Coordinate Descent and Ascent Methods

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Motivation

- Projected-Gradient Methods
  - ✓ Rewrite non-smooth problem as smooth constrained problem:

$$\min_{x \in \mathcal{C}} f(x)$$

  - ✗ Only handles 'simple' constraints, e.g., bound constraints.
  - → Franke-Wolfe Algorithm: minimize linear function over $\mathcal{C}$.
- Proximal-Gradient Methods
  - ✓ Generalizes projected-gradient:

$$\min_{x} f(x) + r(x),$$

    where f is smooth, r is general convex function (proximable).
  - ✗ Dealing with $r(x) = \phi(Ax)$ difficult, even when ϕ simple.
  - → Alternating Direction Method of Multipliers

✴ **TODAY**: We focus on coordinate descent, which is for the case where r is separable and f has some special structure.

Coordinate Descent Methods

- Suitable for large-scale optimization (dimension d is large):
  - Certain smooth (unconstrained) problems.
  - Non-smooth problems with *separable* constraints/regularizers.
    - e.g., ℓ_1-regularization, bound constraints

✴ Faster than gradient descent if iterations d times cheaper.

Problems Suitable for Coordinate Descent

Coordinate update d times faster than gradient update for:

$$h_1(x) = f(Ax) + \sum_{i=1}^{d} g_i(x_i), \quad \text{or} \quad h_2(x) = \sum_{i \in V} g_i(x_i) + \sum_{(i,j) \in E} f_{ij}(x_i, x_j)$$

- f and f_{ij} smooth, convex
- A is a matrix
- $\{V, E\}$ is a graph
- g_i general non-degenerate convex functions

Examples h_1: least squares, logistic regression, lasso, ℓ_2-norm SVMs.

$$\text{e.g.,} \quad \min_{x \in \mathbb{R}^d} \frac{1}{2}\|Ax - b\|^2 + \lambda \sum_{i=1}^{d} |x_i|.$$

Examples h_2: quadratics, graph-based label prop., graphical models.

$$\text{e.g.,} \quad \min_{x \in \mathbb{R}^d} \frac{1}{2} x^T A x + b^T x = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij} x_i x_j + \sum_{i=1}^{d} b_i x_i.$$

Notation and Assumptions

We focus on the convex optimization problem

$$\min_{x \in \mathbb{R}^d} f(x)$$

- ∇f coordinate-wise L-Lipschitz continuous

$$|\nabla_i f(x + \alpha e_i) - \nabla_i f(x)| \leq L|\alpha|$$

- f μ-strongly convex, i.e.,

$$x \mapsto f(x) - \frac{\mu}{2}\|x\|^2$$

  is convex for some $\mu > 0$.

- If f is twice-differentiable, equivalent to

$$\nabla^2_{ii}\, f(x) \leq L, \qquad \nabla^2\, f(x) \succeq \mu \mathbb{I}.$$

Coordinate Descent vs. Gradient Descent

$$\boxed{x^{k+1} = x^k - \frac{1}{L}\nabla_{i_k} f(x^k) e_{i_k}} \qquad \boxed{x^{k+1} = x^k - \alpha \nabla f(x^k)}$$

- Global convergence rate for randomized i_k selection [Nesterov]:

$$\mathbb{E}[f(x^{k+1})] - f(x^*) \leq \left(1 - \frac{\mu}{Ld}\right)[f(x^k) - f(x^*)]$$

- Global convergence rate for gradient descent:

$$f(x^{k+1}) - f(x^*) \leq \left(1 - \frac{\mu}{L_f}\right)[f(x^k) - f(x^*)]$$

- Since $Ld \geq L_f \geq L$, coordinate descent is slower *per iteration*, but d coordinate iterations are faster than one gradient iteration.

Proximal Coordinate Descent

$$\min_{x \in \mathbb{R}^d} F(x) \equiv f(x) + \sum_i g_i(x_i)$$

where f is smooth and g_i might be non-smooth.

- e.g., ℓ_1-regularization, bound constraints

- Apply proximal-gradient style update,

$$x^{k+1} = \textbf{prox}_{\frac{1}{L} g_{i_k}} \left[x^k - \frac{1}{L} \nabla_{i_k} f(x^k) e_{i_k} \right]$$

where

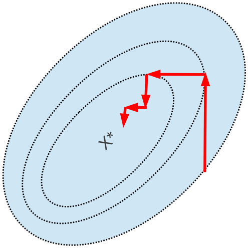
$$\textbf{prox}_{\alpha g}[y] = \underset{x \in \mathbb{R}^d}{\text{argmin}} \frac{1}{2} \|x - y\|^2 + \alpha g(x).$$

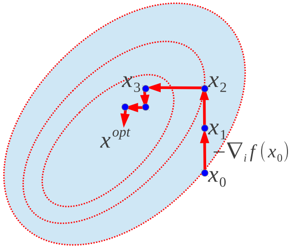
- Convergence for randomized i_k:

$$\mathbb{E}[F(x^{k+1})] - F(x^*) \leq \left(1 - \frac{\mu}{dL}\right) \left[F(x^k) - F(x^*)\right]$$

Sampling Rules

- **Cyclic**: Cycle through i in order, i.e., $i_1 = 1, i_2 = 2$, etc.
- **Uniform random**: Sample i_k uniformly from $\{1, 2, \ldots, d\}$.
- **Lipschitz sampling**: Sample i_k proportional to L_i.
- **Gauss-Southwell**: Select $i_k = \text{argmax}_i \, |\nabla_i f(x^k)|$.
- **Gauss-Southwell-Lipschitz**: Select $i_k = \text{argmax}_i \, \frac{|\nabla_i f(x^k)|}{\sqrt{L_i}}$.



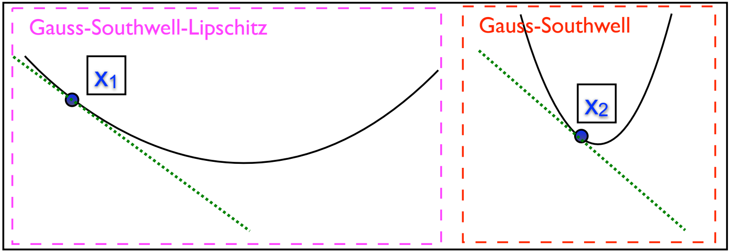


Gauss-Southwell Rules

GSL: $\text{argmax}_i \frac{|\nabla_i f(x^k)|}{\sqrt{L_i}}$

GS: $\text{argmax}_i |\nabla_i f(x^k)|$

Intuition: if gradients are similar, more progress if L_i is small.



- Feasible for problems where A is super sparse or for a graph with mean nNeighbours approximately equals maximum nNeighbours.
- Show GS and GSL up to d times faster than randomized by measuring strong convexity in the 1-norm or L-norm, respectively.

Exact Optimization

$$\boxed{x^{k+1} = x^k - \alpha_k \nabla_{i_k} f(x^k) e_{i_k}, \quad \text{for some } i_k}$$

- Exact coordinate optimization chooses the step size minimizing f:

$$f(x^{k+1}) = \min_{\alpha} \{ f(x^k - \alpha \nabla_{i_k} f(x^k) e_{i_k}) \}$$

- Alternatives:
  - Line search: find $\alpha > 0$ such that $f(x^k - \alpha \nabla_{i_k} f(x^k) e_{i_k}) < f(x^k)$.
  - Select step size based on global knowledge of f, e.g., $1/L$.

Stochastic Dual Coordinate Ascent

- Suitable for large-scale supervised learning (large # loss functions n):
  - Primal formulated as sum of convex loss functions.
  - Operates on the dual.

✴ Achieves faster linear rate than SGD for smooth loss functions.

✴ Theoretically equivalent to SSG for non-smooth loss functions.

The Big Picture...

- Stochastic Gradient Descent (SGD):
  - ✓ Strong theoretical guarantees.
  - ✗ Hard to tune step size (requires $\alpha \to 0$).
  - ✗ No clear stopping criterion (Stochastic Sub-Gradient method (SSG)).
  - ✗ Converges fast at first, then slow to more accurate solution.
- Stochastic Dual Coordinate Ascent (SDCA):
  - ✓ Strong theoretical guarantees that are comparable to SGD.
  - ✓ Easy to tune step size (line search).
  - ✓ Terminate when the duality gap is sufficiently small.
  - ✓ Converges to accurate solution faster than SGD.

Primal Problem

$$\text{(P)} \quad \min_{w \in \mathbb{R}^d} \quad P(w) = \frac{1}{n} \sum_{i=1}^{n} \phi_i(w^T x_i) + \frac{\lambda}{2} \|w\|^2$$

where $x_1, \ldots, x_n$ vectors in $\mathbb{R}^d$, $\phi_1, \ldots, \phi_n$ sequence of scalar convex functions, $\lambda > 0$ regularization parameter.

Examples: (for given labels $y_1, \ldots, y_n \in \{-1, 1\}$)

- **SVMs**: $\phi_i(a) = \max\{0, 1 - y_i a\}$ (L-Lipschitz)
- **Regularized logistic regression**: $\phi_i(a) = \log(1 + \exp(-y_i a))$
- **Ridge regression**: $\phi_i(a) = (a - y_i)^2$ (smooth)
- **Regression**: $\phi_i(a) = |a - y_i|$
- **Support vector regression**: $\phi_i(a) = \max\{0, |a - y_i| - \nu\}$

Dual Problem

$$\text{(P)} \quad \min_{w \in \mathbb{R}^d} \quad P(w) = \frac{1}{n} \sum_{i=1}^{n} \phi_i(w^T x_i) + \frac{\lambda}{2} \|w\|^2$$

$$\text{(D)} \quad \max_{\alpha \in \mathbb{R}^n} \quad D(\alpha) = \frac{1}{n} \sum_{i=1}^{n} -\phi_i^*(-\alpha_i) - \frac{\lambda}{2} \left\| \frac{1}{\lambda n} \sum_{i=1}^{n} \alpha_i x_i \right\|^2$$

where $\phi_i^*(u) = \max_z (zu - \phi_i(z))$ is the convex conjugate of ϕ_i.

- Different dual variable associated with each example in training set.

Duality Gap

$$\text{(P)} \quad \min_{w \in \mathbb{R}^d} \quad P(w) = \frac{1}{n}\sum_{i=1}^{n} \phi_i(w^T x_i) + \frac{\lambda}{2}\|w\|^2$$

$$\text{(D)} \quad \max_{\alpha \in \mathbb{R}^n} \quad D(\alpha) = \frac{1}{n}\sum_{i=1}^{n} -\phi_i^*(-\alpha_i) - \frac{\lambda}{2}\left\|\frac{1}{\lambda n}\sum_{i=1}^{n} \alpha_i x_i\right\|^2$$

- Define $w(\alpha) = \frac{1}{\lambda n}\sum_{i=1}^{n} \alpha_i x_i$, then it is known that $w(\alpha^*) = w^*$.
- $P(w^*) = D(\alpha^*)$, which implies $P(w) \geq D(\alpha)$ for all w, α.
- Duality gap is defined by $P(w(\alpha)) - D(\alpha)$:
  - → Upper bound on the primal sub-optimality: $P(w(\alpha)) - P(w^*)$.

SDCA Algorithm

(1) Select a training example i at random.

(2) Do exact line search in the dual, i.e., find $\Delta\alpha_i$:

$$\text{maximize } -\phi_i^*(-(\alpha_i^{(t-1)}+\Delta\alpha_i))-\frac{\lambda n}{2}\|w^{(t-1)}+(\lambda n)^{-1}\Delta\alpha_i x_i\|^2$$

(3) Update the dual variable $\alpha^{(t)}$ and the primal variable $w^{(t)}$:

$$\alpha^{(t)} \leftarrow \alpha^{(t-1)} + \Delta\alpha_i e_i$$

$$w^{(t)} \leftarrow w^{(t-1)} + (\lambda n)^{-1}\Delta\alpha_i x_i$$

- ✴ Terminate when duality gap is sufficiently small.
- ✴ There are ways to get the rate without a line search that use the primal gradient/subgradient directions.

SGD vs. SDCA

- Alternative to SGD/SSG.
- If primal is smooth, get faster linear rate on duality gap than SGD.
- If primal is non-smooth, get sublinear rate on duality gap.
  - → SDCA has similar update to SSG on primal.
  - ✗ SSG sensitive to step-size.
  - ✓ Do line search in the dual with coordinate ascent.
- SDCA may not perform as well as SGD for first few epochs (full pass)
  - SGD takes larger step size than SDCA earlier on, helps performance.
  - Using modified SGD on first epoch followed by SDCA obtains faster convergence when regularization parameter $\lambda >> \log(n)/n$.

Comparison of Rates

Lipschitz loss function (e.g., hinge-loss, $\phi_i(a) = \max\{0, 1 - y_i a\}$):

Algorithm	convergence type	rate
SGD	primal	$\tilde{O}(1/(\lambda\varepsilon_p))$
online EG (Collins et al., 2008) (for SVM)	dual	$\tilde{O}(n/\varepsilon_d)$
Stochastic Frank-Wolfe (Lacoste-Julien et al., 2012)	primal-dual	$\tilde{O}(n + 1/(\lambda\varepsilon))$
SDCA	primal-dual	$\tilde{O}(n + 1/(\lambda\varepsilon))$ or faster

Smooth loss function (e.g., ridge-regression, $\phi_i(a) = (a - y_i)^2$):

Algorithm	convergence type	rate
SGD	primal	$\tilde{O}(1/(\lambda\varepsilon_p))$
online EG (Collins et al., 2008) (for LR)	dual	$\tilde{O}((n + 1/\lambda)\log(1/\varepsilon_d))$
SAG (Le Roux et al., 2012) (assuming $n \geq 8/(\lambda\gamma)$)	primal	$\tilde{O}((n + 1/\lambda)\log(1/\varepsilon_p))$
SDCA	primal-dual	$\tilde{O}((n + 1/\lambda)\log(1/\varepsilon))$

- Even if α is ε_d-sub-optimal in the dual, i.e.,

$$D(\alpha) - D(\alpha^*) \leq \varepsilon_d,$$

  the primal solution $w(\alpha)$ might be far from optimal.

- Bound on duality-gap is upper bound on primal sub-optimality.

- Recent results have shown improvements upon some of the rates in the above tables.

Accelerated Coordinate Descent

- Inspired by Nesterov's accelerated gradient method.
- Uses multi-step strategy, carries momentum from previous iterations.
- For accelerated randomized coordinate descent:
  - e.g., for a convex function: $O(1/k^2)$ rate, instead of $O(1/k)$.

Block Coordinate Descent

$$\boxed{x^{k+1} = x^k - \frac{1}{L}\nabla_{b_k} f(x^k) e_{b_k}, \quad \text{for some block of indices } b_k}$$

- Search along coordinate hyperplane.
- Fixed blocks, adaptive blocks.
- Randomized/proximal CD easily extended to the block case.
    - For proximal case, choice of block must be consistent with block-separable structure of regularization function g.

Parallel Coordinate Descent

- **Synchronous parallelism**:
  - Divide iterate updates between processors (block), followed by synchronization step.
- **Asynchronous parallelism**:
  - Each processor:
    - Has access to x.
    - Chooses an index i, loads components of x that are needed to compute the gradient component $\nabla_i f(x)$, then updates the ith component x_i.
    - No attempt to coordinate or synchronize with other processors.
    - Always using 'stale' x: convergence results restrict how stale.

Discussion

- **Coordinate Descent**:
  - Suitable for large-scale optimization (when d is large).
  - Operates on the primal objective.
  - Faster than gradient descent if iterations d times cheaper.
- **Stochastic Dual Coordinate Ascent**:
  - Suitable for large-scale optimization (when n is large).
  - Operates on the dual objective.
  - If primal is smooth, obtains faster linear rate on duality gap than SGD.
  - If primal is non-smooth, obtain sublinear rate on duality gap.
    - → Do line search in the dual with coordinate ascent.
  - Outperforms SGD when relatively high solution accuracy is required.
  - Terminate when duality-gap is sufficiently small.
- **Variations**: acceleration, block, parallel.