Causal Inference: Classical Approaches

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UBC MLRG

Outline

- Potential Outcomes
- Confounding and Causal DAGs
- Granger Causality
- ICA for Causal Discovery

Associational Inference

- Universe U
- For each unit $u \in U$:
  - Attribute variable X(u)
  - Observed variable $Y(u)$
- Inference:
  - $P(Y = y|X = x)$

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- For each unit $u \in U$:
  - Treatment variable $T(u) \in \{1,0\}$
  - **Potential outcome** $Y_1(u), Y_0(u)$
- Inference:
  - $Y_1(u) - Y_0(u)$

Rubin's Framework

- For each unit $u \in U$:
  - Treatment variable $T(u) \in \{1,0\}$
  - **Potential outcomes** $Y_1(u), Y_0(u)$
    - the outcome that would be observed if treatment was set to $T = 0$ or 1, on the same unit.
    - (before)

  - If $T(u)$ is set to 1
    - $Y_1(u)$ is the **observed outcome**
    - $Y_0(u)$ is the **counterfactual outcome**
    - (after)

Causal Effects

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  - Abbreviated as Y_1 and Y_0
- **Fundamental Problem of Causal Inference**
  - **It is impossible to *observe* both Y_1 and Y_0 on the same unit, and therefore it is impossible to observe the causal effect.**

THE END

Scientific solution to the Fundamental Problem

- Assume temporal stability and causal transience
  - The value of Y_0 does not depend on when $T = 0$ is applied and measured.
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  - Implies the constant effect assumption: $Y_1(u) - Y_0(u)$ is the same for all $u \in U$.
- It's very difficult to argue that these are valid…

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- **Average causal effect** of $T = 1$ (relative to $T = 0$) over U:
  - $\mathbb{E}(Y_1 - Y_0) = \mathbb{E}(Y_1) - \mathbb{E}(Y_0)$
  - Imagine parallel universes with the same population…
  - Can’t observe this.
- Observed data can only give us information about the average of the outcome over $u \in U$ exposed to $T = t$.
  - $\mathbb{E}(Y_1 | T = 1) - \mathbb{E}(Y_0 | T = 0)$

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  - In general, $\mathbb{E}(Y_t) \neq \mathbb{E}(Y_t|T = \mathrm{t})$
  - Independence assumption hold via randomized treatment assignment allows equality to hold, which lets us compute the ACE above.

Other assumptions

- Stable Unit Treatment Value Assumption (SUVTA)
  - No interference: units do not interact with each other.
  - One version of treatment.
- Consistency
  - The potential outcome Y_t is equal to the observed outcome if the actual treatment received is $T = t$.
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- **Ignorability** (aka no unmeasured confounders assumption)
  - $Y_0, Y_1 \perp \mathrm{T} | \mathrm{X}$
  - Among people with the same features X, we can think of treatment T as being randomly assigned.

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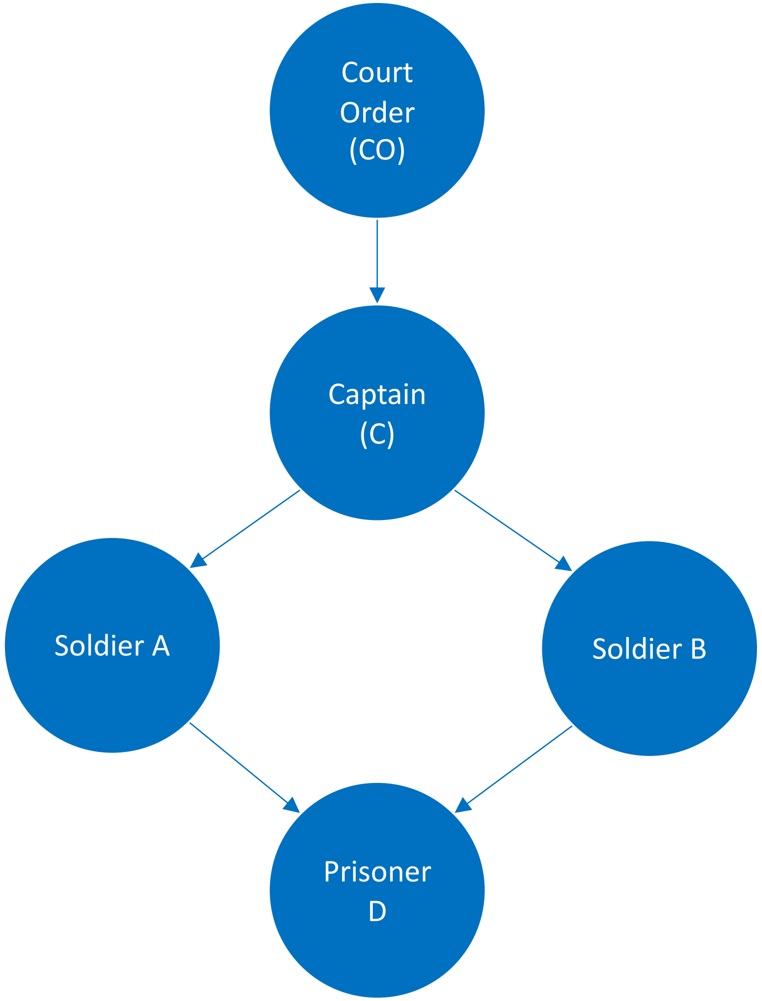
DAGs

- Useful for identifying dependencies and ways to factor and simplify the joint distribution.
- $p(x_1, \ldots, x_n) = \prod_{\{i=1\}}^{n} p(x_i | x_{\{pa(i)\}})$



Firing squad example [Pearl, 2018]

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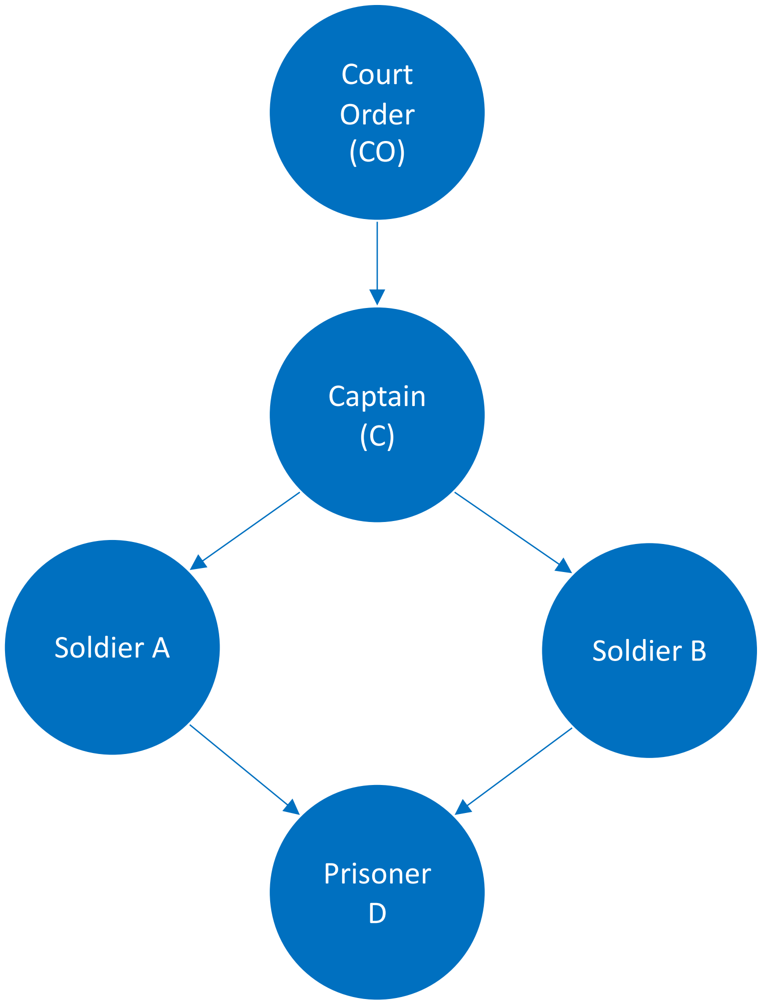
- Two variables A and B are **d-separated** by a set of variables Z if A and B are conditionally independent given Z.
  - $p(A, B|Z) = p(A|Z)p(B|Z)$
  - Chain
  - Fork
  - Inverted fork



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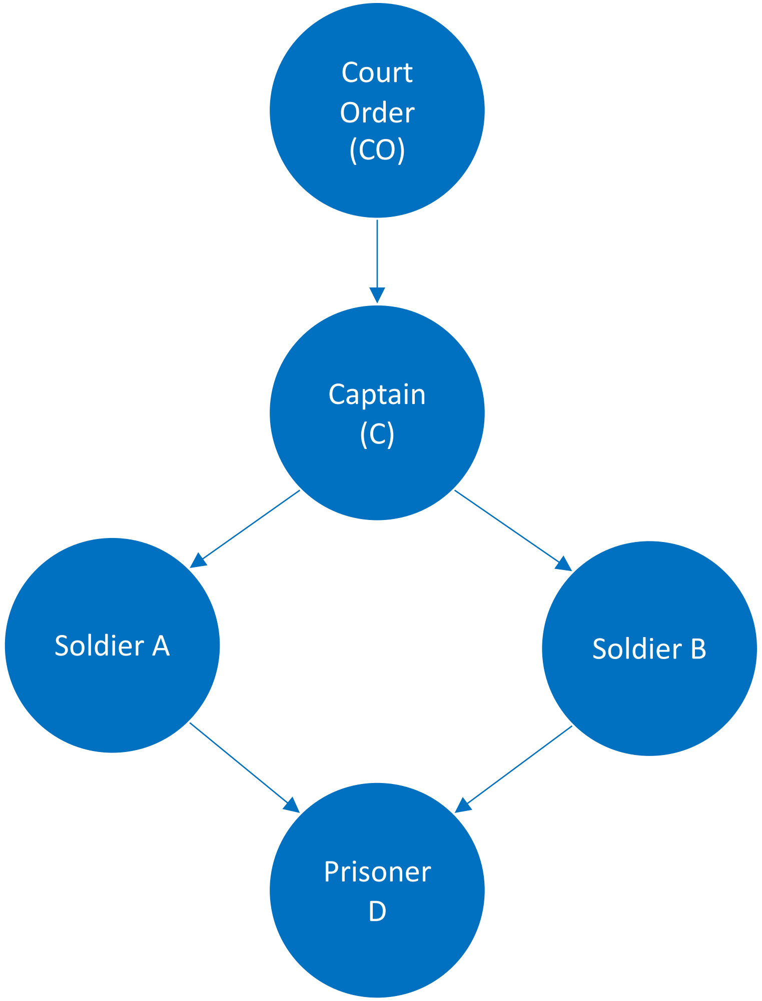
Causal DAGs

- DAGs where directions of the edges represent causal relationships.
- In contrast to Rubin's potential outcome framework, this is a structural approach to causal inference which Pearl advocates.
  - They are shown to be mathematically equivalent.



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Intervention and Pearl’s do-calculus

- $do()$ operator signals an intervention on a variable.
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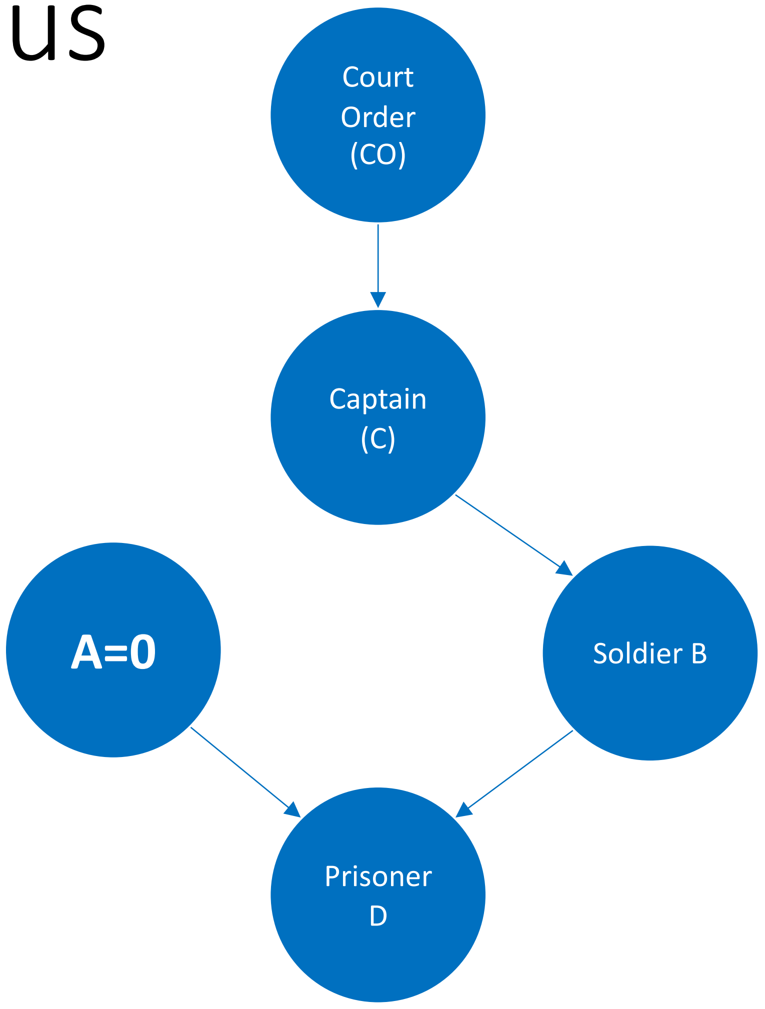

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Intervention and Pearl's do-calculus

- $do()$ operator signals an intervention on a variable.
  - Replace that variable with the actual value that we assign.
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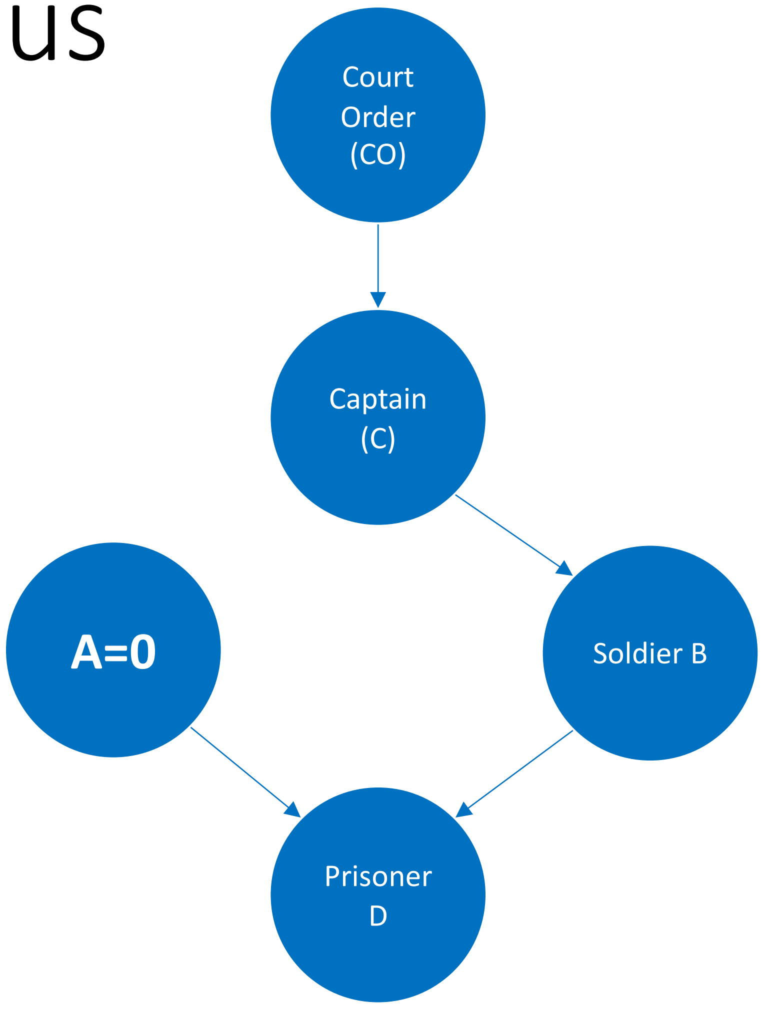
- Instead of $p(D|A = 0)$
- We want $p(D|do(A = 0))$
  - The causal effect of $A = 0$ on D.



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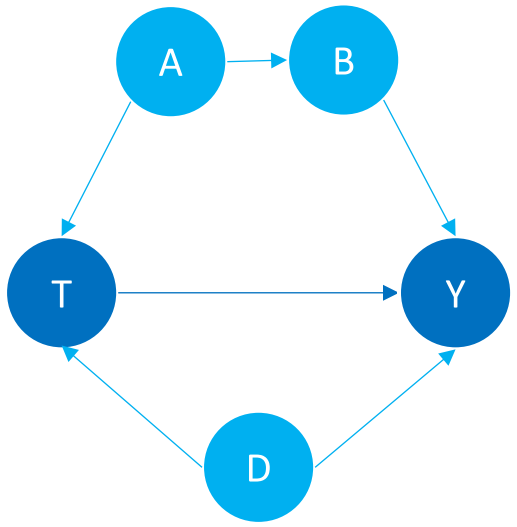
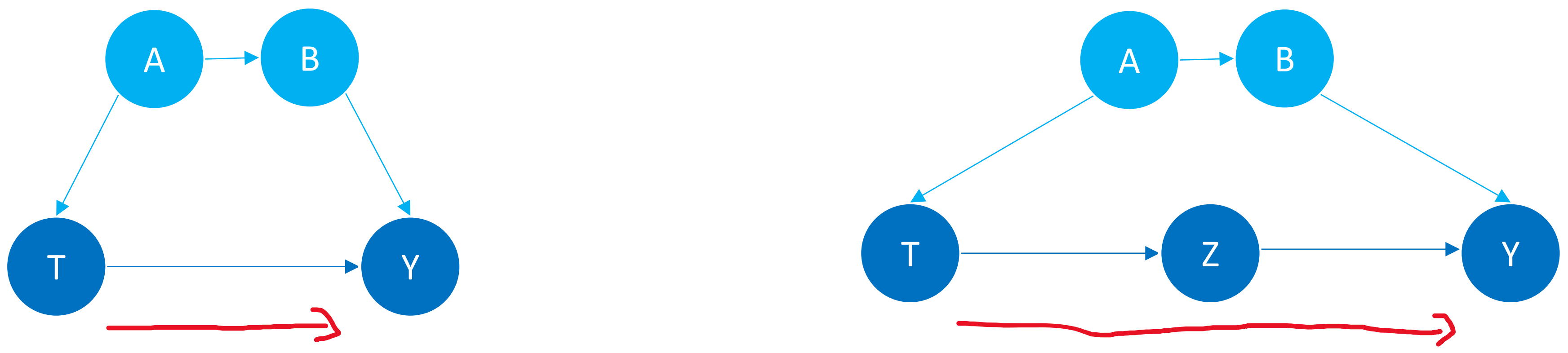
Confounding

- **Confounders**: variables that influences both treatment and outcome.
  - **Want:** identify a set of variables so that ignorability holds.
  - We don’t need to identity specific confounders
  - We just need to be able to control for confounding.
- Need to block backdoor paths from T to Y.



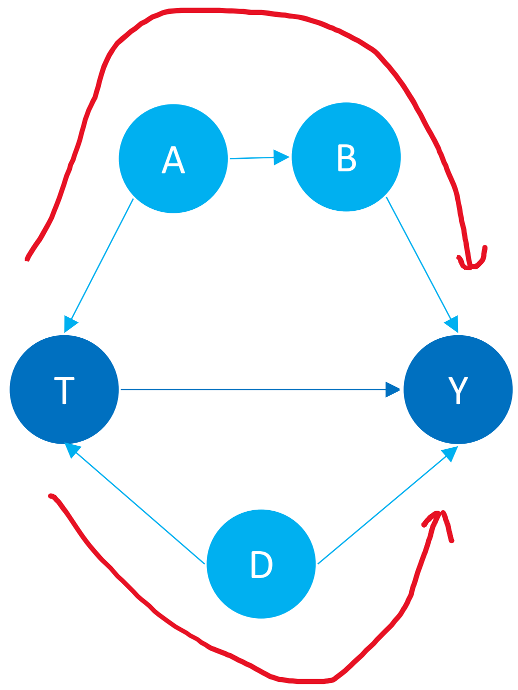
Frontdoor paths



- We are not concerned about frontdoor paths.
- We don't want to control anything along the frontdoor paths.
  - Unless we care about the magnitude of the causal effect…

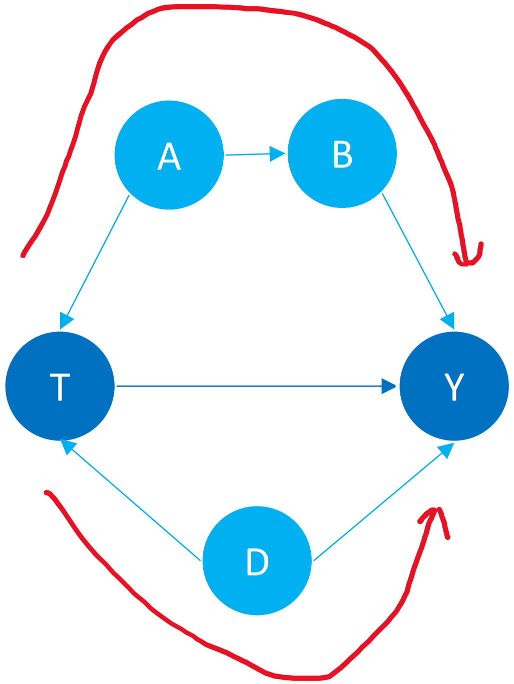
Backdoor paths

- **Begins with a parent of T and ends at Y.**
- Need to control these paths as they confound our causal effect.
- How?
  - Identify the set of variables that blocks all backdoor paths from T to Y.



Backdoor criterion

- A set of variables C satisfies the **backdoor criterion** if
  1. it blocks all backdoor paths from T to Y, and
  2. It does not include any descendants of T.



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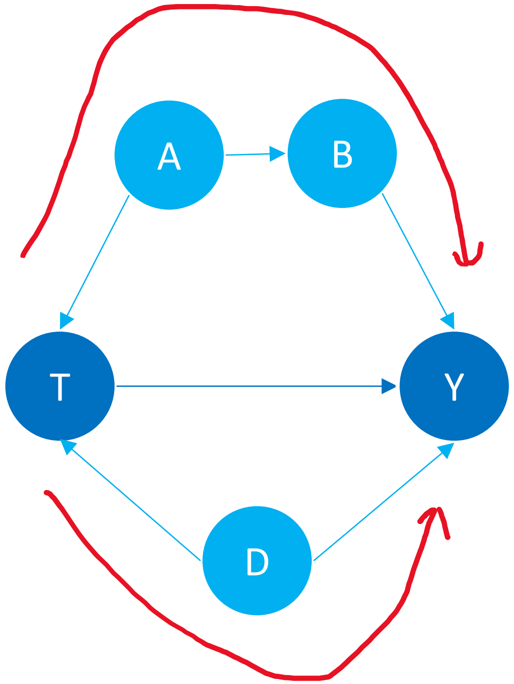

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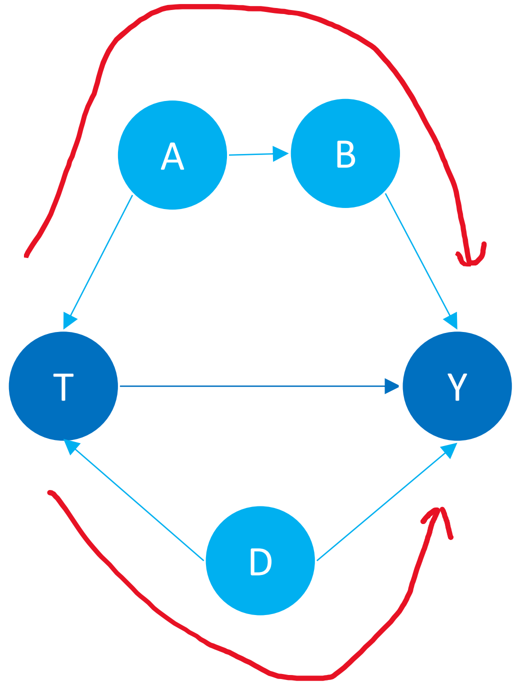

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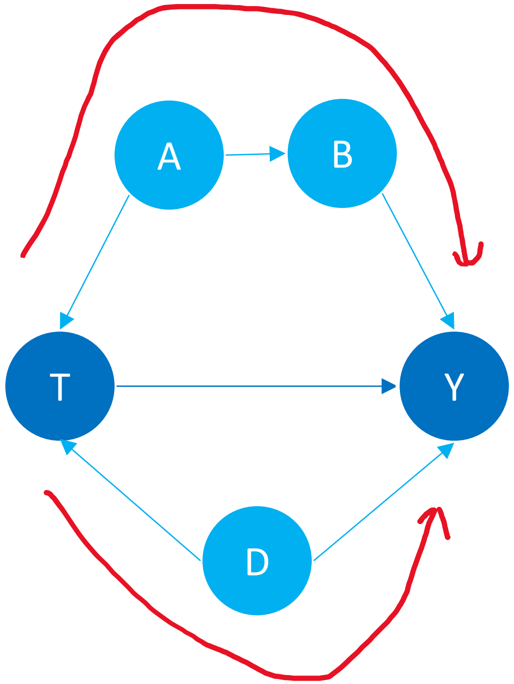

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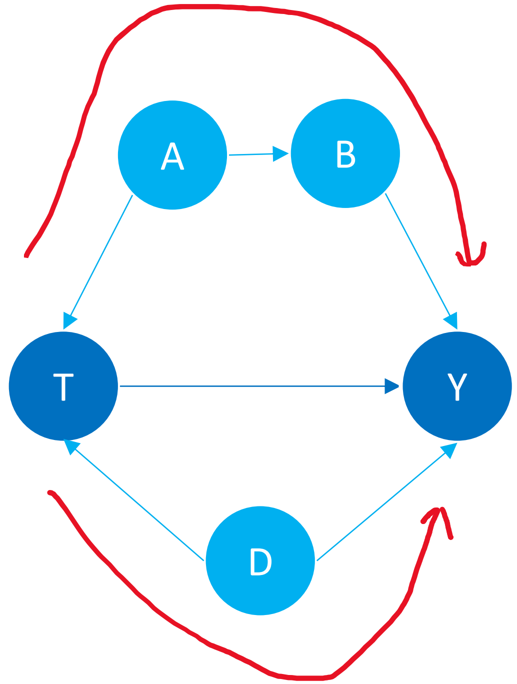
- In Rubin's framework, this is equivalent to the ignorability assumption:
  - Treatment assignment is effectively randomized given C.



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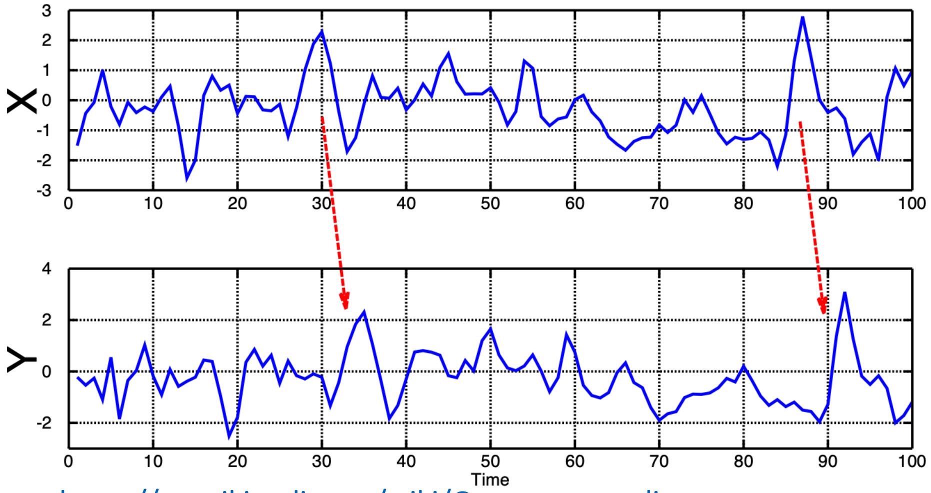

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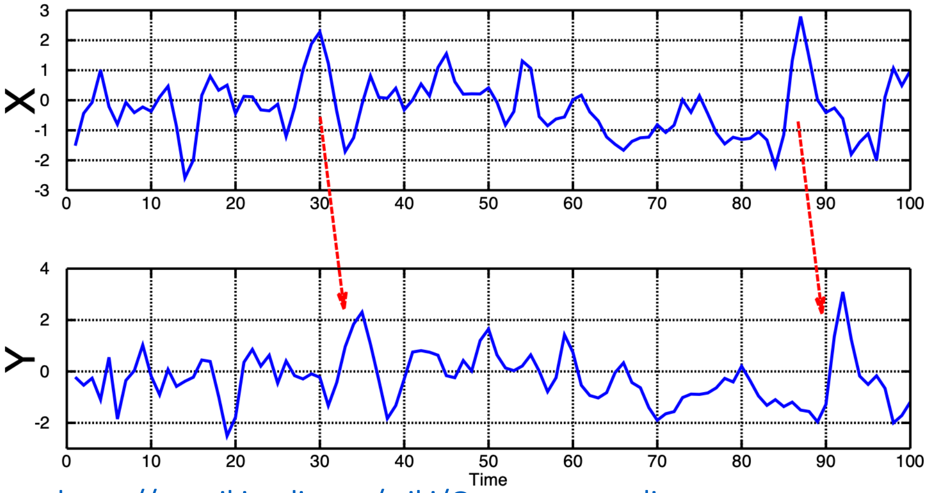

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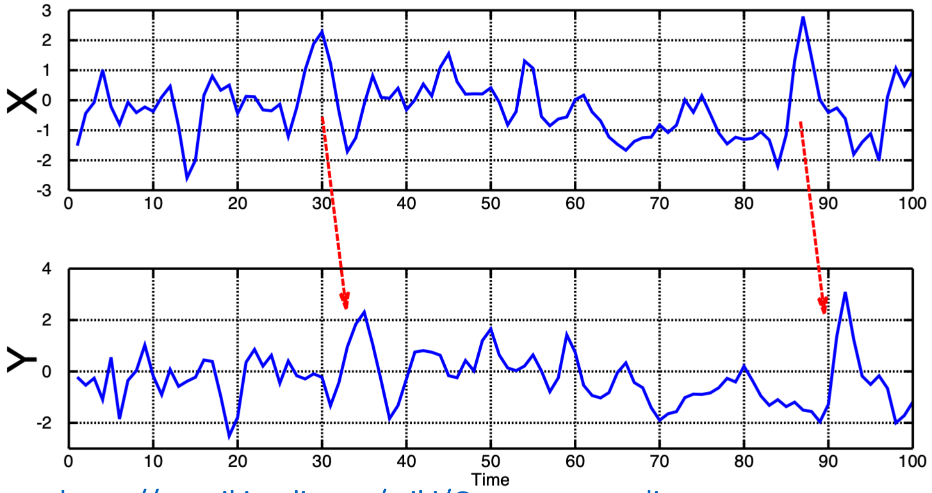

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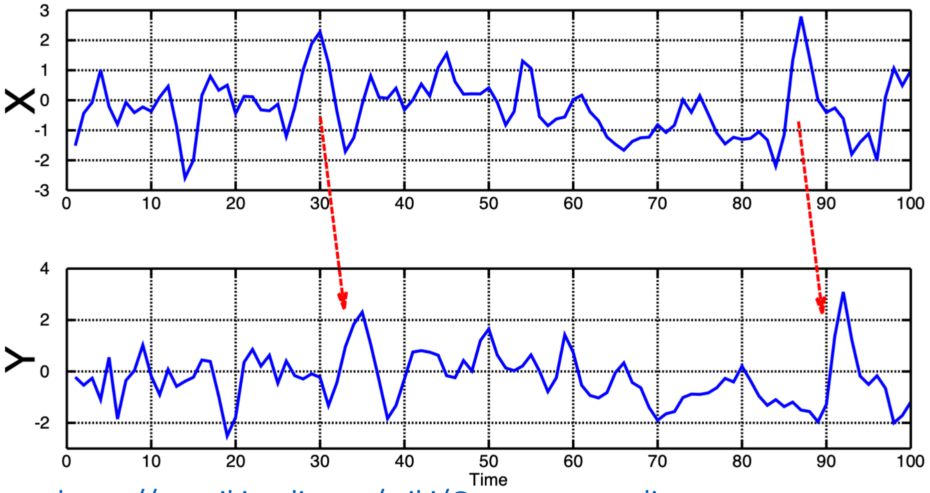

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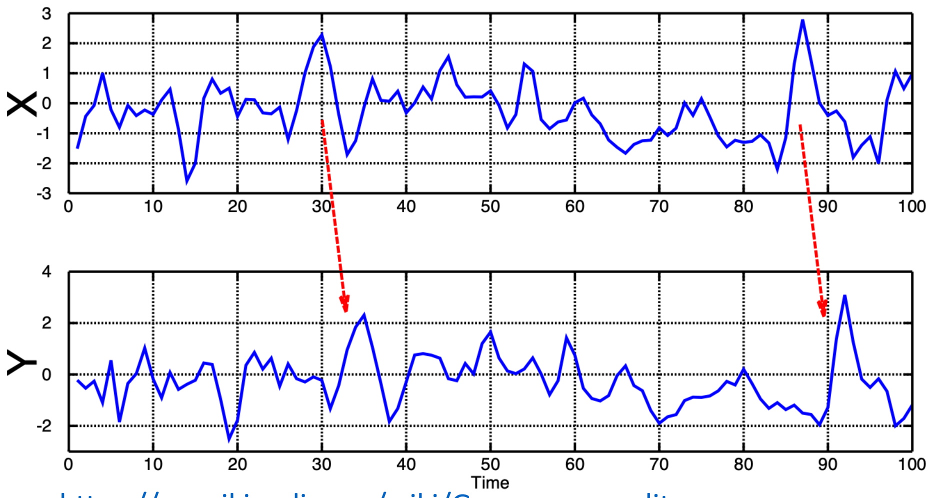

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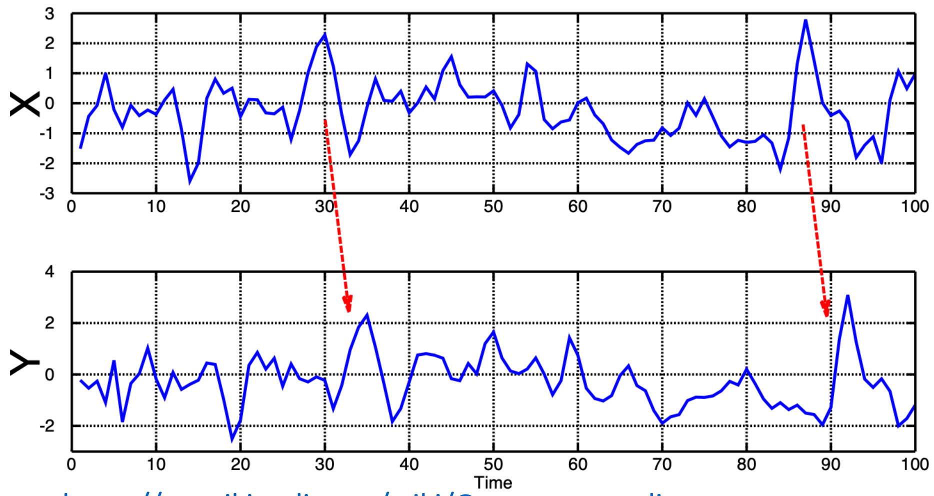
  - H_0: All $b_1 = \cdots = b_p = 0$
  - H_1: At least one is non-zero.
- Null hypothesis: X does not Granger cause Y iff no lagged values of x are retained.



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Pearl

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  - Factor analysis, data compression, etc.
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- ICA: $X = ZW$ usually with $k = d$
  - **Require the components of each z_i to be independent, and at most one can be normally distributed.**
  - Independence is measured by non-normality.
  - $W = \mathrm{argmax}_{\mathrm{W}} \sum_{i=1}^{n} \sum_{j=1}^{k} kurt(w_j^T x_i)^2$ where $kurt(u) = \mathbb{E}(u^4) - 3\left(\mathbb{E}(u^2)\right)^2$
  - Up to permutation and scaling, we can identify the factors W.

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  - **Causal directions are often unknown.**
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- With non-normality, ICA can be used to find the causal ordering of any number of observed variables based on non-experimental data. (Shimizu et al. 2006)

Causal Order

- Causality (a causal order) from a random variable x_1 to another random variable x_2, denoted by $x_1 \rightarrow x_2$ is confirmed if the following holds:
  - $x_2 = f(x_1, \epsilon_2)$ where **ϵ_2 is some perturbation independently distributed from x_1**.
  - Assume $f(x_1, \epsilon_2) = b_{21}x_1 + \epsilon_2$.
  - Boils down to finding b_{21}.

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  - $x_2 = f(x_1, \epsilon_2)$ where **ϵ_2 is some perturbation independently distributed from x_1**.
  - Assume $f(x_1, \epsilon_2) = b_{21}x_1 + \epsilon_2$.
  - Boils down to finding b_{21}.
- Note that we can't just have uncorrelatedness from x_1 and ϵ_2:
  - Independence: $\mathbb{E}\big(g(x)h(y)\big) = \mathbb{E}(g(x))\mathbb{E}(h(y))$ for any deterministic functions g and h.
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  - If x_1 and ϵ_2 are only uncorrelated, then there could exist unobserved confounder z that affects both x_1 and x_2.
  - Suppose $x_1 = \alpha z + \epsilon_1$ and $x_2 = \beta z + b_{21}x_1 + \epsilon_2$, then $Cov(x_1, x_2) = b_{21}Var(x_1) + \alpha\beta Var(z)$ can be non-zero even if b_{21} is 0.

Finding a Causal Order

- Suppose we have N measurements of x_1 and x_2.
  - $\overline{x_1^2} = \frac{1}{N}\sum_{i=1}^{N} x_{1i}^2$, similarly for $\overline{x_2^2}$ and $\overline{x_1 x_2}$
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$$E\begin{bmatrix} \overline{x_1^2} \\ \overline{x_1 x_2} \\ \overline{x_2^2} \end{bmatrix} = \begin{bmatrix} b_{12}^2 E(x_2^2) + E(\xi_1^2) \\ b_{12} E(x_2^2) \\ E(x_2^2) \end{bmatrix} \quad \text{which we denote by} \quad E[\boldsymbol{m}_2] = \boldsymbol{\sigma}_2(\boldsymbol{\tau}_2)$$

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- Fourth order moments for Model 1 and assume the residuals are NOT normally distributed:

$$E\begin{bmatrix} \overline{x_1^4} \\ \overline{x_1^3 x_2} \\ \overline{x_1^2 x_2^2} \\ \overline{x_1 x_2^3} \\ \overline{x_2^4} \end{bmatrix} = \begin{bmatrix} b_{12}^4 E(x_2^4) + 6b_{12}^2 E(x_2^2)E(\xi_1^2) + E(\xi_1^4) \\ b_{12}^3 E(x_2^4) + 3b_{12}E(x_2^2)E(\xi_1^2) \\ b_{12}^2 E(x_2^4) + E(x_2^2)E(\xi_1^2) \\ b_{12}E(x_2^4) \\ E(x_2^4) \end{bmatrix}$$

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$$T = N\left(\begin{bmatrix}\boldsymbol{m}_2 \\ \boldsymbol{m}_4\end{bmatrix} - \begin{bmatrix}\boldsymbol{\sigma}_2(\hat{\boldsymbol{\tau}}_2) \\ \boldsymbol{\sigma}_4(\hat{\boldsymbol{\tau}}_4)\end{bmatrix}\right)^T \hat{M} \left(\begin{bmatrix}\boldsymbol{m}_2 \\ \boldsymbol{m}_4\end{bmatrix} - \begin{bmatrix}\boldsymbol{\sigma}_2(\hat{\boldsymbol{\tau}}_2) \\ \boldsymbol{\sigma}_4(\hat{\boldsymbol{\tau}}_4)\end{bmatrix}\right),$$

- Measure of model fit $\approx$ distance between data and the model used
- **Compare T for the two models which will imply causal direction.**

Causal ordering - more than 2 variables

- Suppose we have n variables, and we want to find an ordering $i(1), \dots, i(n)$ such that
  - $x_{i(j)} = \sum_{k=1}^{j-1} b_{i(j),i(k)} x_{i(k)} + \epsilon_{i(j)}$ for all $j = 1, \dots, n$, with nonzero coefficients and $\epsilon_{i(j)}$ non-normal, independent from $x_{i(k)}$ for $k < j$.
  - "$x_{i(j)}$ can be written as a linear combination of its preceding variables in that order plus an (mutually) independent error."
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- And there exists a unique permutation to make W lower triangular if the coefficients in B are non-zero.
- We can also estimate B, depending whether we care about the coefficients or just the causal ordering.

Summary (and References)

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Thank you