UGM Crash Course:
Conditional Inference and Cutset Conditioning

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August 19th, 2015

Conditional UGM

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  - i.e., we have observations, $P(x_2, x_3|x_1) = \frac{P(x_1,x_2,x_3)}{P(x_1)}$.

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  - **Demo 3 (Tree UGM)**:
    - What happens to the rest of the water system if the source node is in state 4? If we use a model with multiple sources and observe that one of the nodes is in state 4, which source is more likely to also be in an unsafe state?

Undirected Graphical Models (UGMs)

- We are focusing on pairwise UGMs with discrete states,

$$P(X) = \frac{\prod_{i=1}^{N} \phi_i(x_i) \prod_{(i,j)\in E} \phi_{i,j}(x_i, x_j)}{Z},$$

  where we've decomposed X into 'parts' $x_i \in \{1, 2, \ldots, S\}$.
- We are considering exact methods for 3 tasks:
  1. Decoding: Compute the optimal configuration,

$$\max_X P(X).$$

  2. Inference: Compute partition function and marginals,

$$Z(X) = \sum_{X'} P(X'), \quad P(X_i = s) = \sum_{X' | X_i = s} P(X').$$

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- With conditioning, think of reducing problem dimension.

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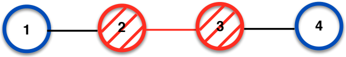
  - Tree inference applies, just apply it to each tree separately.

$$P(x_1, x_4 | x_2, x_3)$$



$$= \frac{P(x_1, x_2, x_3, x_4)}{P(x_2, x_3)}$$

$$= \frac{\frac{1}{Z}\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\phi_{1,2}(x_1, x_2)\phi_{2,3}(x_2, x_3)\phi_{3,4}(x_3, x_4)}{\sum_{x_1', x_4'} \frac{1}{Z}\phi_1(x_1')\phi_2(x_2)\phi_3(x_3)\phi_4(x_4')\phi_{1,2}(x_1', x_2)\phi_{2,3}(x_2, x_3)\phi_{3,4}(x_3, x_4')}$$

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$$= \frac{\phi_1(x_1)\phi_4(x_4)\phi_{1,2}(x_1, x_2)\phi_{3,4}(x_3, x_4)}{\sum_{x_1', x_4'} \phi_1(x_1')\phi_4(x_4')\phi_{1,2}(x_1', x_2)\phi_{3,4}(x_3, x_4')}$$

$$= \frac{\tilde{\phi}_1(x_1)\tilde{\phi}_4(x_4)}{\sum_{x_1', x_4'} \tilde{\phi}_1(x_1')\tilde{\phi}_4(x_4')},$$

where $\tilde{\phi}_1(x_1) = \phi_1(x_1)\phi_{1,2}(x_1, x_2)$, and $\tilde{\phi}_4(x_4) = \phi_4(x_4)\phi_{1,4}(x_1, x_4)$.

- Absorb potentials coming from 1-2 edge and 3-4 edge.

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$$
... = \frac{\phi_1(x_1)\phi_4(x_4)\phi_{1,2}(x_1,x_2)\phi_{3,4}(x_3,x_4)}{\sum_{x_1',x_4'}\phi_1(x_1')\phi_4(x_4')\phi_{1,2}(x_1',x_2)\phi_{3,4}(x_3,x_4')} = \frac{\tilde{\phi}_1(x_1)\tilde{\phi}_4(x_4)}{\sum_{x_1',x_4'}\tilde{\phi}_1(x_1')\tilde{\phi}_4(x_4')}
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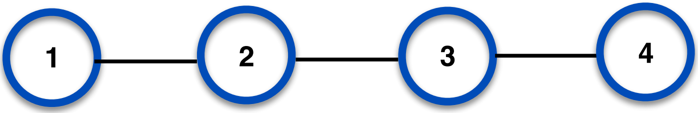
  4. No terms left depending on observed nodes.



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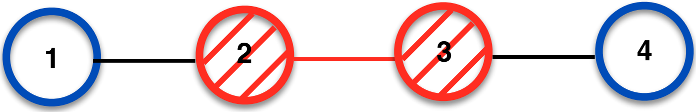

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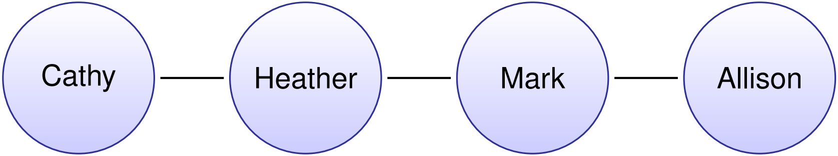

	Node Marginals	
Student	**Right**	**Wrong**
Cathy	0.36	0.64
Heather	0.84	0.16
Mark	0.49	0.51
Allison	0.88	0.12

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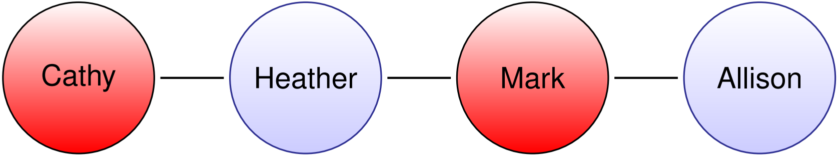
Assume Cathy and Mark get the question wrong.



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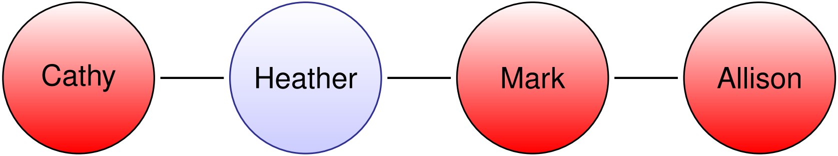
Assume Cathy, Mark **AND** Allison get the question wrong.



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- Heather is ‘independent’ of Allison.

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  - In other words, decoding can often be misleading.

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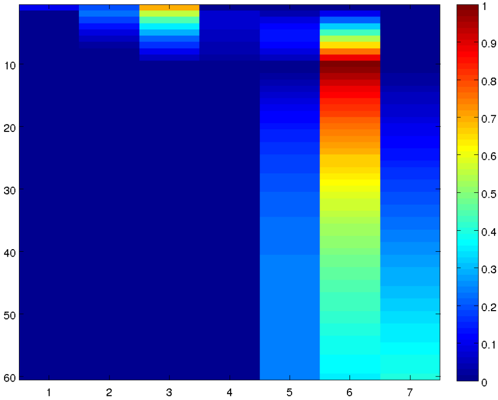

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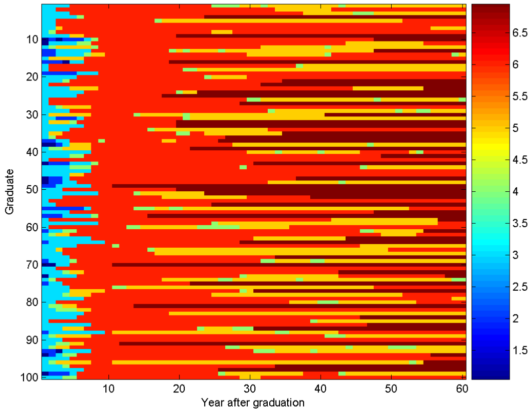

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- However, often our model may have loops.
- Decoding/inference/sampling for general UGMs with loops NP-hard.
- We can exploit graph structure to yield polynomial-time algorithm.

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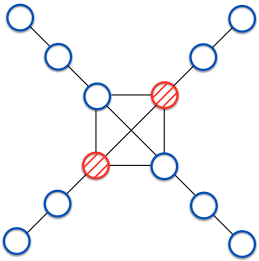

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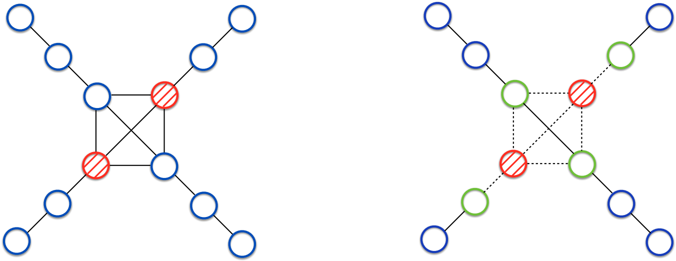

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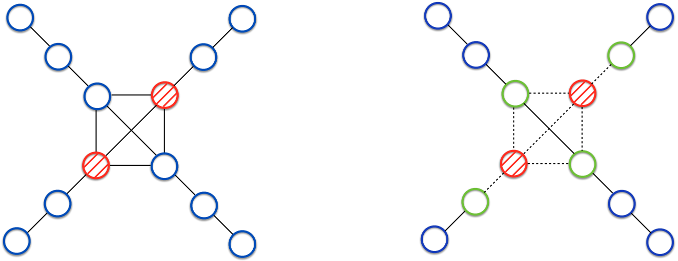

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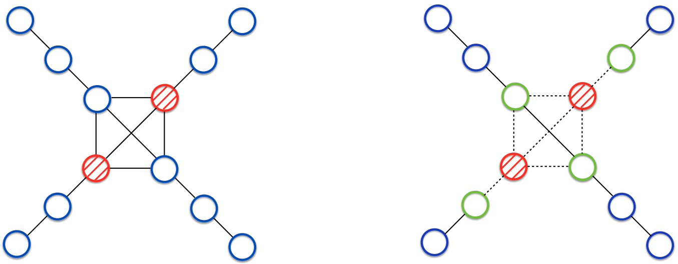

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- As long as we condition on at least two nodes in the above example, the loop will be broken in the *conditional* UGM.

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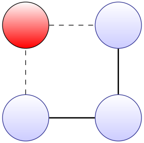
- Find the optimal decoding of the forest, for every possible assignment to the cutset variables.
- Compute the potential of the cutset variables combined with the best decoding given the cutset, and return the one with the highest potential.

Cutset Conditioning for Decoding

1. Select a cutset.



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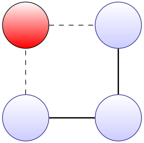
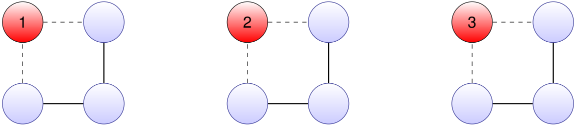


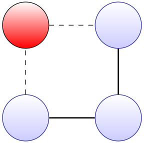
2. Find optimal decoding for each value of cutset: $\{1, 2, 3\}$.



$\color{red}{1} - 3 - 2 - 2$ $\quad$ $\color{red}{2} - 3 - 1 - 3$ $\quad$ $\color{red}{3} - 1 - 1 - 1$

Cutset Conditioning for Decoding

1. Select a cutset.



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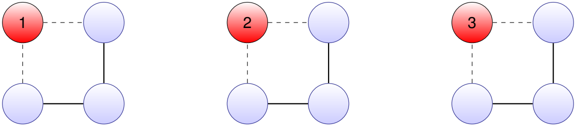
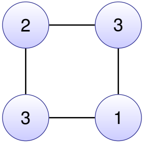
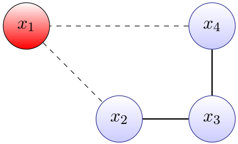


$\color{red}{1} - 3 - 2 - 2 \qquad \color{red}{2} - 3 - 1 - 3 \qquad \color{red}{3} - 1 - 1 - 1$

3. Compute potentials and return one with highest potential.



Cutset Conditioning for Inference



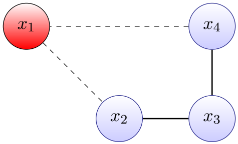
- **Normalizing constant**: Add up normalizing constants from the conditional UGMs (multiplied by the node and edge potentials that are missing from the conditional UGM) under each possible value s_i of the cutset variable x_1, i.e.,

$$Z = \sum_i \tilde{Z}(s_i) \cdot \phi_1(s_i),$$

where

$$\tilde{Z}(s_i) = \sum_{x'_2, x'_3, x'_4} \tilde{\phi}_2^{(i)}(x'_2)\phi_3(x'_3)\tilde{\phi}_4^{(i)}(x'_4)\phi_{2,3}(x'_2, x'_3)\phi_{3,4}(x'_3, x'_4).$$

Cutset Conditioning for Inference



- **Marginals**: Compute conditional marginals under each assignment s_i to the cutset variables ($x_1 := X_{cutset}$):

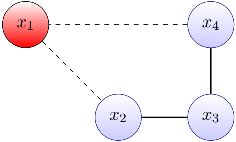
$$P(X_{cutset} = s_i)$$

$$= \sum_{X' | X_{cutset} = s_i} P(X') \quad \left(= \sum_{X'} P(X') \, \mathcal{I}[x'_1 = s_i] \right)$$

$$= \frac{\phi_1(s_i)}{Z} \sum_{x'_2, x'_3, x'_4} \phi_{1,2}(s_i, x'_2) \phi_{1,4}(s_i, x'_4) \prod_{j \neq 1} \phi_j(x'_j) \prod_{\substack{j,k \in E, \\ j \neq 1}} \phi_{j,k}(x'_j, x'_k)$$

$$= \frac{\phi_1(s_i) \tilde{Z}(s_i)}{Z}$$

Cutset Conditioning for Inference



- **Conditionals**: Compute conditionals under each assignment s_i to the cutset variables ($x_1 := X_{cutset}$):

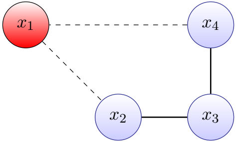
$$P(x_2, x_3, x_4 | x_1 = s_i)$$

$$= \frac{P(s_i, x_2, x_3, x_4)}{P(X_{cutset} = s_i)}$$

$$= \frac{\phi_1(s_i)\phi_{1,2}(s_i, x_2)\phi_{1,4}(s_i, x_4) \prod_{i \neq 1} \phi_i(x_i) \prod_{\substack{(i,j) \in E \\ i \neq 1}} \phi_{i,j}(x_i, x_j)}{\phi_1(s_i)\tilde{Z}(s_i)}$$

$$= \frac{\tilde{\phi}_2(x_2)\phi_3(x_3)\tilde{\phi}_4(x_4)\phi_{2,3}(x_2, x_3)\phi_{3,4}(x_3, x_4)}{\tilde{Z}(s_i)}$$

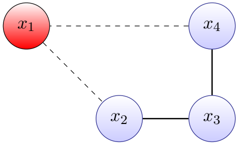
Cutset Conditioning for Inference



- To compute marginals outside of the cutset:

$$P(x_2, x_3, x_4) = \sum_i P(x_1 = s_i) P(x_2, x_3, x_4 | x_1 = s_i)$$

Cutset Conditioning for Inference



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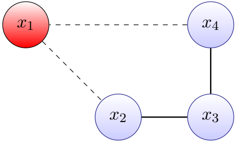
- This is summing two quantities (the cutset marginals and the conditionals) that we already know how to calculate over all assignments of the cutset variable.

Cutset Conditioning for Sampling



- **Sampling**: First compute the weights of the different possible values of the cutset variables, $\tilde{Z}(s_i)$.

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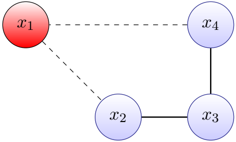

- **Sampling**: First compute the weights of the different possible values of the cutset variables, $\tilde{Z}(s_i)$.
- Generate random value, normalized distribution of weights:

$$x_1' \sim P\left(\frac{\tilde{Z}(s_i)}{Z}\right)$$

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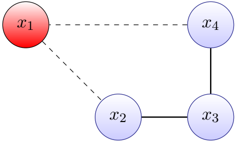

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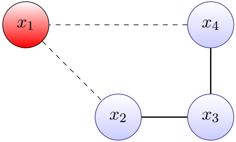

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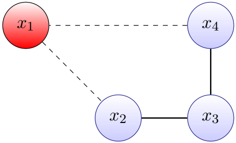

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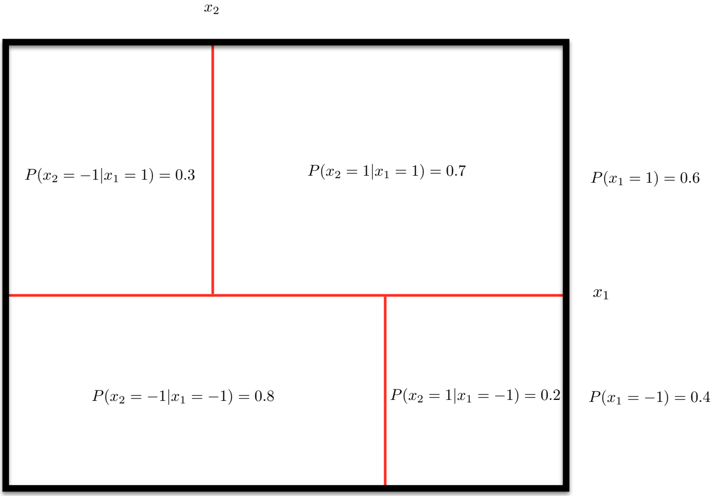
  - $P(x) = P(x_2, x_3, x_4|x_1)P(x_1)$
  - 2 bouts of sampling (ancestral sampling)

Ancestral sampling involves sampling proportional to a ‘new normalized’ distribution:

$$P(x_2 = 1, x_1 = -1) = P(x_2 = 1 | x_1 = -1) P(x_1 = -1) = 0.08$$



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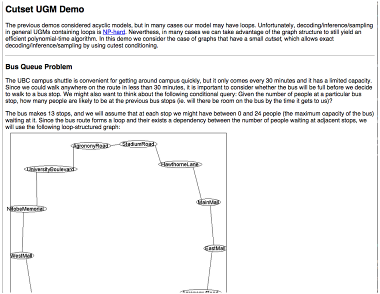
  - For example, if we want to do decoding and the cutset needed to make the conditional UGM a forest has k elements, we will need to do decoding in s^k forests (where s is the number of states).
- Thus, cutset conditioning is only practical when the graph structure allows us to find a small cutset.

Homework

- Go through the Cutset and Junction demos in UGM.



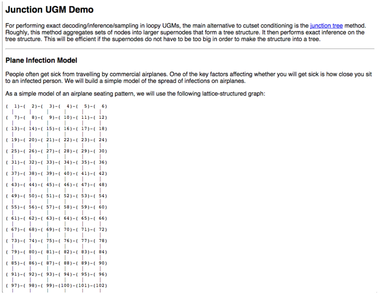
Cutset UGM Demo

The previous demos considered acyclic models, but in many cases our model may have loops. Unfortunately, decoding/inference/sampling in general UGMs containing loops is NP-hard. Neverthess, in many cases we can take advantage of the graph structure to still yield an efficient polynomial-time algorithm. In this demo we consider the case of graphs that have a small *cutset*, which allows exact decoding/inference/sampling by using cutest conditioning.

Bus Queue Problem

The UBC campus shuttle is convenient for getting around campus quickly, but it only comes every 30 minutes and it has a limited capacity. Since we could walk anywhere on the route in less than 30 minutes, it is important to consider whether the bus will be full before we decide to walk to a bus stop. We might also want to think about the following conditional query: Given the number of people at a particular bus stop, how many people are likely to be at the previous bus stops (ie. will there be room on the bus by the time it gets to us)?

The bus makes 13 stops, and we will assume that at each stop we might have between 0 and 24 people (the maximum capacity of the bus) waiting at it. Since the bus route forms a loop and their exists a dependency between the number of people waiting at adjacent stops, we will use the following loop-structured graph:



Junction UGM Demo

For performing exact decoding/inference/sampling in loopy UGMs, the main alternative to cutset conditioning is the junction tree method. Roughly, this method aggregates sets of nodes into larger supernodes that form a tree structure. It then performs exact inference on the tree structure. This will be efficient if the supernodes do not have to be too big in order to make the structure into a tree.

Plane Infection Model

People often get sick from travelling by commercial airplanes. One of the key factors affecting whether you will get sick is how close you sit to an infected person. We will build a simple model of the spread of infections on airplanes.

As a simple model of an airplane seating pattern, we will use the following lattice-structured graph:

```
(  1)-(  2)-(  3)-(  4)-(  5)-(  6)
(  7)-(  8)-(  9)-( 10)-( 11)-( 12)
( 13)-( 14)-( 15)-( 16)-( 17)-( 18)
( 19)-( 20)-( 21)-( 22)-( 23)-( 24)
( 25)-( 26)-( 27)-( 28)-( 29)-( 30)
( 31)-( 32)-( 33)-( 34)-( 35)-( 36)
( 37)-( 38)-( 39)-( 40)-( 41)-( 42)
( 43)-( 44)-( 45)-( 46)-( 47)-( 48)
( 49)-( 50)-( 51)-( 52)-( 53)-( 54)
( 55)-( 56)-( 57)-( 58)-( 59)-( 60)
( 61)-( 62)-( 63)-( 64)-( 65)-( 66)
( 67)-( 68)-( 69)-( 70)-( 71)-( 72)
( 73)-( 74)-( 75)-( 76)-( 77)-( 78)
( 79)-( 80)-( 81)-( 82)-( 83)-( 84)
( 85)-( 86)-( 87)-( 88)-( 89)-( 90)
( 91)-( 92)-( 93)-( 94)-( 95)-( 96)
( 97)-( 98)-( 99)-(100)-(101)-(102)
```

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  - The conditional UGM will be defined on the subgraph of the original graph corresponding to the unobserved nodes.
  - Subgraphs may have simpler structure than original graph.
- We can deal with loops using cutset conditioning.
  - If cutset is small, exactly decode/infer/sample.
  - If cutset is large, cutset conditioning is not practical.