Basics Concepts of Riemannian Manifolds

MLRG summer, 2021

Outline

Linear algebra

Multivariate/vector calculus

Manifold

Chart/coordinate

Riemannian vector, dual vector, Riemannian metric

Geodesic, retraction

Geometric meaning of the matrix determinant

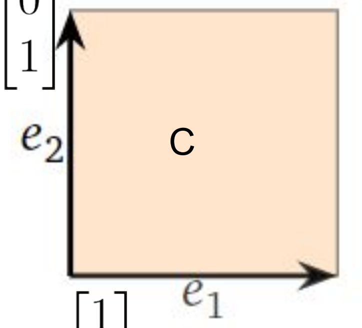
Source: https://textbooks.math.gatech.edu/ila/determinants-volumes.html

An invertible matrix T can be viewed as a map

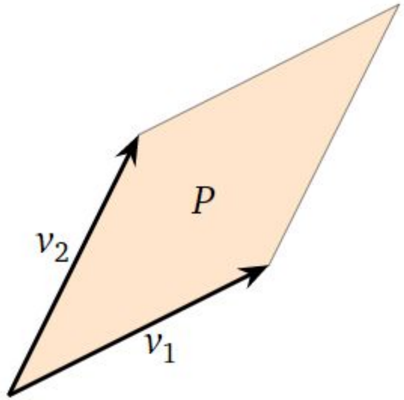
$$\mathbf{T} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}$$

$$T : \mathcal{R}^n \to \mathcal{R}^n$$

$$\mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$



$$\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$$



$$\mathbf{v}_i = T(\mathbf{e}_i) = \mathbf{T}\mathbf{e}_i$$

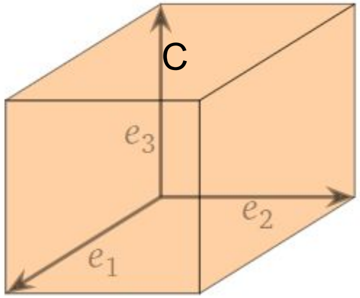
claim: $\mathrm{vol}(\mathbf{P}) = |\det(T)|\mathrm{vol}(\mathbf{C})$

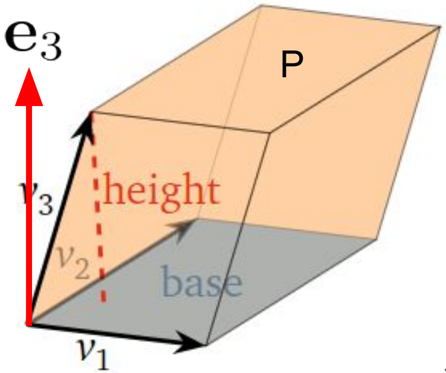
An informal geometric proof $\mathrm{vol}(\mathbf{P}) = |\det(T)|\mathrm{vol}(\mathbf{C})$

We consider the n = 3 case.

$$\mathbf{v}_i = T(\mathbf{e}_i) = \mathbf{T}\mathbf{e}_i$$

$$\mathrm{vol}(\mathbf{C}) = 1 \qquad \mathbf{T} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$$





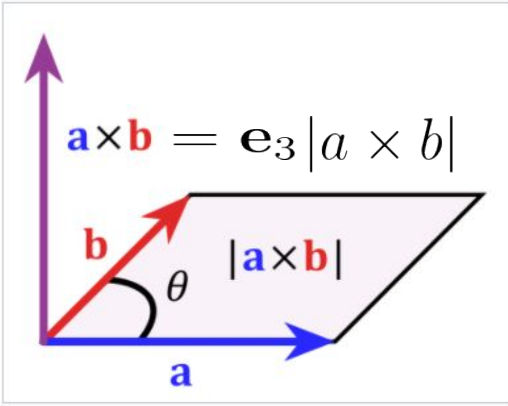
T

Cross product: $\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} \mathbf{e}_1 & v_{1,1} & v_{2,1} \\ \mathbf{e}_2 & v_{1,2} & v_{2,2} \\ \mathbf{e}_3 & v_{1,3} & v_{2,3} \end{bmatrix}$

0 0

base: $|\mathbf{v}_1 \times \mathbf{v}_2|$

height: $\mathbf{e}_3 \cdot \mathbf{v}_3$

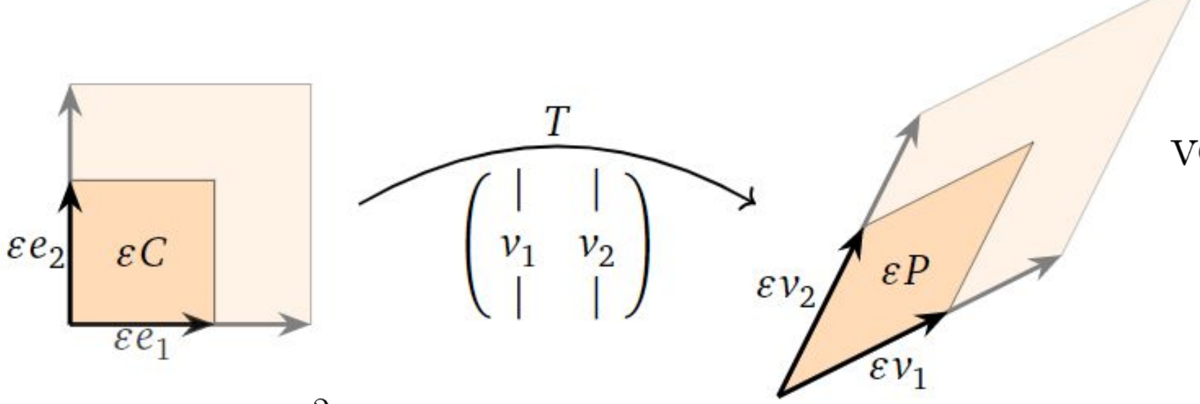


$$\mathrm{vol}(\mathbf{P}) = |(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3|$$

$$(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = \det \begin{bmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{bmatrix}$$

an invertible matrix T

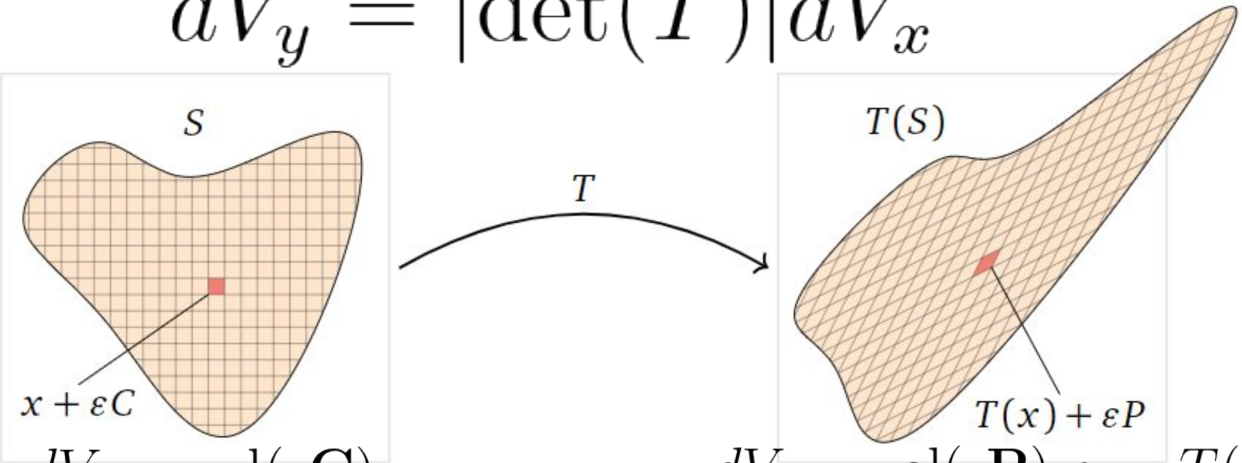
$$\mathrm{vol}(\mathbf{P}) = |\det(T)|\mathrm{vol}(\mathbf{C})$$



$$\mathrm{vol}(\epsilon\mathbf{C}) = \epsilon^2$$

$$\mathrm{vol}(\epsilon\mathbf{P}) = \epsilon^2|\det(T)|$$

$$dV_y = |\det(T)|dV_x$$



$$dV_x = \mathrm{vol}(\epsilon\mathbf{C})$$

$$dV_y = \mathrm{vol}(\epsilon\mathbf{P}), y = T(x)$$

Integration by substitution

$$\int_{\psi(\mathbf{S})} f(y)dV_y = \int_{\mathbf{S}} f \circ \psi(x) |\det(\nabla\psi)| dV_x$$

$$\psi : \mathcal{R}^n \to \mathcal{R}^n$$

ψ deterministic, one-to-one map

continuously diff.

We approximate map ψ using its linearization $T = \nabla\psi$

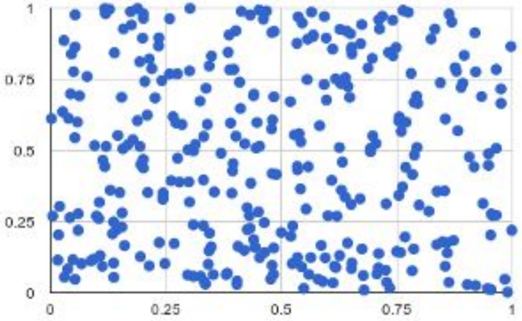
Classical examples: Work (line integral), Flux (surface integral)

ML example: normalizing flows $y = \psi(x), x \sim p(x)$

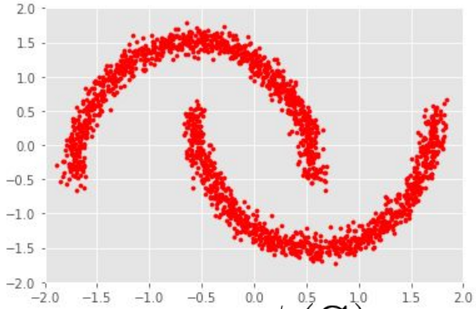
$$1 = \int p_y(y)dy = \int p_y(\psi(x))|\det(\nabla\psi)|dx$$

$$1 = \int p(x)dx$$

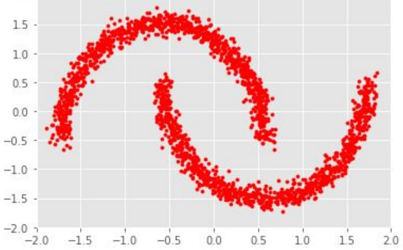
$$p_y(y) = p(x)|\det(\nabla\psi(x))|^{-1}, x = \psi^{-1}(y)$$

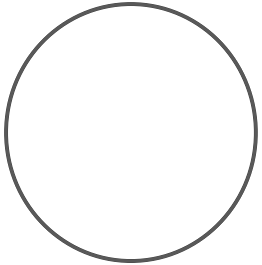


$x \in \mathbf{S}$



$y \in \psi(\mathbf{S})$





$$x^2 + y^2 = 1$$

Manifolds & charts

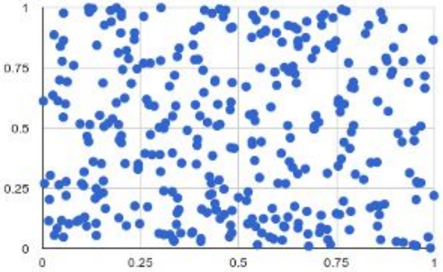
A curved space locally looks like a flat/Euclidean space

$\psi(\mathbf{S})$ An open subset of a manifold

$\mathbf{S}$ An open subset in a Euclidean space

$\phi := \psi^{-1}$ a (local) chart/coordinate

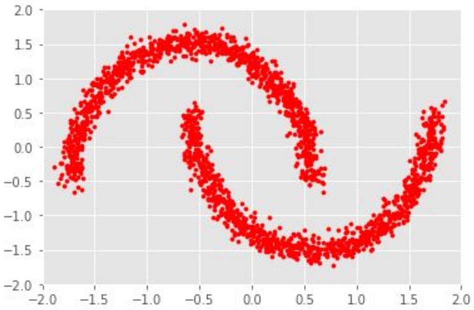
$\mathbf{S}$



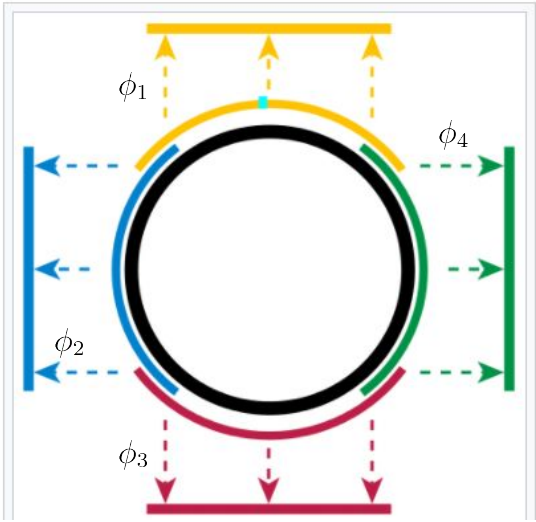
ψ

ϕ

$\psi(\mathbf{S})$

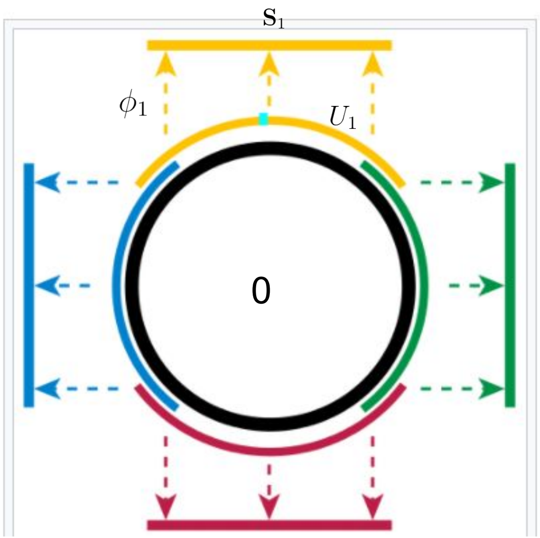


$$U := \psi(\mathbf{S})$$



Free parameters in a chart

1-dim manifold



We will assume ϕ_1 and U_1 are known.

$$U_1 = \{(x,y) | x^2 + y^2 = 1, y > 0, -1 < x < 1\}$$

$$\mathbf{S}_1 = \{x | -1 < x < 1\} \subset \mathcal{R}^1 \subset \mathcal{R}^2$$

$$\phi_1((x,y)) = x \in \mathbf{S}_1$$

$$\psi_1(t) := \phi_1^{-1}(t) = (t, \sqrt{1-t^2}) \in U_1, t \in \mathbf{S}_1$$

$$x^2 + y^2 = 1$$

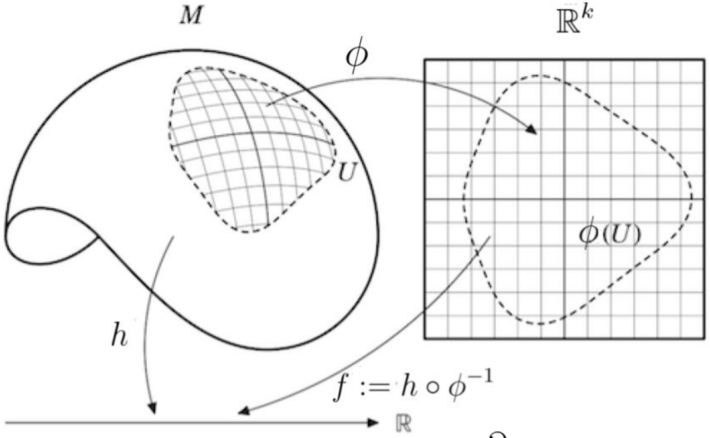
In set U_1, y is fully determined by x. Therefore, we only consider x is the single free parameter

# of free parameters implicitly defines the dimension of a manifold

We ONLY use free parameters to represent a vector in the manifold

We only compute gradients w.r.t. free parameters. There are some exceptions for matrix manifolds

Riemannian vectors and metric



Given a chart, x=[x_1,...x_k] represents

a set of free variables.

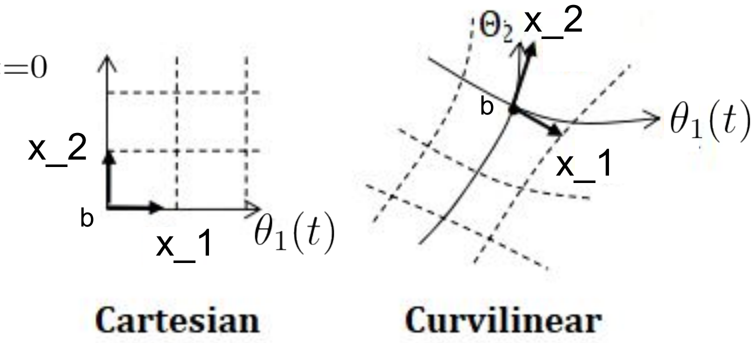
A basis of a tangent vector space at point b=[b_1,...b_k] is denoted by $\{\frac{\partial}{\partial x_i}\}_{i=1}^{k}$

A basis vector $\frac{\partial}{\partial x_1}$ can be viewed as the tangent vector of the coordinate curve/direction $\theta_1(t)$

$$\frac{\partial}{\partial x_1} := \nabla_t \theta_1(t)\Big|_{t=0} = \nabla_t [b_1 + t, b_2, \cdots, b_k]\Big|_{t=0}$$

$$\langle \frac{\partial}{\partial x_1}, v \rangle_{\mathbf{F}} := \mathbf{e}_1^T \mathbf{F} \mathbf{v}$$

Metric $\mathbf{F}$



In the Euclidean case, F is the identity matrix

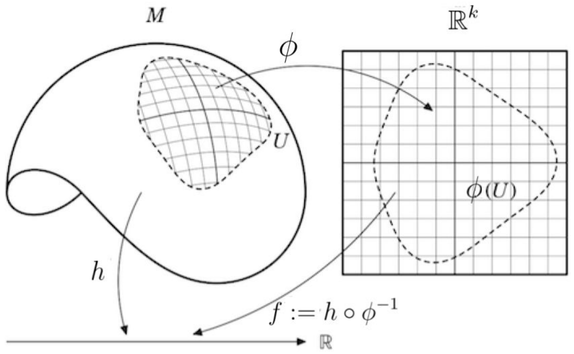
Gradients of a scalar function h

$$\frac{\partial}{\partial x_1} := \nabla_t [b_1 + t, b_2, \ldots, b_k] \Big|_{t=0}$$

Directional derivative

$$\frac{\partial}{\partial x_1}(f) := \nabla_t f([b_1 + t, b_2, \cdots, b_k]) \Big|_{t=0}$$

A scalar function h defined on a k-dim manifold can be expressed as f(x) given a chart, where x=[x_1,...x_k] are free variables.



$$\langle \frac{\partial}{\partial x_1}, \hat{\mathbf{g}}_f \rangle_{\mathbf{F}} := \frac{\partial}{\partial x_1}(f) = \frac{\partial}{\partial x_1} f(x)$$

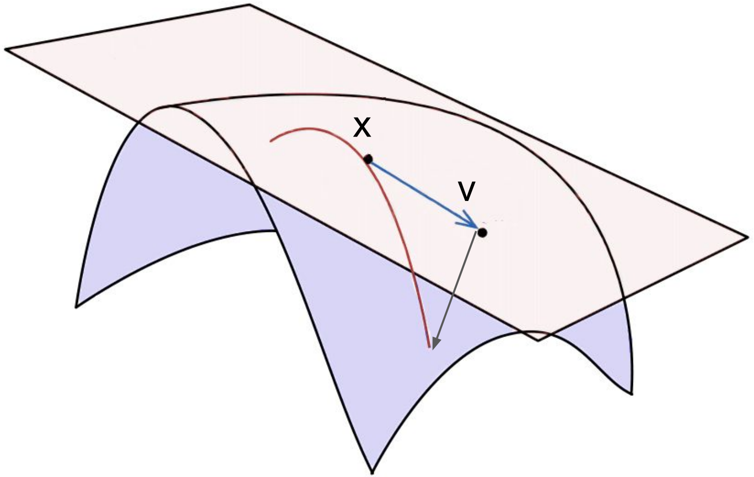
$$\langle \frac{\partial}{\partial x_1}, \hat{\mathbf{g}}_f \rangle_F := \mathrm{e}_1^T \mathbf{F} \hat{\mathbf{g}}_f$$

Riemannian metric	$\mathbf{F}$
Riemannian gradient (vector)	$\hat{\mathbf{g}}_f$
Euclidean gradient (dual vector)	$\mathbf{g}_f$

$$\hat{\mathbf{g}}_f = \mathbf{F}^{-1} \mathbf{g}_f$$

$$\mathbf{g}_f := [\frac{\partial}{\partial x_1} f(x), \cdots, \frac{\partial}{\partial x_k} f(x)]$$

Geodesic & Manifold Exponential Map & Retraction Maps



Given a starting point x and a Riemannian direction/gradient v,
A geodesic: a “straight line” on the manifold.
Input: t; Output: a point on the manifold

$$L(t) = \mathbf{x} + t\mathbf{v}$$ a straight line in the Euclidean case

Given a starting point x and step-size t=1
The exponential map:
Input: v; Output: a point on the manifold

Retraction: an approximation of the exponential map