Optimal Control and Dynamical Systems

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UBC MLRG

Introduction

Introduction

Control theory is the study and practice of manipulating dynamical systems.

- Inseparable from data science - sensor measurements (data)
- Characteristics of this data is different from a statistical learning setting.

Example - PID temperature controller

Figure 1: https://bit.ly/2Zk2JKE

- A **P**roportional-**I**ntegral-**D**erivative controller is a feedback control mechanism.

Example - PID temperature controller

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- A **P**roportional-**I**ntegral-**D**erivative controller is a feedback control mechanism.
- A temperature controller takes measurements from a temperature sensor.
- Its output is connected to a control element such as a heater or a fan.

Example - MCAS

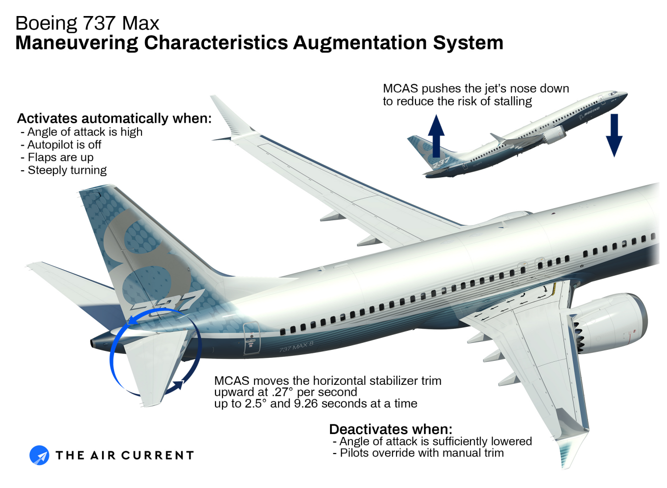
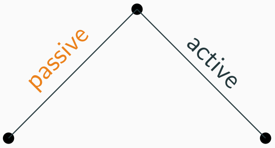


Figure 2: https://bit.ly/3iYLkyI

Types of control



- **Passive control** does not require input energy.
  - Cheap, simple, reliable.
  - May not be sufficient.
  - Example: stop signs at traffic intersections.

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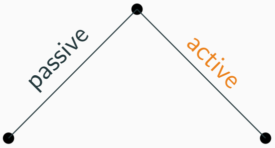

- **Passive control** does not require input energy.
  - Cheap, simple, reliable.
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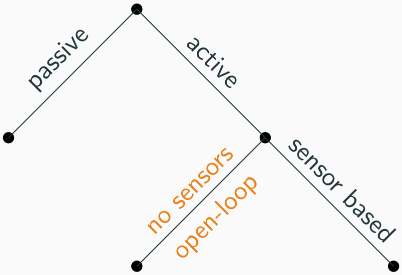
  - Example: stop signs at traffic intersections.
- **Active control** requires input energy.
  - Further categorized based on whether sensors are used.

Types of control



- **Open-loop control** relies on a pre-programmed control sequence.
  - Example: traffic lights.

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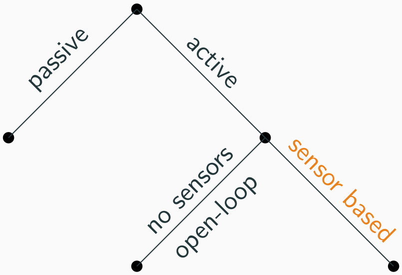
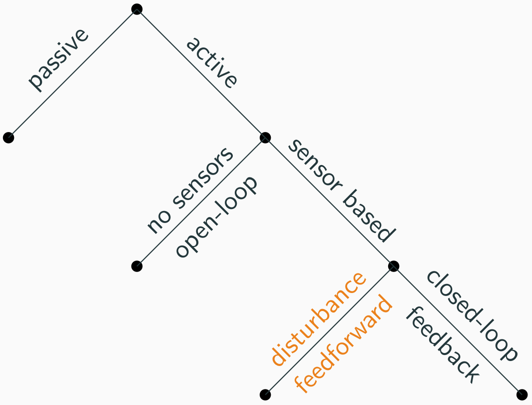


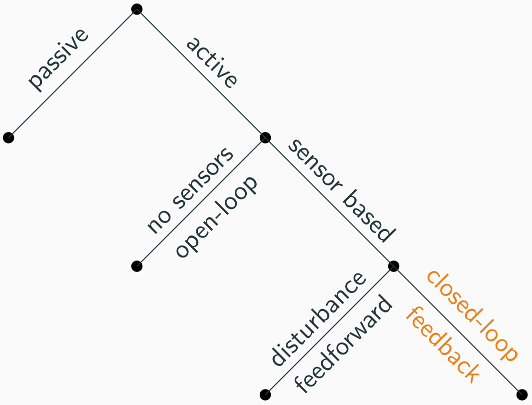
- **Open-loop control** relies on a pre-programmed control sequence.
    - Example: traffic lights.
- **Sensor-based control** uses sensor measurements to inform the control law.

Types of control



- **Disturbance feedforward control** measures external disturbances to the system, then feeds this into an open-loop control law.
    - Example: Preemptive road closure near a stadium before a concert.

Types of control



- **Disturbance feedforward control** measures external disturbances to the system, then feeds this into an open-loop control law.
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- **Closed-loop control** measures the system directly, then feeds the sensor measurements back.
  - Example: Sensors in the roadbed.

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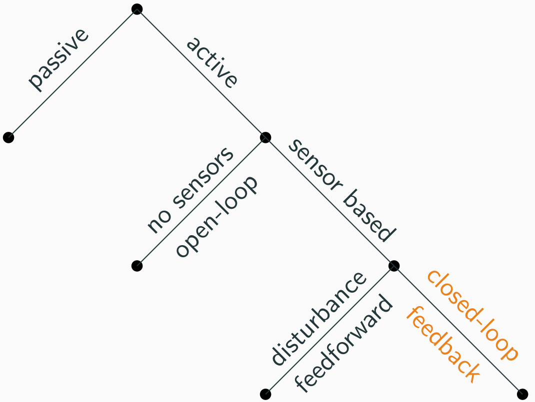

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- **Closed-loop control** measures the system directly, then feeds the sensor measurements back.
  - Example: Sensors in the roadbed.
  - This will be our main focus.

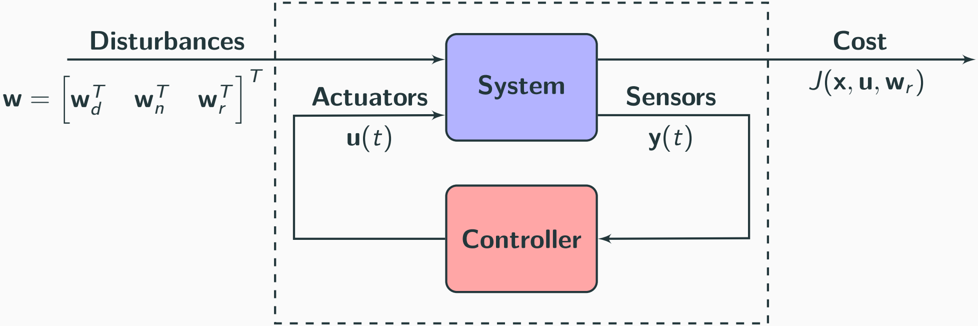
Outline

We will follow Chapter 8 in Brunton and Kutz [2019],

- Closed-loop feedback control (Section 8.1)
- Stability and eigenvalues (Section 8.2)
- Controllability (Section 8.3)
- Reachability (Section 8.3)
- Optimal full-state control: LQR (Section 8.4)

Closed-loop feedback control

Closed-loop feedback control



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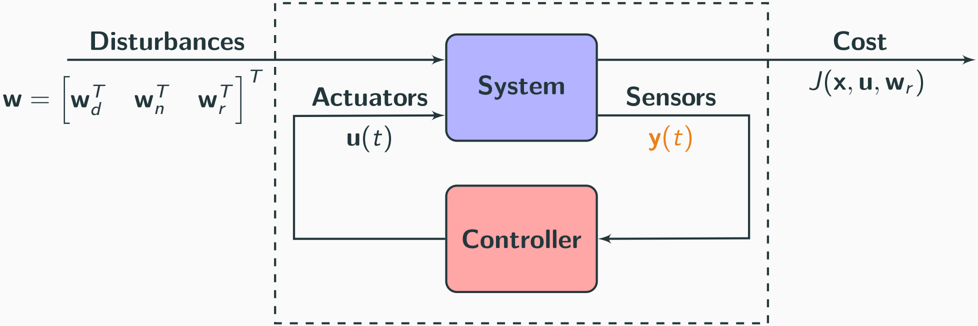
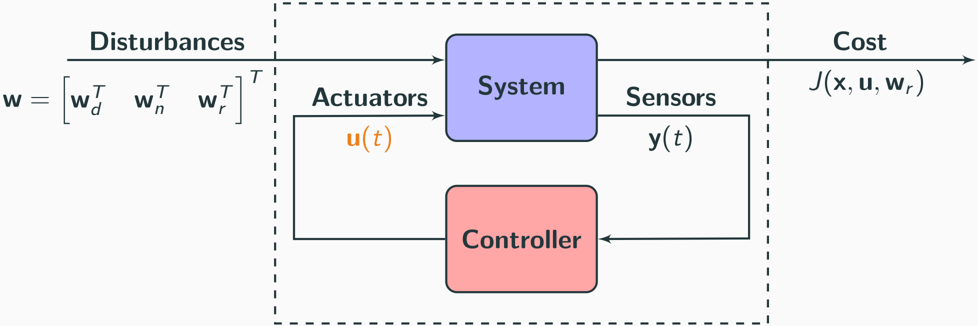


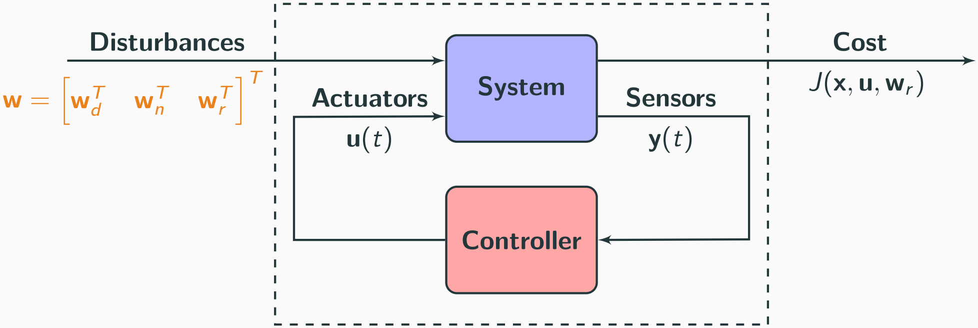
- $\mathbf{y}(t)$ sensor measurements

Closed-loop feedback control



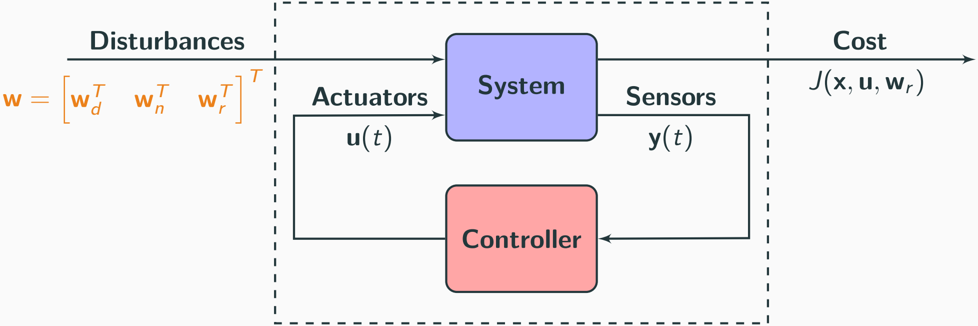
- $\mathbf{y}(t)$ sensor measurements
- $\mathbf{u}(t)$ actuation signal

Closed-loop feedback control



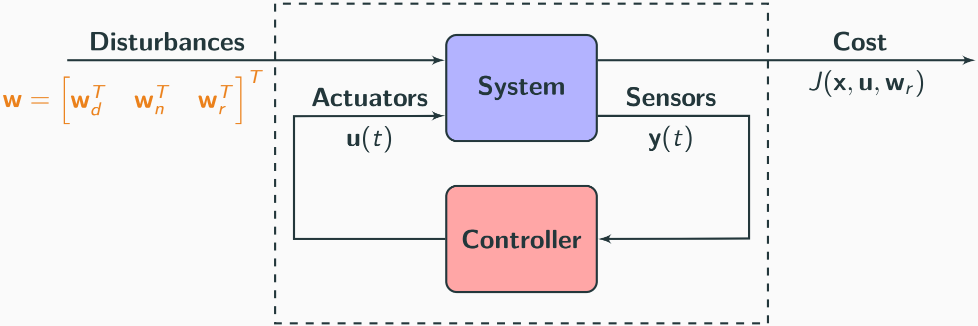
- $\mathbf{w}_d$ disturbances to the system

Closed-loop feedback control



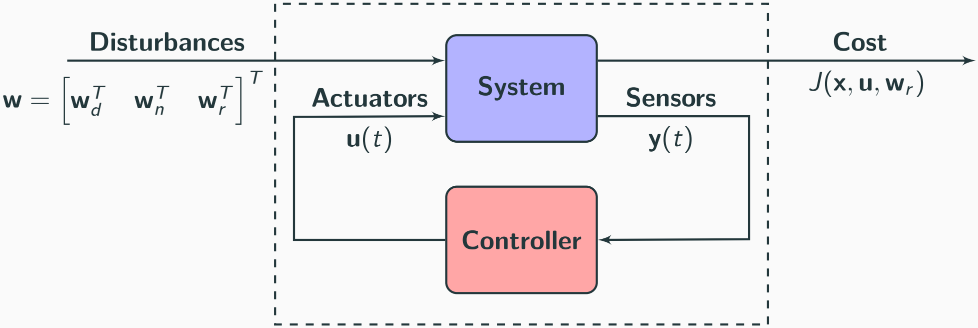
- $\mathbf{w}_d$ disturbances to the system
- $\mathbf{w}_n$ measurement noise

Closed-loop feedback control



- $\mathbf{w}_d$ disturbances to the system
- $\mathbf{w}_n$ measurement noise
- $\mathbf{w}_r$ reference trajectory

Closed-loop feedback control



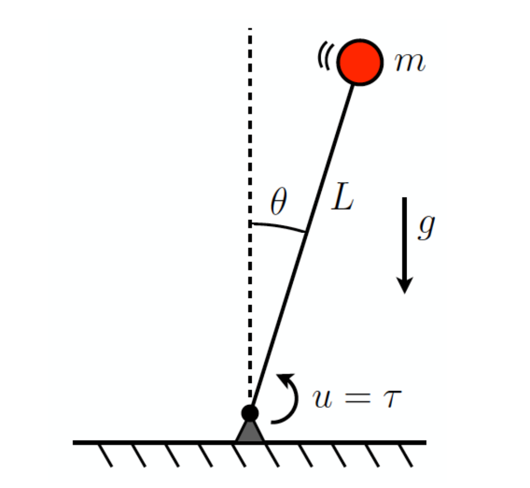
Together, this forms a dynamical system given by

$$\dot{\mathbf{x}} := \frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}_d), \qquad \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{w}_n),$$

and the goal is to construct a **control law**

$$\mathbf{u} = \mathbf{k}(\mathbf{y}, \mathbf{w}_r) \quad \text{such that the cost } J \text{ is minimized.}$$

Example: Inverted pendulum



Benefits of feedback control

Compared to open-loop control, closed-loop feedback makes it possible to

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- Stabilize an unstable system.
- Compensate for external disturbances.
- Correct for unmodeled dynamics.

Stability and eigenvalues

Linearization of nonlinear dynamics

Our nonlinear dynamical system is given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}_d), \qquad \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{w}_n),$$

and the goal is to construct a **control law**

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Linearization of nonlinear dynamics

For simplicity, let's ignore the external disturbances $\mathbf{w}$, which gives

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Near a fixed point $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ where $\mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = \mathbf{0}$, we can use a Taylor expansion to obtain the following linearization

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},$$

where $\mathbf{A} = \nabla \mathbf{f}_{\mathbf{x}}(\bar{x}, \bar{u})$, $\mathbf{B} = \nabla \mathbf{f}_{\mathbf{u}}(\bar{x}, \bar{u})$, $\mathbf{C} = \nabla \mathbf{g}_{\mathbf{x}}(\bar{x}, \bar{u})$, and $\mathbf{D} = \nabla \mathbf{g}_{\mathbf{u}}(\bar{x}, \bar{u})$.

Unforced linear system - without control

Linear system

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Now suppose

- In the absence of control: $\mathbf{u} = \mathbf{0}$
- and with measurements of the full state: $\mathbf{y} = \mathbf{x}$,

our dynamical system becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

and the solution $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0).$$

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where the matrix exponential is given by the infinite power series

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^2 t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k t^k.$$

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- When $\mathbf{A}$ is diagonalizable, $e^{\mathbf{A}t}$ can be computed by leveraging $\mathbf{A}$'s eigendecomposition:
  - $\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1} \implies e^{\mathbf{A}t} = \mathbf{Q}e^{\boldsymbol{\Lambda}t}\mathbf{Q}^{-1}$
- When $\mathbf{A}$ is not diagonalizable, write $\boldsymbol{\Lambda}$ in Jordan form and compute the matrix exponential with simple extensions.

Unforced linear system - without control

If we write the states as $\mathbf{x} = \mathbf{Q}\mathbf{z}$, then

$$\begin{aligned}
\dot{\mathbf{z}} &= \mathbf{Q}^{-1}\dot{\mathbf{x}} \\
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Our dynamical system simplifies from $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ to $\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z}$, with solution

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The eigenvalues in $\boldsymbol{\Lambda}$ also tell us about the stability of the system.

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Unforced linear system - stability

$$\mathbf{x}(t) = \mathbf{Q}e^{\mathbf{\Lambda}t}\mathbf{Q}^{-1}\mathbf{x}(0).$$

- In general, the eigenvalues may be complex numbers: $\lambda = a + ib$.
- Using Euler's formula: $e^{\lambda t} = e^{at}(\cos(bt) + i\sin(bt))$.

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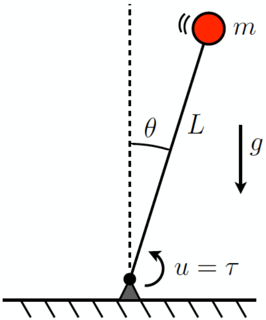
- Therefore, if **all the eigenvalues** λ_k **have negative real part**, i.e. $a < 0$, then the system is **stable** and $\mathbf{x} = 0$ as $t \to \infty$.
- If for any λ_k we have $a > 0$, then the system will diverge in this direction, which is very likely for a random initial condition.

Example: Stability of the inverted pendulum



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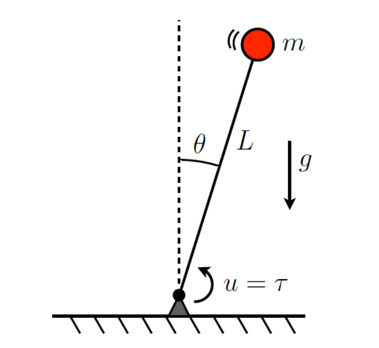

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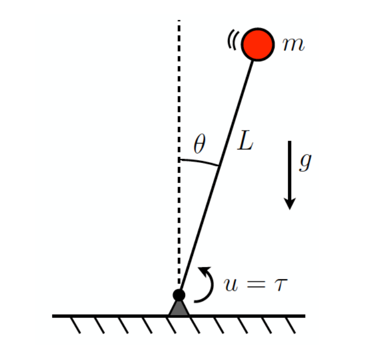

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Writing the system as a first-order differential equation,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \implies \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{L}\sin(x_1) + u \end{bmatrix}.$$

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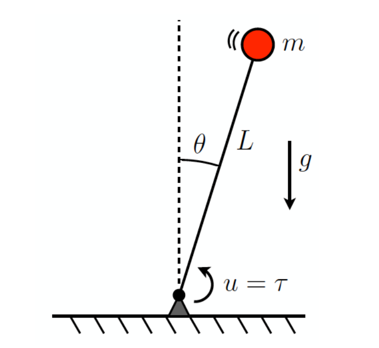

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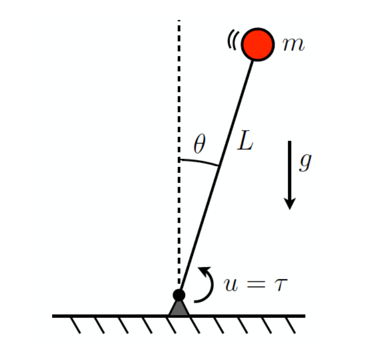
Taking the Jacobian of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ yields

$$\frac{\mathbf{df}}{\mathbf{dx}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(x_1) & 0 \end{bmatrix}, \quad \frac{\mathbf{df}}{\mathbf{du}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Stability of the inverted pendulum



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Linearizing at the pendulum up $(x_1 = \pi, x_2 = 0)$ fixed point,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

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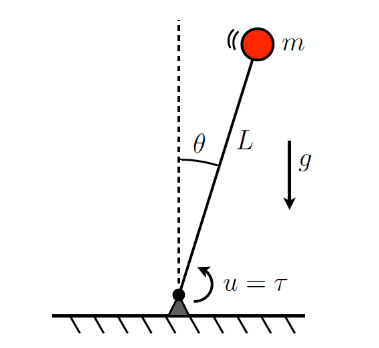

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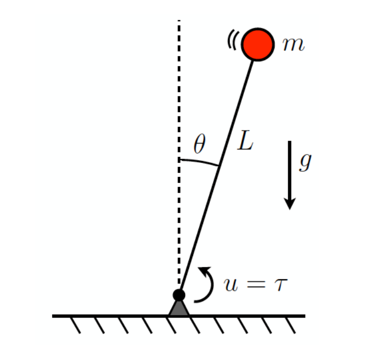
- Pendulum up (“inverted”): $\lambda = \pm\sqrt{g/L}$, positive real part $\implies$ instability.

Stability of the inverted pendulum



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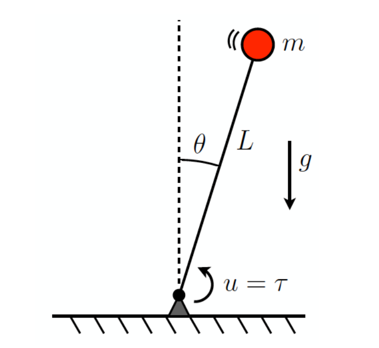

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- Pendulum up ("inverted"): $\lambda = \pm\sqrt{g/L}$, positive real part $\implies$ instability.
- Pendulum down: $\lambda = 0 \pm i\sqrt{g/L}$, stable.
- Good news: if we use closed-loop feedback control $\mathbf{u} = -\mathbf{Kx}$, we may be able to stabilize it!

Controllability

Controllability

Linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{x}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^q$, $\mathbf{A} \in \mathbb{R}^{n \times m}$, and $\mathbf{B} \in \mathbb{R}^{n \times q}$.

Controllability:

- When can we use feedback control to manipulate the system into what we want?

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- If we can control the system, how do we design the control law $\mathbf{u} = -\mathbf{K}\mathbf{x}$ to drive the system to the desired behaviour?

Controllability

Linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{x}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^q$, $\mathbf{A} \in \mathbb{R}^{n \times m}$, and $\mathbf{B} \in \mathbb{R}^{n \times q}$.

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With feedback control, we can write the dynamical system as

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

and hopefully we can use $\mathbf{K}$ such that we can place the eigenvalues wherever we want.

Controllability matrix

The controllability of a linear system in the form $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$ is determined entirely by the column space of the controllability matrix:

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

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  - Columns of $\mathcal{C}$ span all of $\mathbb{R}^n$.

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- Reachability of $\mathbb{R}^n$:
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Controllability - Example I

Consider the following system:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Is this system controllable?

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We can also check the controllability matrix, which is in this case

$$\mathcal{C} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

and the two columns are linearly dependent.

Controllability - Example II

What about allowing two knobs? Consider the following system:

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Yes. Both states can be independently controlled by u_1 and u_2.

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The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

which spans all of $\mathbb{R}^2$.

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What about when the states are coupled? Consider the following system:

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Maybe not obvious, but **Yes.** Even though we only have a single actuation, we can actually control x_1 through controlling x_2 since the states are coupled.

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which again spans all of $\mathbb{R}^2$.

The PBH test for controllability

The Popov-Belevitch-Hautus test

The system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is controllable if and only if the column rank of $\begin{bmatrix} (\mathbf{A} - \lambda \mathbf{I}) & \mathbf{B} \end{bmatrix}$ is equal to n for all $\lambda \in \mathbb{C}$.

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    - Or just take a random vector...

The Gramian - degrees of controllability

- The rank tests only give yes or no answers.
- But some states can be easier to control than others.

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The controllability Gramian

$$\mathbf{W}(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau \in \mathbb{R}^{n\times n},$$

which is often evaluated at infinite time,

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- The eigendecomposition of $\mathbf{W}$ also tells us how much we can steer the system in the direction of the eigenvectors.

Reachability

The Cayley-Hamilton theorem and reachability

Reachability: it's possible to steer the system to any arbitrary state $\mathbf{x}(t) = \xi \in \mathbb{R}^n$ in finite time with some actuation signal $\mathbf{u}(t)$.

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The Cayley-Hamilton theorem

Every square matrix $\mathbf{A}$ satisfies its own characteristic equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$
$$\implies \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_2\mathbf{A}^2 + a_1\mathbf{A} + a_0\mathbf{I} = 0.$$

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This allows us to express $\mathbf{A}^n$ as a linear combination of the lower-order powers:

$$\mathbf{A}^n = -a_{n-1}\mathbf{A}^{n-1} - \cdots - a_2\mathbf{A}^2 - a_1\mathbf{A} - a_0\mathbf{I}.$$

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More importantly, we can do this for any power greater than n:

$$\mathbf{A}^{k\geq n} = \sum_{j=0}^{n-1} \alpha_j \mathbf{A}^j.$$

The Cayley-Hamilton theorem and reachability

The Cayley-Hamilton theorem allows us to express the infinite power series $e^{\mathbf{A}t}$ as a finite sum:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^2 t^3 + \dots$$

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What does this have to do with reachability?

With control and zero initial condition $\mathbf{x}(0) = \mathbf{0}$, the solution to the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau.$$

So a state $\xi \in \mathbb{R}^n$ being reachable just means there exists $\mathbf{u}(t)$ such that

$$\xi = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau.$$

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A state $\xi \in \mathbb{R}^n$ is reachable if there exists $\mathbf{u}(t)$ such that

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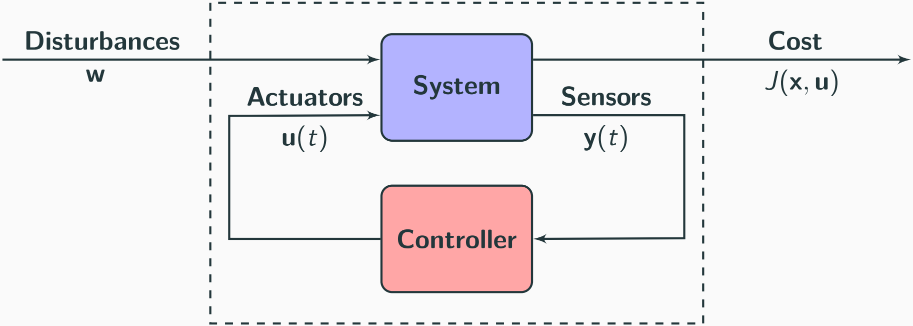
- Therefore, the only way for all of $\mathbb{R}^n$ to be reachable is when the columns of $\mathcal{C}$ spans $\mathbb{R}^n$.
- If $\mathcal{C}$ has rank n, then we can design $\mathbf{u}(t)$ to reach any state $\xi \in \mathbb{R}^n$.

Optimal full-state control: LQR

Optimal control



- Recall that if the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is controllable, then it's possible to arbitrarily manipulate the eigenvalues through a full-state feedback control law $\mathbf{u} = -\mathbf{K}\mathbf{x}$.

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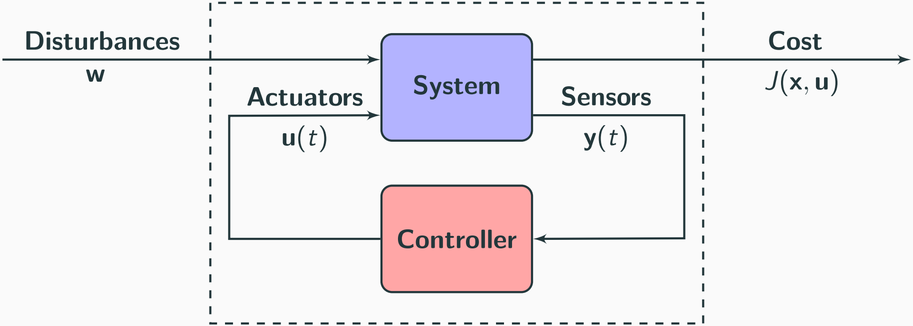

- Recall that if the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is controllable, then it's possible to arbitrarily manipulate the eigenvalues through a full-state feedback control law $\mathbf{u} = -\mathbf{K}\mathbf{x}$.
- If we choose $\mathbf{u}$ to make the system arbitrarily stable, this can lead to
  - Expensive control expenditure $J(\mathbf{x}, \mathbf{u})$.
  - Over-react to noise and disturbances.

Optimal control: LQR

- **Optimal control**: choosing the best gain matrix $\mathbf{K}$ to stabilize the system with minimum effort.
- Seek balance between stability and aggressiveness of control.

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$$J(t) = \int_0^t \underbrace{\mathbf{x}(\tau)^T \mathbf{Q} \mathbf{x}(\tau)}_{\text{cost of deviations of } \mathbf{x}} + \underbrace{\mathbf{u}(\tau)^T \mathbf{R} \mathbf{u}(\tau)}_{\text{cost of control}} d\tau$$

- $\mathbf{Q} \succeq 0$ - can achieve zero deviation.

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- $\mathbf{Q} \succeq 0$ - can achieve zero deviation.
- $\mathbf{R} \succ 0$ - but control effort is always needed.

Optimal control: LQR

- **Optimal control**: choosing the best gain matrix $\mathbf{K}$ to stabilize the system with minimum effort.
- Seek balance between stability and aggressiveness of control.

Consider the cost function

$$J(t) = \int_0^t \underbrace{\mathbf{x}(\tau)^T \mathbf{Q} \mathbf{x}(\tau)}_{\text{cost of deviations of } \mathbf{x}} + \underbrace{\mathbf{u}(\tau)^T \mathbf{R} \mathbf{u}(\tau)}_{\text{cost of control}} \, d\tau$$

- $\mathbf{Q} \succeq 0$ - can achieve zero deviation.
- $\mathbf{R} \succ 0$ - but control effort is always needed.
- Often diagonal, tuned to weigh the relative importance of the states/control knobs.

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- We now have an optimization problem!!!!!

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Since $J(t)$ is quadratic, there is an analytical solution given by

$$\mathbf{K}_r = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{X},$$

where $\mathbf{X}$ is the solution to an algebraic Riccati equation:

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- Very expensive for high-dimensional systems - $O(n^3)$.
- Reduced-order models: use fewer states.

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- Closed-loop feedback control.
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What we didn't cover:

- How to derive the Riccati equations for LQR. (End of Section 8.4 in [Brunton and Kutz, 2019])
- Full-state estimation and the Kalman filter. (Section 8.5 in [Brunton and Kutz, 2019])

Thank you


WHEN DID YOU BECOME AN EXPERT IN CONTROL THEORY?
LAST NIGHT.
memegenerator.net

Steven L. Brunton. Control Bootcamp.
https://www.youtube.com/playlist?list=PLMrJAkhIeNNR20Mz-VpzgfQs5zrYi085m, 2020.

Steven L. Brunton and J. Nathan Kutz. *Data-Driven Science and Engineering: Machine Learning, Dynamical Systems, and Control*. Cambridge University Press, 2019.