{h}} = \begin{pmatrix} 5/2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

and

$$\boldsymbol{g} = \begin{pmatrix} 8/15 & 0 \\ 0 & 10 \end{pmatrix}.$$

The form of the metric tensor on $\mathcal{W}$ shows that the end-effector is much more mobile in the $x$-direction, than in the $y$-direction. Note also, that the optimal way to move in the $x$-direction consists of a translation *and* a rotation of the center bar. □

The metric tensor can be used to quantify mobility in an arbitrary direction $\boldsymbol{u}$, i.e. by constructing the scalar performance measure

$$\boldsymbol{u}^T\boldsymbol{g}\boldsymbol{u}/\|\boldsymbol{u}\|^2,$$

where $\|\boldsymbol{u}\|$ is Eucidean norm on $\mathcal{W}$. Alternatively, the measure

$$Y = 1/\sqrt{\det \boldsymbol{g}}$$

measures the mobility averaged over all directions. One can also use the condition number of $\boldsymbol{g}$ to obtain a measure for the anisotropy of the manipulator. We refer to [18] for more details.

# 5 Joint Connected Systems of Rigid Bodies

In this section we consider systems of joint connected rigid bodies. The systems consist of free rigid bodies, upon which constraints are placed at the joints. Many simulators specify the system in terms of joint connected rigid bodies [14].

Our formalism for constrained systems can be used to define performance measures for such system in a very general way. We construct the constraints for some types of joints and show how performance measures can be computed. A MATLAB program was written, that takes the specification of a system in terms of rigid bodies with joint constraints, and can compute a large class of performance measures for a given configuration of the system.

Some more details of our description of rigid bodies can be found in the appendix.

## 5.1 Defining a System

For a rigid body, we specify a set of homogeneous configuration matrices $\boldsymbol{T}_1^J, \ldots, \boldsymbol{T}_k^J$ that determine the positions and orientations of the $k$ joints on the body. This formulation is more general than required for most joints. For example, for a revolute joint it suffices to specify the point of attachment and a direction of the rotation axis. However, this way we can treat all joints uniformly.

The matrices $\boldsymbol{T}^J$ are to be interpreted in the reference frame attached to the body.

Bodies are connected at the joints as follows. Suppose we connect bodies 1 and 2, that are in configurations $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$, at joint $J$, located at $\boldsymbol{T}_1^J$ (on body 1) and $\boldsymbol{T}_2^J$ (on body 2). We say that bodies 1 and 2 are connected in the *zero* configuration, if their joint frames coincide, i.e. if

$$\hat{\boldsymbol{T}}_1^J = \hat{\boldsymbol{T}}_2^J,$$

where the "hatted" $\boldsymbol{T}^J$ matrices are the frames of the joints relative to world coordinates. So we have

$$\hat{\boldsymbol{T}}_i^J = \boldsymbol{T}_i \boldsymbol{T}_i^J,$$

with $i = 1, 2$. Define the connection configuration by the matrix

$$\boldsymbol{T}^C = (\hat{\boldsymbol{T}}_1^J)^{-1} \hat{\boldsymbol{T}}_2^J \ . \tag{13}$$

It equals the $4 \times 4$ identity matrix in the zero configuration.

The matrix $\boldsymbol{T}^C$ ranges over what we call the connection space, and describes the allowed relative motion of the joints.

$$\mathcal{S}_C \subset SE(3). \tag{14}$$

The bodies are connected by specifying their connection spaces on all joint connections. This can be done for example for a revolute joint by specifying an attachment point and an axis direction.

Usually there is a special body, ground, that also may have joint frames, but is immobile, and does not have inertial properties.

The $n$ bodies are described by the $6n$ coordinates $\widehat{\boldsymbol{q}}$, parametrizing $\hat{\mathcal{C}} = SE(3)^{\hat{n}}$. We also have a metric on $\hat{\mathcal{C}}$,

$$ds^2 = \widehat{\boldsymbol{dq}}^T \hat{\boldsymbol{h}} \widehat{\boldsymbol{dq}}.$$

The forward kinematics is defined by specifying the Jacobian $\hat{\boldsymbol{J}}$. An important workspace is formed by the configuration of an end-effector, which is a single rigid body. If the end-effector is represented by body $b$, the Jacobian takes the form

$$\hat{\boldsymbol{J}} = \begin{pmatrix} 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix},$$

where the first non-zero column is column $6b - 5$.

## 5.2 Computing the Constraint Matrix

The constraint matrix can be constructed explicitly for several types of joints as follows.

Consider a system with $c$ connections. At each connection $l$ there are a number of constraints, $nc(l)$, that contribute to rows $p$ to $p + nc(l) - 1$ of the constraint matrix $\boldsymbol{F}$ defined in eq. 3, where

$$p = \sum_{k=1}^{l-1} nc(k).$$

Let the bodies involved in connection $c$ be $b_1$ and $b_2$ (there are always two bodies involved in a connection, one of which may be the ground). Body $b_i$ contributes to columns $6b_i - 5$ to $6b_i$ of the matrix $\boldsymbol{F}$ if $b_i$ does not represent the ground. If it represents the ground it does not contribute to $\boldsymbol{F}$. The total contribution to $\boldsymbol{F}$ of connection $i$ is thus given by two matrices $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, of dimension $nc(i) \times 6$ to be added to $\boldsymbol{F}$ at locations as specified above, with the understanding that $\boldsymbol{F}_i$ does not contribute if $b_i$ represents the ground.

The actual constraints are placed on the connection configuration, as described above. With the notation of that section, we write

$$\boldsymbol{T}_{b_i} = \begin{pmatrix} \boldsymbol{R}_i & \boldsymbol{t}_i \\ \boldsymbol{0} & 1 \end{pmatrix},$$

$$\boldsymbol{T}_{b_i}^J = \begin{pmatrix} \boldsymbol{R}_i^J & \boldsymbol{t}_i^J \\ \boldsymbol{0} & 1 \end{pmatrix},$$

and

$$\boldsymbol{T}^C = \begin{pmatrix} \boldsymbol{R}^C & \boldsymbol{t}^C \\ \boldsymbol{0} & 1 \end{pmatrix}.$$

Some algebra yields

$$\boldsymbol{R}^C = \boldsymbol{R}_1^{J^T} \boldsymbol{R}_1{}^T \boldsymbol{R}_2 \boldsymbol{R}_2^J, \tag{15}$$

and

$$\boldsymbol{t}^C = \boldsymbol{R}_1^{J^T} \boldsymbol{R}_1^T (\boldsymbol{R}_2 \boldsymbol{t}_2^J + \boldsymbol{t}_2 - \boldsymbol{R}_1 \boldsymbol{t}_1^J - \boldsymbol{t}_1) \tag{16}$$

At this point we use the angular velocity basis for $\mathcal{TC}$, as defined in eq. 29.

The constraints will be built from the derivatives of $\boldsymbol{R}^C$ and $\boldsymbol{T}^C$ with respect to the coordinates of the bodies $b_1$ and $b_2$, which appear in the $6n$ dimensional coordinate vector

$$\boldsymbol{q} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{x}_{b_i} \\ \boldsymbol{e}_{b_i} \\ \vdots \\ \boldsymbol{x}_n \\ \boldsymbol{e}_n \end{pmatrix}.$$

They can be computed from eq. 15 and eq.16.

We can now compute the resulting constraint matrices $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ as defined above for several types of joint connections. We shall illustrate this with two examples, a weld joint and a revolute joint.

- **Weld joint**

  No motion on the joint is allowed at all. There are thus 6 constraints. The constraints are $\boldsymbol{t}^C = \boldsymbol{0}$, 3 constraints, and $\boldsymbol{R}^C = \mathbb{1}$, 3 constraints. The last 3 constraints can be written as

  $$R_{12}^C = R_{13}^C = R_{23}^C = 0.$$

Thus

$$\boldsymbol{F}_i = \begin{pmatrix} \frac{\partial \boldsymbol{t}^C}{\partial \boldsymbol{x}_{b_i}} & \frac{\delta \boldsymbol{t}^C}{\delta \boldsymbol{\omega}_{b_i}} \\ \boldsymbol{0}_{3\times 3} & \tilde{\boldsymbol{F}}_i \end{pmatrix}, \tag{17}$$

with

$$\tilde{\boldsymbol{F}}_i = \begin{pmatrix} \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^1_{b_i}}\right)_{12} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^2_{b_i}}\right)_{12} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^3_{b_i}}\right)_{12} \\ \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^1_{b_i}}\right)_{13} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^2_{b_i}}\right)_{13} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^3_{b_i}}\right)_{13} \\ \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^1_{b_i}}\right)_{23} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^2_{b_i}}\right)_{23} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^3_{b_i}}\right)_{23} \end{pmatrix}.$$

We refer to the appendix for the definition of "differentiating" to $\boldsymbol{\omega}$.

- **Revolute joint**

  The only allowed motion is rotation around the $z$-axis, i.e. the connection configuration should describe a pure rotation around the $z$-axis. The constraints are $\boldsymbol{t}^C = \boldsymbol{0}$, 3 constraints, and $\boldsymbol{R}^C = \mathrm{Rot}_z$, a rotation around the $z$-axis, 2 constraints. The last 2 constraints can be written as

$$R^C_{13} = R^C_{23} = 0.$$

  Thus

$$\boldsymbol{F}_i = \begin{pmatrix} \frac{\partial \boldsymbol{t}^C}{\partial \boldsymbol{x}_{b_i}} & \frac{\delta \boldsymbol{t}^C}{\delta \boldsymbol{\omega}_{b_i}} \\ \boldsymbol{0}_{2\times 3} & \tilde{\boldsymbol{F}}_i \end{pmatrix}, \tag{18}$$

  with

$$\tilde{\boldsymbol{F}}_i = \begin{pmatrix} \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^1_{b_i}}\right)_{13} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^2_{b_i}}\right)_{13} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^3_{b_i}}\right)_{13} \\ \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^1_{b_i}}\right)_{23} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^2_{b_i}}\right)_{23} & \left(\frac{\delta \boldsymbol{R}^C}{\delta \omega^3_{b_i}}\right)_{23} \end{pmatrix}.$$

## 5.3 Metrics on Configuration Space

Two special types of metric on $\mathcal{C}$ are especially important. We call these the kinematic and the dynamic metric. They are defined as follows.

1. **Kinematic metric**

   Consider an infinitesimal motion of the constrained system. The norm of this displacement has a contribution from each joint connection. The contribution of a connection is defined by specifying a metric tensor $\boldsymbol{a}$ on the local connection space $\mathcal{S}_C \subset SE(3)$, as defined in eq. 14. Usually this will be a kinematic metric as given in eq. 30, up to a constant multiplicative factor.

2. **Dynamic Metric**

   Consider an infinitesimal motion of the constrained system. The norm of this displacement has a contribution from each rigid body. The contribution is defined by specifying a metric $\boldsymbol{a}$ on the local configuration space $SE(3)$. Usually this will be a dynamic metric (essentially the generalized inertia matrix) for this body as given in eq. 32.

These two types of metrics can be mixed, i.e. the total norm of a motion can have contributions from both.

We will explicitly construct the metric tensor $\hat{\boldsymbol{h}}$ on $\hat{\mathcal{C}}$, from which the metric on $\mathcal{C}$ then follows simply by projection, i.e. as in eq.5.

In the case of the dynamic metric, each body $b$ contributes a submatrix at rows and columns $6b-5,\ldots,6b$ to the metric tensor $\hat{\boldsymbol{h}}$. This matrix is just the generalized inertia matrix of this body.

The kinematic metric is a little more involved. Consider a system with $c$ connections. Let the bodies involved in connection $c$ be $b_1$ and $b_2$, as in subsection 5.2. The contribution $\Delta\hat{\boldsymbol{h}}$ to the $6n \times 6n$ metric tensor $\hat{\boldsymbol{h}}$ is generated from the line element

$$\Delta ds^2 = \boldsymbol{dq}_1^T \boldsymbol{B}_{11} \boldsymbol{dq}_1 + \boldsymbol{dq}_2^T \boldsymbol{B}_{22} \boldsymbol{dq}_2 + 2\boldsymbol{dq}_1^T \boldsymbol{B}_{12} \boldsymbol{dq}_2, \tag{19}$$

with $\boldsymbol{dq}_i$ being the configuration of body $b_i$. The $6 \times 6$ matrices $\boldsymbol{B}_{ij}$ contribute to the appropriate rows and columns of $\hat{\boldsymbol{h}}$, e.g. $\boldsymbol{B}_{12}$ should be added to rows $6b_1-5,\ldots,6b_1$ and columns $6b_2-5,\ldots,6b_2$ of $\hat{\boldsymbol{h}}$.

We denote the metric chosen on the connection configuration by $\boldsymbol{a}$, which is a $6 \times 6$ matrix. (Note that this matrix will be different for different connections, in general.)

The matrices $\boldsymbol{B}_{ij}$ are constructed as follows. Consider an infinitesimal motion described by

$$\boldsymbol{dq}_i = \begin{pmatrix} \boldsymbol{dt}_i \\ \boldsymbol{d\omega}_i \end{pmatrix},$$

where $i = 1,2$, in the angular velocity basis (see section 9.2.2). The change in the connection configuration (defined in eq.13), which is the motion on $SE(3)$ is given by

$$\boldsymbol{dT}^C = \begin{pmatrix} \boldsymbol{dR}^C & \boldsymbol{dt}^C \\ \boldsymbol{0} & 0 \end{pmatrix}.$$

Using the definition of $\boldsymbol{T}^C$ given in eq. 13 and eqs.15-16, we find

$$\begin{aligned}\boldsymbol{dt}^C &= -(\boldsymbol{R}_1^J)^T \boldsymbol{R}_1^T(\boldsymbol{dt}_1 - \boldsymbol{dt}_2) + (\boldsymbol{R}_1^J)^T \left(\frac{\delta \boldsymbol{R}_1}{\delta\omega_{b_1}^m}\right)^T (\boldsymbol{R}_2\boldsymbol{t}_2^J + \boldsymbol{t}_2 - \boldsymbol{t}_1) d\omega_{b_1}^m + \\ &(\boldsymbol{R}_1^J)^T \boldsymbol{R}_1^T \left(\frac{\delta \boldsymbol{R}_2}{\delta\omega_{b_2}^m}\right) \boldsymbol{t}_2^J d\omega_{b_2}^m\end{aligned} \tag{20}$$

and

$$\boldsymbol{d\Omega}^C \stackrel{\text{def}}{=} (\boldsymbol{R}^C)^T \boldsymbol{dR}^C = -(\boldsymbol{R}_2^J)^T \boldsymbol{d\Omega}_2 \boldsymbol{R}_2^J + (\boldsymbol{R}_2^J)^T \boldsymbol{R}_2^T \boldsymbol{R}_1 d\Omega_1((\boldsymbol{R}_2^J)^T \boldsymbol{R}_2^T \boldsymbol{R}_1)^T \tag{21}$$

Using eqs. 27 and 28, we obtain

$$\boldsymbol{d\omega}^C = -(\boldsymbol{R}_2^J)^T \boldsymbol{R}_2^T \boldsymbol{R}_1 \boldsymbol{d\omega}_{b_1} + (\boldsymbol{R}_2^J)^T \boldsymbol{d\omega}_{b_2} \tag{22}$$

Writing for the metric on the connection configuration

$$\boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_{tt} & \boldsymbol{a}_{t\omega} \\ (\boldsymbol{a}_{t\omega})^T & \boldsymbol{a}_{\omega\omega} \end{pmatrix},$$

the contribution of this connection to the line element, as given in eq. 19 is now obtained by substituting eqs. 20 and 22 into

$$\Delta ds^2 = \begin{pmatrix} \boldsymbol{dt}^C \\ \boldsymbol{d\omega}^C \end{pmatrix}^T \boldsymbol{a} \begin{pmatrix} \boldsymbol{dt}^C \\ \boldsymbol{d\omega}^C \end{pmatrix},$$

and collecting the matrices $\boldsymbol{B}_{ij}$.

# 6 Implementation

The results given above have been implemented in a MATLAB program, which is available from the authors.

The user specifies a number of *systems*, labeled $1, \ldots, N_s$. A system is a collection of rigid bodies, labeled $1, \ldots, N_b$, plus the "ground", labeled 0. The ground has an inertial frame attached to it, called the ground frame. The bodies are defined in the ground frame. For each body, except the ground, the user specifies its mass, and its moment of inertia, as defined in eq.23. Each body, including the ground, has a number of "joints", which are attachment points of other bodies (that may or may not be used). A joint is labeled $1, \ldots, N_j$, and is defined by a $4 \times 4$ configuration matrix $\boldsymbol{T}^J$, which the user specifies. The system is put together by giving a list of connections, labeled $1, \ldots, N_c$, which consist of a joint-pair, plus a specification of the type of the connection.

Once the systems are specified, the user needs to specify the $\hat{\mathcal{C}}$ space metric. The metric can be given as a $6n \times 6n$ matrix (where $n$ is the number of rigid bodies in this system), it can also be set to be a mixture of the "kinematic" and "dynamic" metrics, as defined in section 5.3. The forward kinematics is defined by providing the Jacobian locally, as discussed in section 3, eq. 6.

After this, performance measures can be computed. We note that the measures can also be computed in singular postures, where the dimensionality of $\mathcal{W}$ changes.

Generally a function is given a system argument, a positive integer, and an argument $\boldsymbol{T}$, a $4 \times 4n$ matrix, defining the configuration of each rigid body. It is the user's responsibility to generate these correctly.

# 7 Applications

We have applied these results to two realistic examples, the SARCOS arm, [16] and the Platonic Beast, a multilegged walking machine [12].

## 7.1 The SARCOS ARM

The SARCOS arm [16] is a redundant arm with seven degrees of freedom. We are interested here in the ability of the arm to reconfigure itself, while leaving the end-effector fixed. Performance measures for reconfigurability were investigated for a planar mechanism by us in [18], and were called redundancy measures there.

We shall show here how a redundancy measure can also be interpreted as a "normal" mobility measure for a constrained system. Suppose we keep the end-effector fixed, in some generic configuration. The arm now has one degree of freedom left. We quantify its mobility in terms of the mobility of, for example, the "elbow", which we take to be the joint connecting links three and four. Denoting the Euclidean coordinates of the elbow by $\hat{\boldsymbol{x}}$, and denoting the seven joint angles by $\hat{\boldsymbol{q}}$, we obtain a seven dimensional extended configuration space $\hat{\mathcal{C}}$, and a three dimensional extended work space $\hat{\mathcal{W}}$, and we have six constraints on the end-effector. The configuration space $\mathcal{C}$ and the work space $\mathcal{W}$ are both one dimensional. We choose the kinematic metric on $\hat{\mathcal{C}}$, i.e. we measure distance$^2$ by the sum of the squares of the joint angle differences. The metric on $\hat{\mathcal{W}}$ is just the Euclidean metric.

The redundancy measure is defined as the ratio of the distance that the elbow moves and the distance covered in configuration space to generate this motion. This is just the generalized Yoshikawa measure for the constrained system, as defined in [18].

In figure 3 we show the optimal posture and in figure 4 we show the worst posture for reaching the same point in workspace. [1] We note that the optimal posture has the elbow bent more than the worst posture, to make it more mobile.

## 7.2 The Platonic Beast

The Platonic Beast is a novel, spherically symmetric, walking machine [12]. Kinematically, it can be considered a tetrahedron, with four identical three degree of freedom limbs attached to the vertices (see figure 5). We consider standing postures with the beast resting on three limbs, with the feet non-sliding. In such a standing posture we then consider the mobility of the body. In figure 5 we show the posture that maximizes the generalized Yoshikawa measure, i.e. the mobility averaged over all directions, for the body. The forward kinematics is defined here as the mapping form the joint angles of the tree legs to the configuration of

[1] The joint angles are as follows in degrees, using the Denavit-Hartenberg convention; figure 3: (0.0, -33.0, 0.0, -87.0, 0.0, 0.0, -90.0+48); figure 4: (0.0, -46.0, 0.0, -60.0, 0.0, 0.0, -90.0)

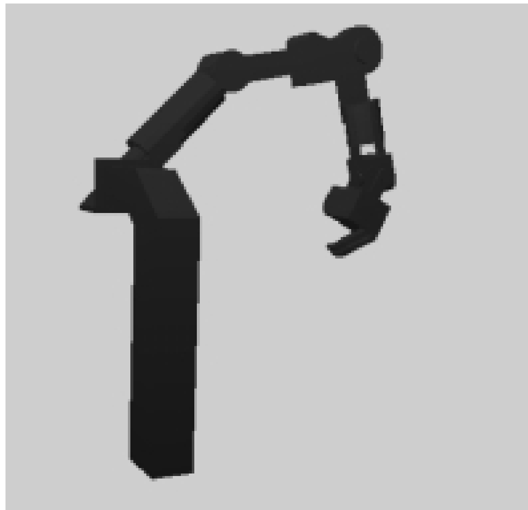

Figure 3: SARCOS arm with maximal elbow mobility.

Figure 4: SARCOS arm with minimal elbow mobility.

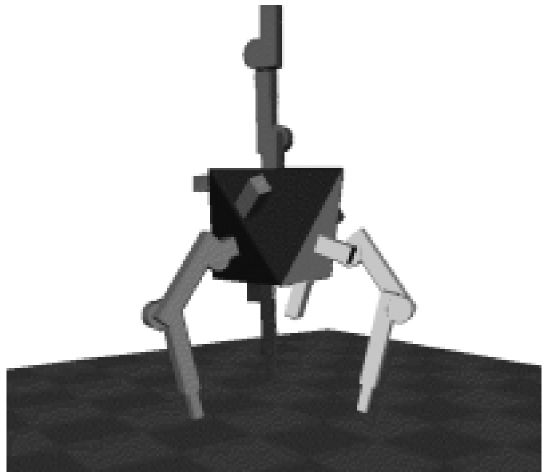

Figure 5: Optimal posture of Beast, maximizing body mobility.

the body. The posture shown is the absolute maximum, i.e. it is the best standing posture possible.

A different type of body mobility is obtained by considering the ability to aim a camera attached to the top of the body. In this case the forward kinematics is the mapping from the joint angles of the legs to the direction of the camera, i.e. $S(2)$. Without limits on the joint angles, the optimal posture is degenerate, with the feet placed on an infinitely small triangle inside the body[2]. In figure 6 we show the optimal posture, where the joint limits are now $-70 < \theta^1 < 70$, $0 < \theta^2 < 45$, $0 < \theta^1 < 90$.

# 8 Conclusions

We have developed a theory for quantifying the performance of very general mechanical systems with constraints. The ability of the manipulator, the requirements of the task, and the constraints are all specified as geometrical properties of descriptor space, the work space and their mappings. This allows us to construct an important geometrical object, the induced metric tensor, that contains information on the ability of the manipulator to perform the task in the work space.

Our formulation can be used easily for systems with constraints. This allows us to consider systems with closed loops, as well as systems of joint connected rigid bodies. We demonstrate this with some examples of performance measures for two robots in typical situations: the SARCOS arm forming a closed kinematic loop while holding a fixed object, and the Platonic Beast walking robot.

[2]Directing the camera corresponds to reorienting this infinitesimal triangle, which can be done with zero movements of the joint angles.

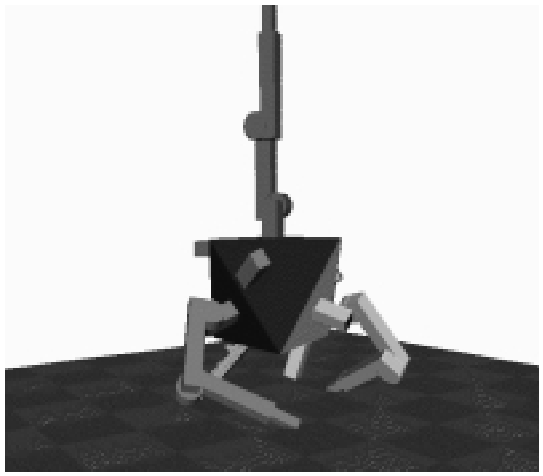

Figure 6: Optimal posture of Beast, maximizing mobility on $S(2)$, with joint constraints.

The construction of the induced metric has been implemented in a MATLAB program, that allows the computation of the performance metric for any configuration of a mechanical system specified as a collection of joint connected rigid bodies. While specific performance measures have been proposed and used before, we believe this is the first software which computes measures for a general class of multibody systems and allows the specification of new performance measures. Such measures can be a useful tool for the design of task strategies for a given manipulator, as well as for the design and selection of manipulators.

# 9 Appendix

In this appendix, we review some properties of rigid body kinematics and we define our conventions and notation.

## 9.1 Defining a Rigid Body

A rigid body is defined in a reference coordinate frame, also called the ground-frame, as some mass density field $\mu(\boldsymbol{x})$ on $E^3$, Euclidean space. On $E^3$ we assume a right-handed Cartesian coordinate system, and the coordinates of a point are denoted by $\boldsymbol{x}$, with components $x^i$, $i = 1, \ldots, 3$. This configuration of the rigid body is called the reference configuration, which we always describe in the center of mass, i.e.

$$\int_{E^3} \boldsymbol{x}\mu(\boldsymbol{x})d^3x = \mathbf{0}.$$

The mass of the body is defined as

$$m = \int_{E^3} \mu(\boldsymbol{x}) d^3x.$$

Its moment of inertia (in the reference configuration, in the ground-frame) is defined as the symmetric rank two tensor

$$I_{ij} = \int_{E^3} \mu(\boldsymbol{x})(\boldsymbol{x}.\boldsymbol{x}\delta_{ij} - x_i x_j) d^3x, \tag{23}$$

with $\delta_{ij}$ the components of the unit matrix.

## 9.2 Motion of a Rigid Body

### 9.2.1 Configuration of a Body

The configuration of a rigid body is described by a $4 \times 4$ homogeneous transformation matrix $\boldsymbol{T}$, which has the form

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{pmatrix},$$

where $\boldsymbol{R}$ is a $3 \times 3$ rotation matrix (i.e. $\boldsymbol{R}^T\boldsymbol{R} = \mathbb{1}$, and $\det(\boldsymbol{R}) = 1$), and $\boldsymbol{t}$ is the position vector of the center of mass of the body, as described in the ground-frame. A point $\boldsymbol{x}$ on the body in the reference configuration in the ground-frame will be located at $\hat{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$ for the body in configuration $\boldsymbol{T}$. The body moves in $SE(3) \equiv SO(3) \times \mathbb{R}^3$. We shall always parametrize the position of the center of mass by its Cartesian coordinates, $t^i$. The orientation of the body, as described by $\boldsymbol{R}$ is parametrized by the coordinates $e^i$, which we will not need to specify explicitly here. Euler angles and Roll-Pitch-Yaw angles are typical examples. The $\boldsymbol{t}$ and $\boldsymbol{e}$ together are denoted by the 6-tuple

$$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{e} \end{pmatrix}.$$

### 9.2.2 Velocity of a Body

Assume the body is in configuration $\boldsymbol{T}$, with $SE(3)$ coordinates $\boldsymbol{p}$, and moves an infinitesimal amount $\boldsymbol{dp}$. Its 6-velocity lies in $\mathcal{T}SE(3)$ (the tangent space to $SE(3)$). The movement of its center of mass is $\boldsymbol{dt}$. We take the basis for covariant vectors and tensors in $\mathcal{T}E^3$ to be

$\boldsymbol{dt}^i = \boldsymbol{d}t^i$.[3] The structure of $\mathcal{T}SO(3)$ is a little more complicated. Consider

$$\boldsymbol{dR} = \frac{\partial \boldsymbol{R}}{\partial e^i} de^i.$$

Define

$$\boldsymbol{d\Omega} = \boldsymbol{R}^T \boldsymbol{dR}$$

and the associated 1-form $\boldsymbol{d\omega}$ with components

$$(d\omega)^i = -\frac{1}{2}\epsilon^{ijk}(d\Omega)_{jk}, \tag{24}$$

with $\epsilon_{ijk}$ the components of the Levi-Civita tensor, i.e. $\epsilon_{ijk}$ is completely antisymmetric in $ijk$ and $\epsilon_{123} = 1$. Note that $\boldsymbol{d\omega}$ is *not* the differential of $\boldsymbol{\omega}$, i.e. $d\omega$ is one symbol. Similarly for $\boldsymbol{d\Omega}$. This is why we write $(d\omega)^i$ instead of $d\omega^i$. The inverse of eq. 24 reads

$$(d\Omega)_{ij} = -\epsilon_{ijk}(d\omega)^k.$$

The 3 1-forms $\boldsymbol{d}e^i$ form a coordinate basis $\boldsymbol{de}^i = \boldsymbol{d}e^i$ for $\mathcal{T}SO(3)$. An alternative (non-coordinate) basis is given by

$$(\boldsymbol{d\omega})^i = \Lambda^i_j \boldsymbol{de}^j,$$

where

$$\Lambda^i_j = -\frac{1}{2}\epsilon_{ikl} R_{mk} \frac{\partial R_{ml}}{\partial e^j}.$$

Note that we use the Einstein summation convention for repeated indices, even if they are both upper or lower. This is allowed since the indices refer to Euclidean space, and simplifies some expressions.

[3] This means the following. Consider an infinitesimal movement in the $i$ direction in a manifold $\mathcal{M}$ by amount $dx^i$, i.e. moving from the point labeled by the coordinate $\mathbf{x}$ to the point labeled by the coordinate

$$\mathbf{x} + \begin{pmatrix} 0 \\ \vdots \\ dx^i \\ \vdots \\ 0 \end{pmatrix}.$$

This looks like an infinitesimal line segment, or an infinitesimal vector. There are $n$ of these (with $n$ the dimension of the manifold), generated by $dx^i$ for $i = 1, \ldots, n$. We denote these $n$ infinitesimal vectors by $\boldsymbol{dx}^i$, and expand any infinitesimal vector (also called a 1-form) in them. This is called the coordinate basis for the tangent space $\mathcal{T}\mathcal{M}$. Any linear combination of the $\boldsymbol{dx}^i$ can also function as a basis, but cannot in general be constructed from a coordinate system as above. The angular velocity basis on $SO(3)$ is a non-coordinate basis, for example.

This is called the angular velocity basis. So a 1-form $\boldsymbol{v}$ can be written as

$$\boldsymbol{v} = \hat{v}_i \boldsymbol{de}^i = v_i \boldsymbol{d\omega}^i,$$

where $\hat{v}_i$ are the components of $\boldsymbol{v}$ in the coordinate basis, and $v_i$ are its components in the angular velocity basis. They are related by

$$\hat{v}_i = \Lambda_i^j v_j.$$

We define

$$\frac{\delta}{\delta\omega^j} = (\Lambda^{-1})_{ij} \frac{\partial}{\partial e^i}. \tag{25}$$

Note that this operator is not a partial derivative, which reflects the fact that $\boldsymbol{d\omega}$ is not a total derivative. The operator $d\omega^i \frac{\delta}{\delta\omega^i}$, when applied to a function on $SO(3)$ yields the change in that function under an infinitesimal rotation of angle $\sqrt{\boldsymbol{d\omega}^T \boldsymbol{d\omega}}$ around the axis defined by $\boldsymbol{d\omega}$. A useful formula is

$$\left(\frac{\delta R}{\delta\omega^j}\right)_{ln} = -R_{lm}\epsilon_{mnj}. \tag{26}$$

Other useful formulae are

$$\begin{aligned} \hat{\boldsymbol{\Omega}} &= \boldsymbol{R}^T \boldsymbol{\Omega} \boldsymbol{R} \\ \hat{\boldsymbol{\omega}} &= \boldsymbol{R}^T \boldsymbol{\omega}, \end{aligned} \tag{27}$$

which relates rotations of $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$, and

$$\boldsymbol{R}(\boldsymbol{dR})^T = \boldsymbol{R}\boldsymbol{d\Omega}\boldsymbol{R}^T. \tag{28}$$

We shall now use the basis

$$\boldsymbol{dp} = \begin{pmatrix} \boldsymbol{dt} \\ \boldsymbol{d\omega} \end{pmatrix} \tag{29}$$

on $\mathcal{T}SE(3)$, unless otherwise indicated. These are sometimes called "twist coordinates" (e.g. [10]).

## 9.3 Metrics on $SE(3)$

The length $ds^2$ of an infinitesimal vector $\boldsymbol{dq}$ (in any kind of basis) is taken to be

$$ds^2 = h_{ab} dq^a dq^b,$$

where indices run from $1, \ldots, 6$. The metric tensor $\boldsymbol{h}$ is assumed to be positive definite, so the manifold is Riemannian. Several choices can be made for $\boldsymbol{h}$. Two important choices, the kinematic and the dynamic metrics, will be discussed now.

### 9.3.1 Kinematic Metric

The distance $ds^2$ is taken as the sum of the Euclidean displacement of the center of mass (i.e. $\boldsymbol{dt}^T\boldsymbol{dt}$) and the square of the angle of rotation multiplied with some constant with the dimension *length*$^2$ (which is of course inevitable, as rotations and translations have different dimensions).

The resulting line element is ($\lambda$ being a length scale)

$$ds^2 = \delta_{ij}dt^i dt^j + \frac{\lambda^2}{2}\mathrm{tr}(\frac{\partial \boldsymbol{R}^T}{\partial e^n}\frac{\partial \boldsymbol{R}}{de^m})de^n de^m, \tag{30}$$

which takes the following form in the angular velocity basis,

$$ds^2 = \delta_{ij}dt^i dt^j + \lambda^2\delta_{ij}d\omega^i d\omega^j. \tag{31}$$

### 9.3.2 Dynamic Metric

The dynamic metric is obtained by taking $ds^2/d\tau^2$ ( with $\tau$ denoting time) to be twice the kinetic energy of the rigid body. In this case, in the angular velocity basis,

$$\boldsymbol{h} = \begin{pmatrix} m\mathbb{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix},$$

and

$$ds^2 = m\boldsymbol{dt}^T\boldsymbol{dt} + \boldsymbol{d\omega}^T\boldsymbol{I}\boldsymbol{d\omega}. \tag{32}$$

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