# Spectral Methods

# Outline

- Example Applications
- Spectral Methods - PCA
- Latent Variable Models - Gaussian Mixture Model
- Tensor factorization
- Eigen Analysis
- Conclusion

# Applications

- Gaussian mixture models
- Hidden Markov Models
- Community Detection
- Topic Models
- Recommender systems
- Feature Learning

# Latent Variable Models

Difficulties in learning:

- Identifiability
- Maximum likelihood is NP-hard
- Practice:EM, Variational Bayes have no consistency guarantees.
- Efficient computational and sample complexities

# PCA - Spectral method on covariance matrices

## Optimization problem

For (centered) points $x_i \in \mathbb{R}^d$ , find projection $P$ with $\text{Rank}(P) = k$ s.t.

$$\boxed{\min_{P \in \mathbb{R}^{d \times d}} \frac{1}{n} \sum_{i \in [n]} \|x_i - Px_i\|^2.}$$

**Result:** If $S = \text{Cov}(X)$ and $S = U \Lambda U^\top$ is eigen decomposition, we have $P = U_{(k)} U_{(k)}^\top$, where $U_{(k)}$ are top-$k$ eigen vectors.

# Gaussian mixture models

- $k$ Gaussians: each sample is $x = Ah + z$.
- $h \in [e_1, \ldots, e_k]$, the basis vectors. $\mathbb{E}[h] = w$.
- $A \in \mathbb{R}^{d \times k}$: columns are component means.
- Let $\mu := Aw$ be the mean.
- $z \sim \mathcal{N}(0, \sigma^2 I)$ is white Gaussian noise.

# Gaussian mixture models

$$\boxed{\mathbb{E}[(x-\mu)(x-\mu)^\top] = \sum_{i\in[k]} w_i(a_i-\mu)(a_i-\mu)^\top + \sigma^2 I.}$$

Aim: Given the points x, learn A
Conventional Method: Expectation Maximization
Problem: Converges to local minima
Idea: Use higher order moments

# Higher order moments for GMM

For the GMM example,

$$\mathbb{E}[x \otimes x \otimes x] = \sum_i w_i a_i \otimes a_i \otimes a_i + \sigma^2 \sum_i (\mu \otimes e_i \otimes e_i + \ldots)$$

$$M_3 = \sum_i w_i a_i \otimes a_i \otimes a_i$$

$$M_2 = \sum_i w_i a_i \otimes a_i.$$

# Tensor factorization

Multilinear transformation of tensor

$$\boxed{M_3(B, C, D) := \sum_i w_i (B^\top a_i) \cdot (C^\top a_i) \cdot (D^\top a_i)}$$

If the columns of A are orthogonal,

$$M_3(I, a_1, a_1) = \sum_i w_i \langle a_i, a_1 \rangle^2 a_i = w_1 a_1$$

$a_i$ are eigenvectors of tensor $M_3$

# Whitening

Problem: A is not orthogonal in general

Solution:

Find whitening matrix $W$ s.t. $W^\top A = V$ is an orthogonal matrix.

$$T = M_3(W,W,W) = \sum_i w_i (W^\top a_i)^{\otimes 3} = \sum_{i \in [k]} w_i \cdot v_i \otimes v_i \otimes v_i$$

# Whitening

$$M_2 = U \mathsf{Diag}(\tilde{\lambda}) U^\top \quad W = U \mathsf{Diag}(\tilde{\lambda}^{-1/2})$$

V is an orthogonal matrix; T is an orthogonal tensor.

$$T(I, v_1, v_1) = \sum_i \lambda_i \langle v_i, v_1 \rangle^2 v_i = \lambda_1 v_1$$

$v_i$ are eigenvectors of tensor $T$.

# Tensor power method

- Randomly initialize the power method. Run to convergence to obtain $v$ with eigenvalue $\lambda$.
- Deflate: $T - \lambda v \otimes v \otimes v$ and repeat.

Is there convergence? Does the convergence depend on initialization?

# Matrix EigenAnalysis

Eigen vectors are fixed points: $Mv = \lambda v$

Uniqueness (Identifiability): Iff. $\lambda_i$ are distinct.

Power method: $\boxed{v \mapsto \frac{M(I,v)}{\|M(I,v)\|}.}$

$v_1$ is the only local optimum

Let initialization $v = \sum_i c_i v_i$.

If $c_1 \neq 0$, power method converges to $v_1$

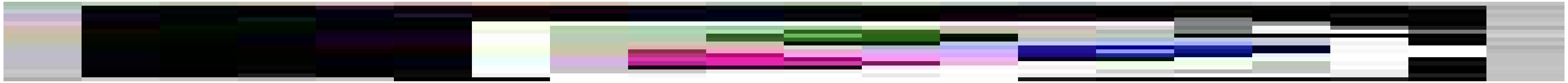
# Tensor EigenAnalysis

Bad news:

- Decomposition may not always exist for general tensors.
- Finding the decomposition is NP hard in general

For an orthogonal tensor, no spurious local optima!

$\{v_i\}$ are the only local optima.



Converges to $v_i$ for which $v_i|c_i| = \max!$ could be any of them

# Tensor EigenAnalysis

- Matrix power method - Linear convergence;
- Tensor power method - Quadratic convergence
- Matrix power method: Requires gap between largest and second-largest eigenvalue
- Tensor power method: Requires gap between largest and second-largest $\lambda_i c_i$
- Tensor Power method - robust to noise

# Putting it together

- Gaussian mixture: $x = Ah + z$, where $\mathbb{E}[h] = w$.
- $z \sim \mathcal{N}(0, \sigma^2 I)$.

$$M_2 = \sum_i w_i a_i \otimes a_i, \quad M_3 = \sum_i w_i a_i \otimes a_i \otimes a_i.$$

- Obtain whitening matrix $W$ from SVD of $M_2$.
- Use $W$ for multilinear transform: $T = M_3(W, W, W)$.
- Find eigenvectors of $T$ through power method and deflation.

# Conclusion

- Good method for guaranteed convergence to global minima (not guaranteed by EM)
- Numerous applications to latent variable models
- Scalability issues: requires computing SVDs of large matrices. Storage and decomposition of large tensors. In practice: use SGD techniques. Don’t know if there are guarantees.
- Weak robustness results
- Higher sample complexity