Tensor Completion for Estimating Missing Values in Image Data

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[1] Ji Liu et al., 2009.

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- Signal: Order 1 Tensor
- Image: Order 2 Tensor
- Video: Order 3 Tensor
- MRI Video: Order 4 Tensor

Solution

- Expand low-rank matrix completion to tensor completion

Figure 1: The left figure contains 80% missing entries shown as white pixels and the right figure shows its reconstruction using the low rank approximation.

Eating Our Vegetables

$$D_\tau(X) = U\Sigma_\tau V^\top \quad (1)$$

$$\Sigma_\tau(X) = diag(max(\sigma_i - \tau, 0)) \quad (2)$$

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Masked sample set: X_Ω

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Masked sample set: X_Ω
Tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_n}$ and a tensor unfold operation defined as:

$$unfold_k(\mathcal{X}) := X_{(k)} \in \mathbb{R}^{I_k \times (I_1 \times \ldots \times I_{k-1} \times I_{k+1} \times \ldots \times I_n)} \tag{3}$$

and the reverse:

$$fold_k(X_{(k)}) = \mathcal{X} \tag{4}$$

Trace Norm

The rank operation is discrete and non-convex.

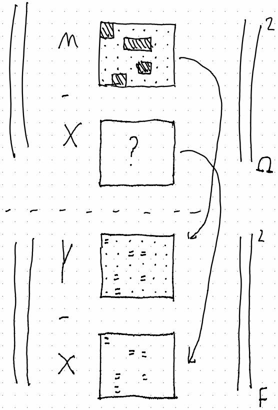
Trace Norm

The rank operation is discrete and non-convex. Thus trace norm is used as a method:

$$\|X\|_{tr} = \sum_i \sigma_i(X) \tag{5}$$

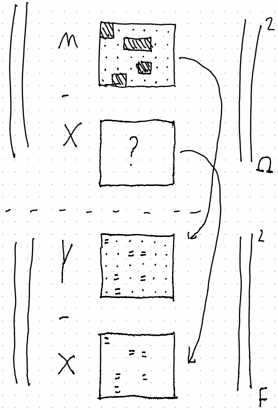
Tightest convex envelope for rank of matrices. (loose reason: L1 enforces sparsity)

Matrix Completion



We have a matrix M at Ω. We want to estimate a low-rank X.

Matrix Completion



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$$\min_{X} : \frac{1}{2}\|X - M\|_{\Omega}^{2}$$
$$s.t.\, rank(X) \leq r$$

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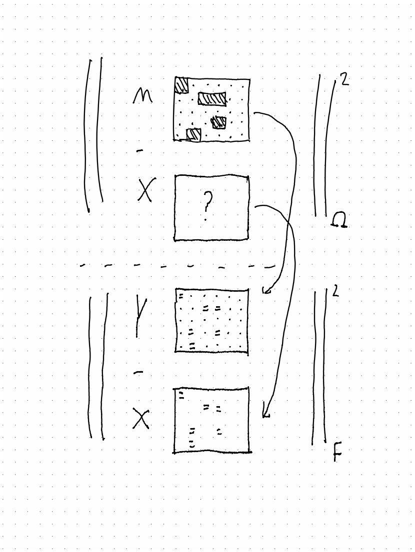

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Tensor Completion

$\mathcal{T}_\Omega$ is what we know

$$\min_{\mathcal{X},\mathcal{Y}} : \frac{1}{2}\|\mathcal{X} - \mathcal{Y}\|_F^2$$
$$s.t. \|\mathcal{X}\|_{tr} \leq c$$
$$\mathcal{Y}_\Omega = \mathcal{T}_\Omega$$

Tensor Completion

$$\min_{\mathcal{X},\mathcal{Y}} : \frac{1}{2n} \sum_{i=1}^{n} \|X_{(i)} - Y_{(i)}\|_F^2$$
$$s.t. \|\mathcal{X}\|_{tr} \leq c$$
$$\mathcal{Y}_\Omega = \mathcal{T}_\Omega$$

What is the tensor trace norm?

Tensor Completion

$$\min_{\mathcal{X},\mathcal{Y}} : \frac{1}{2n} \sum_{i=1}^{n} \|X_{(i)} - Y_{(i)}\|_F^2$$
$$\text{s.t.} \frac{1}{n} \sum_{i=1}^{n} \|X_{(i)}\|_{tr} \leq c$$
$$\mathcal{Y}_\Omega = \mathcal{T}_\Omega$$

Average of each unfolded trace norm.

Tensor Completion

Loosening it up:

$$\min_{\mathcal{X},\mathcal{Y},M_i} : \frac{1}{2n}\sum_{i=1}^{n}\|M_i - Y_{(i)}\|_F^2$$

$$\text{s.t.} \frac{1}{n}\sum_{i=1}^{n}\|M_i\|_{tr} \leq c$$

$$M_i = X_{(i)} \text{for } i = 1, 2, \ldots, n$$

$$\mathcal{Y}_\Omega = \mathcal{T}_\Omega$$

Tensor Completion

Relaxing equality:

$$\min_{\mathcal{X},\mathcal{Y},M_i} : \frac{1}{2n}\sum_{i=1}^{n}\|M_i - Y_{(i)}\|_F^2$$

$$\text{s.t. } \frac{1}{n}\sum_{i=1}^{n}\|M_i\|_{tr} \leq c$$

$$\|M_i - X_{(i)}\|_F^2 \leq d_i, \text{for } i = 1, 2, \ldots, n$$

$$\mathcal{Y}_\Omega = \mathcal{T}_\Omega$$

Tensor Completion

Converting to the dual:

$$\min_{\mathcal{X},\mathcal{Y},M_i} : \frac{1}{2n}\sum_{i=1}^{n}\|M_i - Y_{(i)}\|_F^2$$
$$+\frac{\gamma}{n}\sum_{i=1}^{n}\|M_i\|_{tr}$$
$$+\frac{1}{2n}\sum_{i=1}^{n}\alpha_i\|M_i - X_{(i)}\|_F^2$$
$$\text{s.t } \mathcal{Y}_\Omega = \mathcal{T}_\Omega$$

Tensor Completion

Tossing in more weights:

$$\min_{\mathcal{X},\mathcal{Y},M_i} : \frac{1}{2n}\sum_{i=1}^{n}\beta_i\|M_i - Y_{(i)}\|_F^2$$
$$+\frac{1}{n}\sum_{i=1}^{n}\gamma_i\|M_i\|_{tr}$$
$$+\frac{1}{2n}\sum_{i=1}^{n}\alpha_i\|M_i - X_{(i)}\|_F^2$$
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$$+\frac{1}{2n}\sum_{i=1}^{n}\alpha_i\|M_i - X_{(i)}\|_F^2$$
$$\text{s.t}\ \mathcal{Y}_\Omega = \mathcal{T}_\Omega$$

$\mathcal{X}$ is totally free

Block Coordinate Descent

- divide into $n + 2$ blocks
- break up

$\mathcal{X}$

$$\min_{\mathcal{X}} : \frac{1}{2}\sum_{i=1}^{n} \alpha_i \|M_i - X_{(i)}\|_F^2 \tag{6}$$

Solution is the weighted mean.

$\mathcal{X}$

$$\min_{\mathcal{X}} : \frac{1}{2}\sum_{i=1}^{n} \alpha_i \|M_i - X_{(i)}\|_F^2 \tag{6}$$

Solution is the weighted mean.

$$\mathcal{X} = \frac{\sum_{i=1}^{n} \alpha_i fold_i(M_i)}{\sum_{i=1}^{n} \alpha_i} \tag{7}$$

$\mathcal{Y}$

$$\min_{\mathcal{Y}} : \frac{1}{2} \sum_{i=1}^{n} \beta_i \| M_i - Y_{(i)} \|_F^2$$
$$\text{s.t } \mathcal{Y}_\Omega = \mathcal{T}_\Omega$$

Solution is the weighted mean with a mask

$$\mathcal{Y}_{\bar{\Omega}} = \left(\frac{\sum_{i=1}^{n} \alpha_i fold_i(M_i)}{\sum_{i=1}^{n} \alpha_i} \right)_{\bar{\Omega}} \tag{8}$$

M_i

$$\min_{M_i} : \frac{\beta_i}{2}\|M_i - Y_{(i)}\|_F^2 + \frac{\gamma_i}{n}\|M_i\|_{tr} + \frac{\alpha_i}{2n}\|M_i - X_{(i)}\|_F^2 \tag{9}$$

Solution:

$$M = D_\tau(Z),$$

$$\tau = \frac{\gamma_i}{\alpha_i + \beta_i}, Z_i = \frac{\alpha_i X_{(i)} + \beta_i Y_{(i)}}{\alpha_i + \beta_i} \tag{10}$$

$$D_\tau(X) = U\Sigma_\tau V^\top \tag{11}$$

Compared Methods

- CP/Parafac
- Tucker
- SVD

Results

Table 1: The RSE comparison on the synthetic data of size $40 \times 40 \times 40 \times 40$. P: Parafac model based heuristic algorithm; T: Tucker model heuristic algorithm; SVD: the heuristic algorithm based on the SVD; $\alpha 0$, $\alpha 10$ and $\alpha 50$ denote the proposed LRTC algorithm with three different values of the parameter: $\alpha = 0$, $\alpha = 10$ and $\alpha = 50$, respectively. The top, middle and bottom parts of the table respond to the sample percentage: 3%, 20% and 80%, respectively.

RSE Comparison (10^{-4})

Rank	T	P	SVD	$\alpha 0$	$\alpha 10$	$\alpha 50$
10,11,10,9	725	677	759	321	302	**257**
14,16,15,14	892	901	863	72.5	70.2	**65.3**
20,22,21,19	1665	1302	1474	50.1	46.9	**36.2**
24,25,25,26	2367	1987	2115	40.6	**38.7**	**38.7**
10,11,9,11	371	234	347	16.3	14.2	**12.7**
15,15,16,14	728	530	611	8.92	8.41	**8.23**
21,19,21,20	1093	982	895	**8.48**	8.56	**8.48**
24,25,26,26	1395	1202	1260	40.7	34.3	**13.7**
10,9,11,9	145	45	136	**3.08**	4.01	3.12
15,14,14,16	326	65	217	2.17	**2.05**	2.35
21,20,19,21	518	307	402	1.36	2.06	**1.27**
24,25,25,26	685	509	551	1.41	1.59	**1.04**

Table 2: The RSE comparison on the brain MRI data of size $181 \times 217 \times 181$. P: Parafac model based heuristic algorithm; T: Tucker model heuristic algorithm; SVD: the heuristic algorithm based on the SVD; $\alpha 0$, $\alpha 10$ and $\alpha 50$ denote the proposed LRTC algorithm with three different values of the parameter: $\alpha = 0$, $\alpha = 10$ and $\alpha = 50$, respectively. The top and bottom parts respond to the sample percentage: 20% and 80%, respectively.

RSE Comparison (10^{-4})

Rank	T	P	SVD	$\alpha 0$	$\alpha 10$	$\alpha 50$
21,24,23	311	234	274	210	193	**177**
38,41,37	1259	1001	1322	148	141	**121**
90,93,87	4982	3982	5025	61.0	53.7	**42.8**
21,24,23	12.3	8.64	**5.48**	29.4	12.8	13.5
35,42,36	179	153	99	**4.41**	5.32	5.69
39,48,41	279	345	199	**0.72**	1.05	1.26
45,55,47	606	523	513	1.22	1.35	**1.06**

Pretty Results

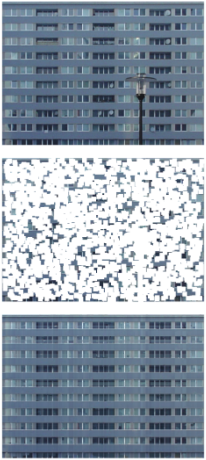


Figure 3: Facade in-painting. The top image is the original image; we select the lamp and satellite dishes together with a large set of randomly positioned squares as the missing parts shown in white in the middle image; the bottom image is the result of the proposed completion algorithm.

Figure 2: The left image (one slice of the MRI) is the original; we randomly select pixels for removal shown in white in the middle image; the right image is the result of the proposed completion algorithm.

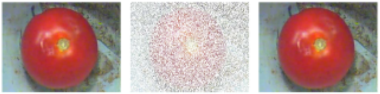


Figure 4: Video completion. The left image (one frame of the video) is the original; we randomly select pixels for removal shown in white in the middle image; the right image is the result of the proposed LTRC algorithm.

BRDF

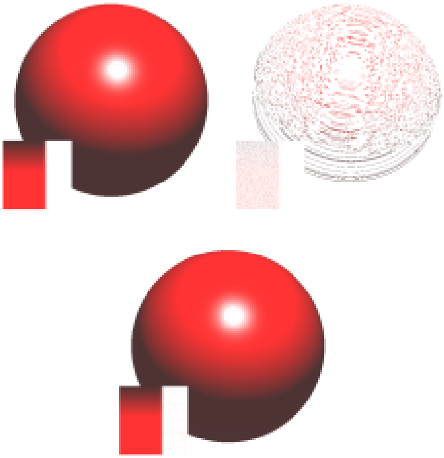


Figure 5: The top left image is a rendering of an original phong BRDF; we randomly select 90% of the pixels for removal shown in white in the top right image; the bottom image is the result of the proposed LRTC algorithm.

Where is this useful?

[2] J. Liu et al., 2013.

Where is this useful?

Images?

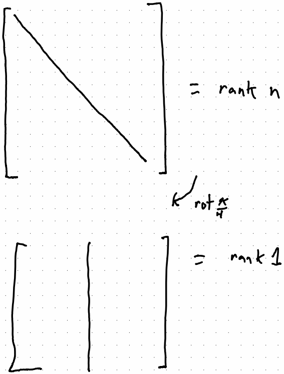
[2] J. Liu et al., 2013.

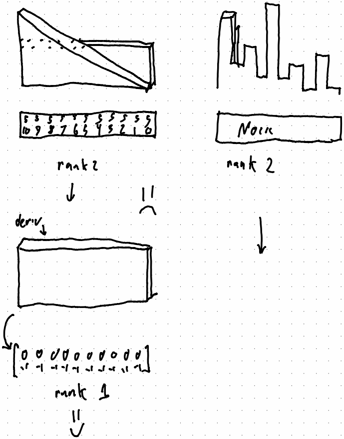
Where is this useful?

Images?
follow-up paper[2]

[2] J. Liu et al., 2013.

Pathological Cases





Shopping for alternatives

[3] H. Liu et al., 2019.

Shopping for alternatives

CoSTCo[3] Takes in indices, spits out value at those indices

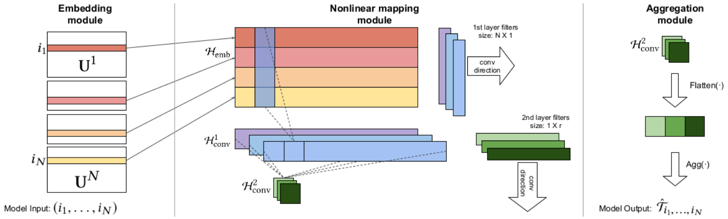


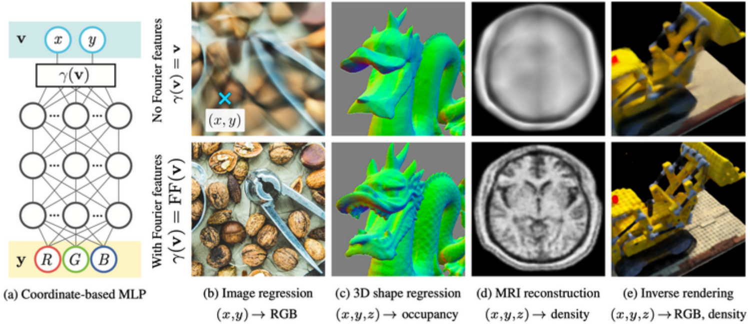
Figure 2: Model architecture of CoSTCo.

Uses relus and 2d convs. Seems interesting for non-spatially dependent data.

[3]H. Liu et al., 2019.

Implicit Functions

Fourier Features[4], NeRF[5] and its many children, some conditioned children[6].



Allow interpolation for free. Implicit neural functions for continuous images[7] and sinusoidal activation functions Implicit Neural Representations with Periodic Activation Functions[8]

[4]Tancik et al., n.d.

[5]Mildenhall et al., 2020.

[6]Yu et al., 2020.

[7]Skorokhodov, Ignatyev, and Elhoseiny, 2020.

[8]Sitzmann et al., 2020.

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