Guaranteed Voronoi Diagrams of Uncertain Sites

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Abstract

In this paper we investigate the Voronoi diagram that is induced by a set of sites in the plane, where each site's precise location is uncertain but is known to be within a particular region, and the cells of this diagram contain those points guaranteed to be closest to a particular site. We examine the diagram for sites with disc-shaped regions of uncertainty, prove that it has linear complexity, and provide an optimal algorithm for its construction. We also show that the diagram for uncertain polygons has linear complexity. We then describe two generalizations of these diagrams for uncertain discs. In the first, which is related to a standard order-k Voronoi diagram, each cell is associated with a subset of k sites, and each point within the cell is guaranteed closer to any of the sites within the subset than to any site not in the subset. In the second, each cell is associated with the smallest subset guaranteed to contain the nearest site to each point in the cell. For both generalizations, we provide tight complexity bounds and efficient construction algorithms. Finally, we examine the Delaunay triangulations that can exist for sites within uncertain discs, and provide an optimal algorithm for generating those edges that are guaranteed to exist in every such triangulation.

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1 Introduction

Suppose we do not know the precise locations of n sites (n points in the plane) and yet we would like to determine, for every point in the plane, the closest site to that point. If we know the approximate location of each site, say, that the ith site lies in a (closed) subset D_i of the plane, then we might be able to answer this question perhaps not for every point but for many points in the plane. Our goal is to find, for each site i, the set of points that are guaranteed to be closer to that site than to any other. In other words, no matter where each site lies (as long as the jth site is in D_j for every j) the closest site to the point is always site i. For some points, we cannot guarantee a closest site. These points form a subset of the plane that we call the 'neutral zone'.

In this paper, we first formally define the partition of the plane into cells of guaranteed closest points and the neutral zone and state some properties of this partition. We then consider the special case when the uncertain regions (i.e. the subsets D_i) are discs and show that the complexity of the partition in this case is linear in the number, n, of sites, and that it can be calculated in $O(n \log n)$ time.

We also consider the case where each D_i is a polygon, and show that the complexity of the resulting partition is linear in the total number of polygon edges.

We then return to disc-shaped regions of uncertainty, and consider two generalizations of these diagrams. In the first, each cell is associated with a subset of k uncertain discs, and each point in the cell is guaranteed closer to each site within the subset than to any site not in the subset. We show that the complexity of this diagram and the time to construct it does not exceed that of the standard order-k Voronoi diagram.

In the second generalization, we eliminate the neutral zone by associating each point in the plane with the smallest subset of uncertain discs that is guaranteed to contain the nearest site to the point. For example, points that may be closest to sites 1 or 2 form the cell for the set $\{1, 2\}$. We show that the complexity of this finer partition is at most $O(n^3)$, provide an example to show that this bound is tight, and present an algorithm for its construction that is optimal up to logarithmic factors.

Finally, we examine the Delaunay triangulations that can exist for sites within uncertain discs, and provide an optimal algorithm for generating those edges that are guaranteed to exist in every such triangulation.

2 Related work

Voronoi diagrams are a fundamental data structure in computational geometry; see [2] for a survey. Voronoi diagrams involving uncertain sites were investigated with respect to the probabilistic concepts of *expected closest site* and *probably closest site* in [3].

The guaranteed Voronoi diagram of a set of uncertain regions is closely related to the standard Voronoi diagram of those regions. Thus our results rely heavily on properties of standard Voronoi diagrams such as diagrams for circles [9] and diagrams for segments [6].

One of the biggest differences between the guaranteed Voronoi diagram and traditional variants of Voronoi diagrams is that the union of the regions associated with uncertain sites does not cover the plane. The guaranteed Voronoi diagram contains a neutral zone that contains those points that are not guaranteed to be closest to any particular site. Zone diagrams [1] also have this property. In zone diagrams, for a point to be in a site's region, it must be closer to the site than to any point in any other site's region. The recursive nature of this definition raises the question of the uniqueness and existence of zone diagrams; a question that Asano et al. [1] answered (positively).

Some properties of guaranteed Voronoi diagrams of uncertain polygons are given in [5], including a proof of the diagrams' computability, though no complexity claims are made.

3 Definitions

The Euclidean distance between points a and b is denoted $d(a, b)$.

Let A and B be objects in the plane. The interior of A is denoted $int(A)$. The convex hull of A is denoted CH(A). *B penetrates A* if $B \cap int(A) \neq \emptyset$. *$A$ encloses B* if $B \subseteq int(A)$. B is *inside-tangent* to A (or A has inside-tangent B) if $B \subseteq A$ and the boundary of A intersects B, with A being tangent to B at the points in this intersection. B is *outside-tangent* to A (or A has outside-tangent B) if $B \cap A$ is a non-empty subset of the boundary of A. Again, A is tangent to B at the points in this intersection.

If A is a disc, then *shrinking A* refers to the process of reducing the radius of A while keeping its centerpoint fixed. If A has inside-tangent B at a point b, then *shrinking A with respect to b* refers to the process of reducing the radius of A while simultaneously moving its centerpoint towards b, so that B remains inside-tangent to A at b. When clear from the context, we may refer to this as *shrinking A with respect to B*.

We denote the standard Voronoi diagram of a set of regions $\mathcal{D}$ by $V(\mathcal{D})$, and the cell corresponding to region i by R_i. An order-k Voronoi diagram is a generalization of the standard Voronoi diagram in which each cell is associated with a subset $\mathcal{D}' \subseteq \mathcal{D}$ of size k such that the distance from any point p within the cell to any site in $\mathcal{D}'$ is not greater than the distance from p to any site in $\mathcal{D} \setminus \mathcal{D}'$. As in the standard Voronoi diagram the distance from a point p to a site S is $\inf_{q \in S} d(p, q)$. The special case of disc-shaped sites has been investigated in [8]. We denote the order-k Voronoi diagram of a set of discs $\mathcal{D}$ by $V^k(\mathcal{D})$, and a cell of this diagram corresponding to the k discs $\{D_{i \in S \subseteq \{1 \ldots n\}}\}$ by R_S. Hence $V(\mathcal{D}) \equiv V^1(\mathcal{D})$, and $R_i \equiv R_{\{i\}}$.

4 Properties

We are given a set of compact (not necessarily connected) regions in the plane $\mathcal{D} = \{D_1, \ldots, D_n\}$, called *uncertain regions*, each containing a site. Let $H(i,j)$ be the set of points in the plane that are guaranteed to be at least as close to site i as site j. That is,

$$H(i,j) = \{p \mid \forall x \in D_i \ \forall y \in D_j \ d(p,x) \leq d(p,y)\} .$$

We denote the boundary of $H(i,j)$ by $\langle i,j \rangle$; formally,

$$\langle i,j \rangle = \{p \mid \max_{x \in D_i} d(p,x) = \min_{y \in D_j} d(p,y)\} .$$

The *cell* for site i, denoted $\widetilde{R}_i$, is

$$\widetilde{R}_i = \bigcap_{j \neq i} H(i,j) . \tag{1}$$

The boundaries of all such cells $\widetilde{R}_i$ form the *guaranteed Voronoi diagram* for the set $\mathcal{D}$, and we denote it by $\widetilde{V}(\mathcal{D})$.

An *edge of* $\langle i,j \rangle$ in $\widetilde{V}(\mathcal{D})$ is a maximal connected set of points $p \in \langle i,j \rangle$ that lie on the boundary of cell $\widetilde{R}_i$.

We can generalize guaranteed Voronoi diagrams to order-k versions. A *guaranteed order-k Voronoi diagram* of uncertain discs $\mathcal{D}$ (denoted $\widetilde{V}^k(\mathcal{D})$) is the diagram where each cell $\widetilde{R}_S$ contains those points that are guaranteed to be at least as close to every site $D_{i \in S}$ as to any site $D_{j \notin S}$, for every subset $S \subseteq \{1 \ldots n\}$ of size k. Hence

$$\widetilde{R}_S = \bigcap_{i \in S, j \notin S} H(i,j) . \tag{2}$$

Some properties of $\widetilde{V}^k(\mathcal{D})$ are easy to show.

If every uncertain region is a single point, $\widetilde{V}^k(\mathcal{D})$ is the standard nearest-point order-k Voronoi diagram for the regions.

Every cell $\widetilde{R}_S$ of $\widetilde{V}^k(\mathcal{D})$ is a subset of the corresponding cell R_S in the standard order-k Voronoi diagram $V^k(\mathcal{D})$.

It is possible for a cell boundary to not be a one-dimensional curve. Consider $D_i = \{(x,0) \mid x \in [0,2]\}$ and $D_j = \{(x,0) \mid x \in [2,4]\}$. In this case, $\langle i,j \rangle$ is the halfplane $\{(x,y) \mid x \leq 1\}$, and $\widetilde{R}_i = \langle i,j \rangle$. To generalize, if D_j intersects $\mathrm{CH}(D_i)$, then $H(i,j) = \langle i,j \rangle$; and if this intersection is not confined to vertices of $\mathrm{CH}(D_i)$, then $H(i,j) = \langle i,j \rangle = \emptyset$. From this point on, we assume that any nonempty intersection of two regions D_i and D_j is not confined to vertices of $\mathrm{CH}(D_i)$.

A site whose cell is empty can still influence the cell of another site. For example, if the interiors of D_i and D_j intersect, then $\widetilde{R}_i = \emptyset$, yet an edge of $\langle k,i \rangle$ for some other site k can still appear in $\widetilde{V}(\mathcal{D})$.

Lemma 4.1 *Point p is in $\widetilde{R}_{S \subseteq \{1 \ldots n\}}$ in $\widetilde{V}^k(\mathcal{D})$ iff there exists a disc C_p centered at p such that (i) C_p contains every $D_{i \in S}$, (ii) C_p has inside-tangent some $D_{j \in S}$, and (iii) no disc $D_{m \notin S} \in \mathcal{D}$ penetrates C_p. In addition, p is on the boundary of $\widetilde{R}_S$ iff C_p has outside-tangent some $D_{m \notin S}$.*

Proof. This follows immediately from the definitions of $\widetilde{R}_S$ and an edge of $\langle i,j \rangle$. □

Lemma 4.2 *Each cell $\widetilde{R}_S \subseteq \widetilde{V}^k(\mathcal{D})$ is connected and convex.*

Proof. Let p and q be arbitrary points in $\widetilde{R}_S$. We show that segment $\overline{pq}$ lies within $\widetilde{R}_S$, thus proving both properties. Lemma 4.1 implies there exist discs C_p centered at p and C_q centered at q that contain every $D_{i \in S}$, and are penetrated by no disc $D_{k \notin S}$. This implies $D_{i \in S} \subseteq C_p \cap C_q$. Now, each point $p' \in \overline{pq}$ is the center of a disc $C_{p'}$ that contains $C_p \cap C_q$, and lies within $C_p \cup C_q$; hence $C_{p'}$ satisfies conditions (i) and (iii) of Lemma 4.1. C_p can be shrunken until it has inside-tangent some $D_{i \in S}$, at which point it will satisfy condition (ii) as well. Thus $p' \in \widetilde{R}_S$. □

Note that unlike $\widetilde{R}_S$, R_S may not be connected. In Figure 4.1, $R_{\{1,2\}}$ consists of just the two points p and q.

5 A mapping from boundary points of $\widetilde{R}_S$ to R_S

We now introduce a function that maps points on the boundary of a region $\widetilde{R}_S$ of an order-k guaranteed

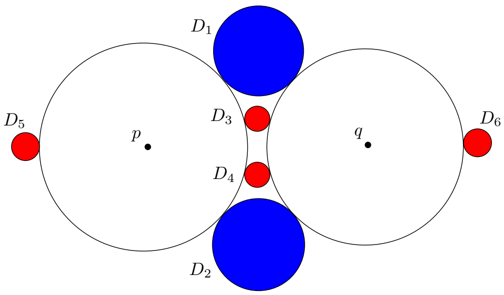

Figure 4.1: $R_{\{1,2\}}$ is not connected

Voronoi diagram of uncertain sites within $\mathcal{D}$ to points on the boundary of the ordinary order-k Voronoi diagram for the regions $\mathcal{D}$. While this mapping exists and its ordering property (Lemma 5.1) holds for arbitrary regions, we will use it to bound the complexity of order-k guaranteed Voronoi diagrams for uncertain discs and uncertain polygons.

We construct a mapping $\delta()$ from boundaries of cells of $\widetilde{V}^k(\mathcal{D})$ to boundaries of cells of $V^k(\mathcal{D})$ in the following way. Consider a point p on the boundary of $\widetilde{R}_{S\subseteq\{1\ldots n\}}$ in $\widetilde{V}^k(\mathcal{D})$, and its disc C_p from Lemma 4.1. Since p is on the boundary of the region, C_p has outside-tangent some[1] $D_{k\notin S}$ at some point b. We now shrink C_p with respect to b, until it has outside-tangent some $D_{i\in S}$. The center of the shrunken C_p is on an edge $\langle\langle i,k\rangle\rangle$ on the boundary of $R_S \subseteq V^k(\mathcal{D})$, and we let $\delta(p)$ be this centerpoint.

Lemma 5.1 *If p, q, and s are points on the boundary of cell $\widetilde{R}_S$ within $\widetilde{V}^k(\mathcal{D})$, and $p \prec_s q$, then $\delta(p) \preceq_{\delta(s)} \delta(q)$.*

Proof. We will first show that if the $\delta(\cdot)$ mapping does not preserve this ordering, then some pair of the line segments $\overline{p\delta(p)}$, $\overline{q\delta(q)}$, $\overline{s\delta(s)}$ must intersect; we will then derive a contradiction from this fact.

For each p on the boundary of $\widetilde{R}_S$, segment $L = \overline{p\delta(p)}$ does not intersect the interior of $\widetilde{R}_S$; for if it did, some point p' interior to L must lie in the interior of $\widetilde{R}_S$, but no disc centered at p' can contain every $D_{j\in S}$ without being penetrated by some $D_{m\notin S}$[2]. Note also that for each $p' \in L$, $C_{p'}$ intersects every region in S, and no smaller disc at p' intersects any region not in S; hence L is within R_S.

Thus if $p \prec_s q$ and $\delta(p) \succ_{\delta(s)} \delta(q)$, then $p \neq q$, $\delta(p) \neq \delta(q)$, and some two of the segments $\overline{s\delta(s)}$, $\overline{p\delta(p)}$, $\overline{q\delta(q)}$ intersect. Without loss of generality, assume $\overline{p\delta(p)}$ intersects $\overline{q\delta(q)}$ at a point v.

By Lemma 4.1, there exists disc C_p which has outside-tangent some $D_{j\notin S}$ at b, such that $\delta(p) \in \overline{pb}$. Similarly, disc C_q exists which has outside-tangent some $D_{k\notin S}$ at d where $\delta(q) \in \overline{qd}$.

The mapping $\delta(\cdot)$ implies there exists disc C_v, centered at v, that has outside-tangent both D_j at b and D_k at d. Now, if $b = d$, then either (i) $v \neq b$, in which case p' and q' must be moving along the same line towards b, and will both stop at the same point, so $\delta(p) = \delta(q)$, a contradiction; or (ii) $v = b$, which implies $\delta(p) = b = d = \delta(q)$, also a contradiction. Hence, $b \neq d$. Now, if some point p' precedes v along $\overline{p\delta(p)}$, then $C_{p'}$ (which must have b on its boundary) will enclose d, a contradiction. Thus $v = p$, and by a symmetric argument, $v = q$; but then $p = q$, again a contradiction. □

6 Uncertain discs

We now consider the case where the uncertain regions are discs; see Figure 6.1.

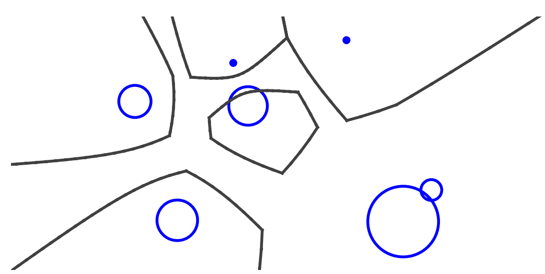

Figure 6.1: Guaranteed (order-1) Voronoi diagram

[1] If more than one region D_j is outside-tangent to C_p (or if D_j is tangent to C_p at more than one point), then we make $\delta(p)$ well-defined by selecting a b according to some total order on possible b's.

[2] Unless D_j and D_m intersect at a single point, in which case the interior of $\widetilde{R}_S$ is empty.

Each disc has a nonnegative radius r_i, and a center α_i. Each $p \in \langle i,j \rangle$ satisfies

$$d(p, \alpha_i) + r_i = d(p, \alpha_j) - r_j \ . \tag{3}$$

Since r_i and r_j are constants, the points p which satisfy (3) lie on an arm of a hyperbola with foci at α_i and α_j. If the discs' radii are both zero, this is the perpendicular bisector of $\overline{\alpha_i \alpha_j}$; otherwise, it is the hyperbolic arm closest to α_i.

The boundary of a cell $\widetilde{R}_i$ within $\widetilde{V}(\mathcal{D})$ may contain two distinct edges of $\langle i,j \rangle$; see Figure 6.2.

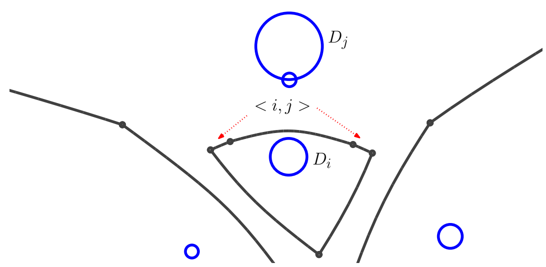

Figure 6.2: Multiple edges of $\langle i,j \rangle$ on the boundary of $\widetilde{R}_i$.

We now show that the number of edges in a guaranteed Voronoi diagram of n discs is $O(n)$. We do this by showing that for each cell $\widetilde{R}_i \in \widetilde{V}(\mathcal{D})$, each edge in $\widetilde{R}_i$ maps to a distinct edge in the corresponding cell of $V(\mathcal{D})$, which is known to have $O(n)$ edges.

Theorem 6.1 *The number of edges in a guaranteed order-k Voronoi diagram of n uncertain discs is $O(k \cdot n)$.*

Proof. We will show that the number of edges in $\widetilde{V}^k(\mathcal{D})$ is at most twice the number of edges in $V^k(\mathcal{D})$. The theorem then follows from the fact that $V^k(\mathcal{D})$ has $O(k \cdot n)$ edges [8].

Consider the edges around $\widetilde{R}_S$ in ccw order. Lemma 4.2 ensures that these edges are connected. We charge each edge E of $\langle i,j \rangle \in \widetilde{V}(\mathcal{D})$ to the edge F of $\langle\langle i,j \rangle\rangle$ on which $\delta(p)$ lies, for p the ccw-first point of E (or any interior point p if E is ccw-infinite). Suppose two distinct edges E_1 and E_2 of $\langle i,j \rangle$ map to the same edge F of $\langle\langle i,j \rangle\rangle$ in $R_{S \supseteq \{i\}} \subseteq V(\mathcal{D})$. Since E_1 and E_2 are distinct but both of $\langle i,j \rangle$, there must exist an edge E' of $\langle i,k \rangle$ ($k \neq j$) between them in the ccw traversal of $\widetilde{R}_i$ that maps to some other edge F' of $\langle\langle i,k \rangle\rangle$ in R_S. This contradicts Lemma 5.1 since all points of F either precede or follow the points of F' in ccw-order.

Thus each edge in $V(\mathcal{D})$ of $\langle\langle i,j \rangle\rangle$ is charged at most twice: once by an edge of $\langle i,j \rangle$, and once by an edge of $\langle j,i \rangle$. Hence $\widetilde{V}^k(\mathcal{D})$, like $V^k(\mathcal{D})$, has $O(n)$ edges. □

We now show how $\widetilde{V}^k(\mathcal{D})$ for a set of discs can be constructed by first constructing $V^k(\mathcal{D})$ for the discs, then performing a linear-time transformation from $V(\mathcal{D})$ to $\widetilde{V}(\mathcal{D})$.

Let $\widetilde{I}_S$ be the sequence of pairs (i,j) corresponding to the edges $\langle i_{\in S}, j_{\notin S} \rangle$ encountered in a ccw traversal of the boundary of $\widetilde{R}_S$, when starting from an edge containing some point p on the boundary. We similary define I_S to be the sequence of pairs (i,j) corresponding to the edges $\langle\langle i_{\in S}, j_{\notin S} \rangle\rangle$ traversed on the boundary of R_S, when starting from the edge containing $\delta(p)$.

Lemma 6.2 *For every cell $\widetilde{R}_S \in \widetilde{V}^k(\mathcal{D})$, $\widetilde{I}_S$ is a subsequence of I_S.*

Proof. If (i,j) is in $\widetilde{I}_S$, then since $\delta(\cdot)$ maps points on edges of $\langle i_{\in S}, j_{\notin S} \rangle$ to points on edges of $\langle\langle i_{\in S}, j_{\notin S} \rangle\rangle$, (i,j) must be in I_S. Furthermore, the order of pairs in $\widetilde{I}_S$ is preserved in I_S since $\delta(\cdot)$ preserves this order by Lemma 5.1. □

Theorem 6.3 *$\widetilde{V}^k(\mathcal{D})$ for n uncertain discs can be constructed in $O(k^2 \cdot n \log n)$ time.*

Proof. We can construct $V^k(\mathcal{D})$ for the disc sites in $O(k^2 \cdot n \log n)$ time [8]. We generate the sequence I_S of sites comprising the boundary of each cell R_S in $V(\mathcal{D})$ from this diagram in linear time by a simple traversal. We then construct the boundary of $\widetilde{R}_i$ by generating and intersecting the sequence of hyperbolic arcs $\langle i,j \rangle$ for each pair $(i,j) \in I_i$. Lemma 6.2 ensures that we consider a correctly ordered supersequence of the arcs bounding $\widetilde{R}_i$. This suffices to construct the boundary of $\widetilde{R}_i$ in time proportional to the length of I_i.

Since each of the $O(k \cdot n)$ edges of $V^k(\mathcal{D})$ appears in two cell boundaries, the running time for the construction of the edges of all cells of $\widetilde{V}^k(\mathcal{D})$ is $O(k \cdot n)$. The time to construct $\widetilde{V}^k(\mathcal{D})$ is thus dominated by the time to construct $V^k(\mathcal{D})$. When $k = 1$, this running time is optimal, since if the disc radii are all zero, $\widetilde{V}(\mathcal{D})$ is the standard Voronoi diagram of n points. □

7 Uncertain polygons

We now turn our attention to the case where the region of uncertainty for each site is a simple polygon. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a set of n polygons. In this case, each $\langle i, j\rangle$ consists of some number of (possibly unbounded) parabolic arcs, each induced by a vertex u of D_i and a vertex[3] or edge (line segment open at its endpoints) v of D_j. We denote such a parabolic arc by $\langle i^u, j^v\rangle$, and define an edge of $\langle i^u, j^v\rangle$ to be a maximal connected set of points $p \in \langle i^u, j^v\rangle$ that lie on the boundary of cell $\widetilde{R}_i$. We define $\langle\!\langle i^u, j^v\rangle\!\rangle$ for $V(\mathcal{D})$ (the standard Voronoi diagram for polygonal sites $\mathcal{D}$) analogously.

Theorem 7.1 *The number of edges in the guaranteed Voronoi diagram of $\mathcal{D}$, a set of n polygons with m total edges, is $O(m)$.*

Proof. We show that the number of edges in $\widetilde{V}(\mathcal{D})$ is at most twice the number of edges in $V(\mathcal{D})$ plus twice the complexity of the furthest point Voronoi diagram of the vertices in D_i summed over all i. The theorem then follows from the fact that $V(\mathcal{D})$ has $O(m)$ complexity [6] and that the total complexity of the furthest point Voronoi diagrams is $O(m)$ [7].

Let E be an edge of $\langle i^u, j^v\rangle$ on the boundary of $\widetilde{R}_i$ and let p be an interior point of E. There then exists a point $\delta(p)$ (defined in Section 5) on an edge of $\langle\!\langle i^w, j^v\rangle\!\rangle$ where w is a vertex or edge of D_i.

Consider the edges around $\widetilde{R}_i$ in ccw order. We charge each edge E of $\langle i^u, j^v\rangle$ to the edge F of $\langle\!\langle i^w, j^v\rangle\!\rangle$ on which $\delta(p)$ lies, for p the ccw-first point of E (or any interior point p if E is ccw-infinite). Now it may happen that a consecutive sequence of edges around $\widetilde{R}_i$ all map to F. (By Lemma 5.1, the edges must be consecutive if they map to the same F.) Let E_1 of $\langle i^{u_1}, j^v\rangle$ and E_2 of $\langle i^{u_2}, j^v\rangle$ be two successive (adjacent) edges in this ccw sequence. The point p shared by E_1 and E_2 lies on an edge of the furthest-point Voronoi diagram of the vertices of D_i that separates the furthest-point regions for u_1 and u_2. We charge the edge E_2 to this edge T of the furthest-point Voronoi diagram. We now show that at most two edges are charged to each T. Every such p intersecting T is the center of a disc C_p that has inside-tangent D_i (at the two farthest vertices u_1, u_2 associated with T) and outside-tangent D_j. Assume by way of contradiction that there are three such points, p_1, p_2, p_3 in order along T. Observe that C_{p_2} is contained within $C_{p_1} \cup C_{p_3}$; thus D_j must be outside-tangent to C_{p_2} at either u_1 or u_2 to avoid penetrating the other two discs. But then $D_i \cap D_j$ is a nonempty subset of $\{u_1, u_2\}$, both vertices of $\mathrm{CH}(D_i)$, a contradiction.

Thus the number of edges on the boundary of cell $\widetilde{R}_i$ is at most the number of edges on the boundary of cell R_i plus twice the number of edges in the furthest-point Voronoi diagram for the vertices of D_i. The theorem then follows since each edge of $V(\mathcal{D})$ bounds two cells R_i and R_j. □

8 Extension to subsets of closest sites

In this section, we look at an extension of the guaranteed Voronoi diagram which assigns every point in the plane to a cell, including points in the neutral zone. We do this by determining, for each point p, what the smallest set of sites is such that one is guaranteed to be a closest site to p (throughout this section, we assume disc-shaped regions of uncertainty). We call the resulting partition of the plane, which we denote by $\widetilde{V}^{\{\}}(\mathcal{D})$, a *guaranteed subset Voronoi diagram* (Figure 8.1).

The cells of $\widetilde{V}^{\{\}}(\mathcal{D})$ are not necessarily connected. In Figure 8.1, for instance, the two shaded regions belong to the same cell.

The guaranteed subset Voronoi diagram can also

[3] In this case, the induced parabola degenerates to a line.

be defined by generalizing equation (1):

$$\widetilde{R}_S = \bigcup_{i\in S}\Big[\bigcap_{j\notin S} H(i,j)\Big] - \bigcup_{S'\subset S} \widetilde{R}_{S'}$$

where $\widetilde{R}_\emptyset = \emptyset$. Note that only the closure of this definition of $\widetilde{R}_S$ is strictly equivalent to the first definition.

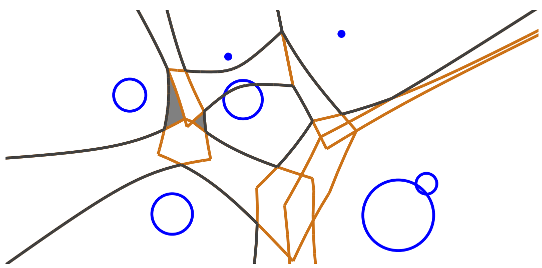

Figure 8.1: Guaranteed subset Voronoi diagram

The following construct will prove useful in generating $\widetilde{V}^{\{\}}(\mathcal{D})$. The *possible cell* for a site i, denoted $\widetilde{P}_i$, is the set of all points for which it is possible that site i is the closest site. Formally,

$$\widetilde{P}_i = \{p | \exists q \in D_i \forall j \in \{1 \ldots n\} \exists s \in D_j d(p,q) \leq d(p,s)\}$$

The possible cell for one particular site is shown in Figure 8.2.

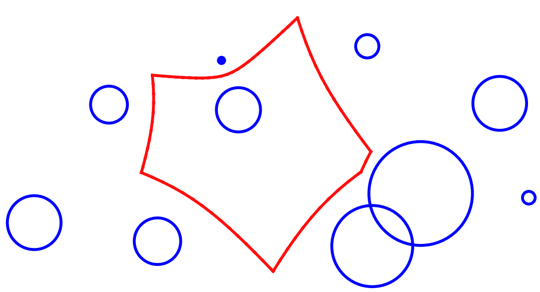

Figure 8.2: The possible cell for a site

The possible cells for a set of sites do not generally have disjoint interiors, unless each uncertain region is a single point, in which case the possible cells are just standard Voronoi cells for point sites.

Lemma 8.1 *Point p is within $\widetilde{P}_i$ iff there exists a disc centered at p that intersects D_i and encloses no other discs.*

Observe that the guaranteed subset Voronoi diagram associates each point in the plane with the subset of sites whose possible cells contain the point. Note also that this diagram is simply the arrangement of the boundaries of each possible cell; therefore, to construct the guaranteed subset Voronoi diagram, it suffices to construct this arrangement.

To construct the possible cells efficiently, we first construct the *minmax Voronoi diagram* of the uncertain discs, which is the additively weighted Voronoi diagram of the disc centers where each disc's weight is set to its (positive) radius. We denote this diagram by $V^m(\mathcal{D})$, and the cell for D_i by R_i^m.

Lemma 8.2 *Every vertex of $\widetilde{P}_i$ lies on an edge of $V^m(\mathcal{D})$.*

Proof. Suppose p is a vertex of $\widetilde{P}_i$. Since $p \in \widetilde{P}_i$, by Lemma 8.1, there exists disc C_i centered at p that has outside-tangent D_i, and contains no other discs. Since p is a vertex of $\widetilde{P}_i$, C_i must have inside-tangent some disc D_j; and since C_i encloses no discs, it follows that $p \in R_j^m$. If p is the point at infinity, then R_j^m is unbounded, and some unbounded edge of $V^m(\mathcal{D})$ must exist containing p. Otherwise, C_i must have inside-tangent some additional disc D_k, and $p \in R_k^m$; hence p is on the edge between R_j^m and R_k^m. □

The *core* of a site i, denoted Γ_i, is the union of the minmax Voronoi cell for D_i with those of any discs intersecting D_i: that is, $\Gamma_i = \bigcup_{D_j\cap D_i\neq\emptyset} R_j^m$.

For convenience, we will surround the uncertain regions by a large circle, and clip the various diagrams ($V^m(\mathcal{D})$, $\widetilde{P}_i$) to this circle. This simplifies our analysis in two ways: it eliminates unbounded edges, and ensures that every vertex has degree of at least three. We will refer to a vertex where a (previously) unbounded edge meets this clipping circle as a *clip vertex*, and the arcs comprising the clipping circle as *clip edges*.

Lemma 8.3 *For each vertex p of $\widetilde{P}_i$, there exists a path within $\widetilde{P}_i$ along edges of $V^m(\mathcal{D})$ from p to some vertex of the core.*

Proof. Suppose p is a vertex of $\widetilde{P}_i$. We first deal with the case where p is not a clip vertex. By Lemma 8.1, there is a circle C_p centered at p that has outside-tangent D_i and inside-tangent some pair of discs D_j and D_k, and encloses no discs. We now move p, maintaining C_p's inside-tangency with D_j and D_k, in the direction that increases the area of intersection between C_p and D_i. If D_i is 'between' D_j and D_k (in the sense of their order of intersection along the boundary of C_p), we will be reducing the radius of C_p. If, on the other hand, D_i lies to one side of both D_j and D_k, we will be increasing the radius of C_p. Observe that in either case, we are moving along edge (j, k) within $V^m(\mathcal{D})$.

We stop moving when some disc D_m becomes inside-tangent to C_p, or when we reach the clipping circle. Suppose the former has occurred. If $m = i$, then p is now at a vertex of R_i^m, a core vertex, and we're done. Otherwise, if D_i is between D_j and D_k, D_m will be between D_j and D_k. If D_m is between D_j and D_i, we replace j with m and resume moving p (along edge (m, k)); otherwise, we replace k with m and resume moving (along edge (j, m)).

We now address the case where p started at, or has moved to, some point on the clip circle. C_p now intersects D_i and has inside-tangent at least one D_j. If C_p has inside-tangent some other D_k as well, then p is on edge (j, k) of $V^m(\mathcal{D})$ (if $k = i$, we have reached a core vertex, and we're done). If D_i is between D_j and D_k, then we move p inward along this edge, as described earlier. Otherwise, without loss of generality suppose D_k is 'farthest' from D_i (in terms of their order of intersection with the boundary of C_p). We ignore D_k, and continue moving along the rim of the clipping circle, maintaining C_p's inside-tangency with D_j, until some D_m becomes inside-tangent C_p, whereupon we repeat the above test.

We can be sure that this process of moving p will terminate, since the area of D_i covered by C_p is strictly increasing, and at some point D_i must become inside-tangent to C_p, at which point the next vertex encountered by p will be a core vertex. $\square$

Let V_i^* be the portion of $V^m(\mathcal{D})$ that lies within $\widetilde{P}_i$ but excludes edges within Γ_i.

Lemma 8.4 *V_i^* contains no cycles.*

Proof. If V_i^* contains a cycle, then it must enclose at least one cell $R_{j \neq i}^m$, and R_j^m cannot be part of Γ_i. Now, assume by way of contradiction that $\alpha_j \notin R_j^m$. Then there exists some disc D_k such that $d(\alpha_j, \alpha_j) + r_j > d(\alpha_j, \alpha_k) + r_k$; but then for any point q, by the triangle inequality, $d(q, \alpha_j) + r_j > d(q, \alpha_j) + d(\alpha_j, \alpha_k) + r_k \geq d(q, \alpha_k) + r_k$, and R_j^m must be empty. Thus $\alpha_j \in V_i^*$. Then $\alpha_j \in \widetilde{P}_i$, and $d(\alpha_i, \alpha_j) - r_i \leq d(\alpha_j, \alpha_j) + r_j$, which implies $d(\alpha_i, \alpha_j) \leq r_i + r_j$. Hence D_i intersects D_j, so $R_j^m \subseteq \Gamma_i$, a contradiction. $\square$

Lemma 8.5 *Each $\widetilde{P}_i$ has $O(n)$ vertices and edges.*

Proof. By Lemma 8.4, each vertex on the boundary of V_i^* is the root of a (possibly empty) tree within V_i^*. By Lemma 8.2, all vertices of $\widetilde{P}_i$ must lie on these edges; and by Lemma 8.3, if an edge contains a vertex p of $\widetilde{P}_i$, then no vertices of $\widetilde{P}_i$ will occur in the subtree rooted at p. If any point on an edge occurs within $\widetilde{P}_i$, then at least one of that edge's endpoints must as well; hence a portion of an edge can appear in at most two trees rooted on the boundary of V_i^*. It follows that there are at most $2n$ vertices in $\widetilde{P}_i$. $\square$

Theorem 8.6 *The number of edges in a guaranteed subset Voronoi diagram of n uncertain discs is $O(n^3)$, and this bound is tight.*

Proof. As noted eariler, $\widetilde{V}^{\{\}}(\mathcal{D})$ is the arrangement of boundaries of possible cells $\widetilde{P}_{i \in 1 \dots n}$. Its complexity is thus the sum of the complexities of the individual possible cells (which, by Lemma 8.5, is $n \cdot O(n)$), plus the number of proper intersection points of edges from pairs of cells $(\widetilde{P}_i, \widetilde{P}_j)$. We can bound this latter term by noting that each point p that lies on the boundaries of $\widetilde{P}_i$ and $\widetilde{P}_j$ is the center of two discs: C_i, which has outside-tangent D_i and inside-tangent some D_k, and C_j, which has outside-tangent D_j and inside-tangent D_m. Since no discs can penetrate C_i or C_j, it follows that $C_i = C_j$. We can assume, without loss of generality, that $r_j > 0$; for if $r_i = r_j = 0$, $\widetilde{P}_i$ cannot properly intersect $\widetilde{P}_j$.

This implies that $j \neq k$, since D_j cannot be both inside- and outside-tangent C_i. Each point p is therefore determined by an ordered triple of distinct discs (D_i, D_j, D_k). The total complexity of $\widetilde{V}^{\{\}}(\mathcal{D})$ is thus $O(n \cdot O(n) + \binom{n}{3}) = O(n^3)$.

We can show that this bound is tight; see Figure 8.3. We place two sets of $n/3$ discs of large radii r clustered tightly around the points $(-r-\epsilon, 0)$ and $(r+\epsilon, 0)$, for small positive ϵ. We place an additional $n/3$ discs with zero radii in a stack near the origin. For each large disc D_i, this stack induces $\Omega(n/3)$ arcs in the boundary of $\widetilde{P}_i$; and for each pair of large discs (D_i, D_j) on opposite sides of the x-axis, the boundaries of $\widetilde{P}_i$ and $\widetilde{P}_j$ will intersect in $\Omega(n/3)$ points. The total number of intersections in $\widetilde{V}^{\{\}}(\mathcal{D})$ is thus $(n/3)^2 \cdot \Omega(n/3) = \Omega(n^3)$. □

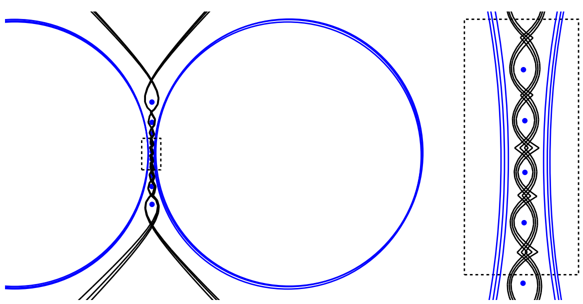

Figure 8.3: $\widetilde{V}^{\{\}}(\mathcal{D})$ complexity for discs is $\Omega(n^3)$ (some edges omitted for clarity)

Theorem 8.7 *The guaranteed subset Voronoi diagram of n uncertain discs can be constructed in time $O((n+k)\log n + \sum_{i=1}^{n} g(i))$, where k is the complexity of the diagram, and $g(i)$ is the number of discs intersecting disc D_i.*

Proof. We start by constructing $V^m(\mathcal{D})$, the min-max Voronoi diagram of the discs, in $O(n \log n)$ time. We assume this diagram exists as a graph, with each edge represented as two directed half-edges, and that these edges are sorted by polar angle around each vertex. We also assume the diagram uses the clipping circle described earlier.

For each edge of $V^m(\mathcal{D})$, we determine if the possible cell for the site, D_i, to its left needs constructing. If so, we determine the vertices comprising the boundary of its core, and perform a depth-first search of each tree within V_i^* rooted at these vertices to find vertices of $\widetilde{P}_i$ (by Lemma 8.2, a vertex of $\widetilde{P}_i$ will lie on edge (j, j') of $V^m(\mathcal{D})$ iff $\langle j, i\rangle \cap \langle j', i\rangle \in (j, j')$). By Lemma 8.3, once we reach an edge that contains a vertex of $\widetilde{P}_i$, we can ignore the rest of that edge's subtree.

Since each vertex has degree of at least three, at most $2k$ tree edges will need to be visited during these searches.

For each D_j that intersects D_i, Γ_i may contain up to two extra vertices that are not incident to edges in V_i^*. Traversing these edges imposes an additional total cost of $O(\sum_{i=1}^{n} g(i))$.

Generating all n possible cells can thus be done in time $O(n \log n + k + \sum_{i=1}^{n} g(i))$. The arrangement of their $k = O(n^2)$ edges can then be generated, using a plane sweep, in time $O(k \log k)$, for the stated total running time. □

Except for a few degenerate cases (e.g., where every disc is nested within another), $k = O(n)$, and the time to construct $\widetilde{V}^{\{\}}(\mathcal{D})$ simplifies to $O(k \log n + \sum_{i=1}^{n} g(i))$. If, in addition, the discs are disjoint, or are guaranteed to intersect at most some small constant number of other discs, then this time further simplifies to $O(k \log n)$. In all cases, by Theorems 8.6 and 8.7, $\widetilde{V}^{\{\}}(\mathcal{D})$ can be constructed in $O(n^3 \log n)$ time.

9 Guaranteed Delaunay edges

The dual of a Voronoi diagram of a set of points is a Delaunay triangulation. Recall that with point sites $\mathcal{D}$, a Delaunay edge exists between D_a and D_b if there exists a point p that is the center of a disc whose boundary contains D_a and D_b, and encloses no other D_c. In this section, we consider the Delaunay triangulation of a set of sites, where each site lies within a disc-shaped region of uncertainty.

We define a *guaranteed Delaunay edge* for uncertain discs $\mathcal{D}$ to be a pair (a, b) such that for all possible vectors of point sites $P = (p_1 \in D_1, \ldots, p_n \in D_n)$, edge (a, b) exists in the Delaunay triangulation of P. See Figure 9.1.

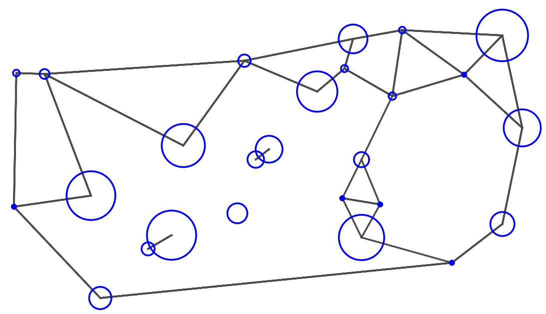

Figure 9.1: Guaranteed Delaunay edges for uncertain discs

It is clear that the set of guaranteed Delaunay edges for a set of uncertain discs form a subgraph of the Delaunay triangulation of the uncertain discs' centerpoints. Thus one approach to generating these edges is to construct this Delaunay triangulation, then test each of its edges to see if it corresponds to a guaranteed Delaunay edge. In fact, we will be able to generate the guaranteed Delaunay edges more efficiently if we instead construct the order-2 Voronoi diagram of the discs, and consider pairs of sites that have a nonempty region as candidates.

For the moment, we will assume that the set of discs $\mathcal{D}$ is *partially disjoint*: no two discs $D_i, D_j \in \mathcal{D}$ exist such that $D_i \subset D_j$. We will see later how to relax this restriction.

In the following analysis, we show how each cell of $V^2(\mathcal{D})$ may generate a *spine* (defined shortly), each of which in turn corresponds to a guaranteed Delaunay edge. Before proceeding, we will require the following definition.

If C is a disc and S' is a partially disjoint subset of a set of objects S, we say that C *is supported by* S' *within* S if every $s' \in S'$ is inside-tangent to C, and no object $s \in S \setminus S'$ penetrates C. When S is clear from the context, we may simply say that C is supported by S'.

Lemma 9.1 *If C is supported by objects $\{X, Y\}$ within S, then for each pair of points $(x \in X, y \in Y)$ there exists a disc $C' \subseteq C$ supported by $\{x, y\}$ within $(S - \{X, Y\}) \cup \{x, y\}$.*

Proof. We can construct C' by shrinking C until its boundary intersects (without loss of generality) x, then shrinking C' with respect to x until its boundary intersects y. $\square$

We can now define the spine $sp(a, b)$ of two partially disjoint discs $D_a, D_b \in \mathcal{D}$ as the set of centerpoints for discs supported by $\{D_a, D_b\}$.

Lemma 9.2 *$sp(a, b)$ is a connected subset of a hyperbolic arc.*

Proof. Every point $p \in sp(a, b)$ satisfies $d(p, \alpha_a) + r_a = d(p, \alpha_b) + r_b$, so p lies on a hyperbolic arc. We now prove that the spine is connected. Place the axes so α_a and α_b are on the x-axis, with α_a to the left of α_b. Each point on the spine now has a unique y-coordinate. Assume by way of contradiction that u, v, and w are points on $sp(a, b)$ such that $u_y < v_y < w_y$, $u, w \in sp(a, b)$, and $v \notin sp(a, b)$.

Let discs C_u,C_v,C_w be the discs centered at u, v, w that have inside-tangent D_a and D_b. Since $v \notin sp(a, b)$, some point $q \in D_{k \notin \{a,b\}} \in \mathcal{D}$ exists that penetrates C_v. Let L be the line through the points of tangency of C_v with D_a and D_b. Observe that the two intersection points of the boundaries of C_u and C_v lie on or above L, while those of C_v and C_w lie on or below L; so if q lies on or above L, it penetrates C_w, and if it lies below L, it penetrates C_u. Thus either u or w is not in $sp(a, b)$; a contradiction. Hence $sp(a, b)$ is connected. $\square$

We are now ready to show that (i) spines correspond to guaranteed Delaunay edges, and (ii) every spine is associated with a cell of $V^2(\mathcal{D})$.

Lemma 9.3 *If (D_a, D_b) is a partially disjoint pair in a set of uncertain discs $\mathcal{D}$, then (a, b) is a guaranteed Delaunay edge iff $sp(a, b)$ is nonempty.*

Proof. Assume (D_a, D_b) is a partially disjoint pair as stated. If there exists a point $p \in sp(a, b)$, then p is the center of a disc supported by $\{D_a, D_b\}$. Lemma 9.1 then implies that for any vector of point sites $P = (p_1 \in D_1, \ldots, p_n \in D_n)$, there exists a disc $D' \subseteq D$ supported by $\{p_a, p_b\}$; hence (a, b) is a guaranteed Delaunay edge.

If (a, b) exists, we place the axes as in Lemma 9.2. Let M_y be the disc centered at the point (x, y) on the hyperbola from Lemma 9.2, so that M_y has inside-tangent discs D_a and D_b. Let a_y (resp., b_y) be the point of tangency of D_a (resp., D_b) with M_y. Let I be the set of discs $\mathcal{D} \setminus \{D_a, D_b\}$ penetrating M_0. If $I = \emptyset$, then M_0 is supported by $\{D_a, D_b\}$, and we are done. Otherwise, without loss of generality, assume $int(M_0) \cap I$ lies below the x-axis (it cannot straddle the x-axis, otherwise there exist points $p^+ \in I$ penetrating all $M_{y>0}$, and $p^- \in I$ penetrating all $M_{y<0}$; so (a, b) does not exist). Let y' be the minimum value such that $I \cap int(M_{y'}) = \emptyset$. Note that $y' > 0$, and both $a_{y'}$ and $b_{y'}$ are on or below the x-axis.

We now prove that no $D_d \in \mathcal{D} \setminus \{D_a, D_b\}$ penetrates $M_{y'}$. Assume by way of contradiction that such D_d exists, with point of intersection $p' \in D_d \cap int(M_{y'})$. Using the same sweep argument as Lemma 9.2, we can show that p' must have first appeared within the upper arc of some $M_{0<y\leq y'}$.

Observe that every disc α whose boundary contains $a_{y'}$ and $b_{y'}$ is centered on the bisector of $a_{y'}$ and $b_{y'}$, and since (a, b) exists, some α must exist which is not penetrated by $I \cup \{p'\}$. But we cannot move the center of $M_{y'}$ closer to the midpoint of $a_{y'}$ and $b_{y'}$ without it penetrating I, nor can we move it farther from the midpoint without it continuing to penetrate p' (since we will never lose the area swept out earlier by the upper arc); hence α does not exist, a contradiction. ☐

Lemma 9.4 *If (D_a, D_b) are partially disjoint discs from $\mathcal{D}$, and $sp(a, b)$ is nonempty, then $R_{\{a,b\}}$ is a cell within $V^2(\mathcal{D})$.*

Proof. Suppose (D_a, D_b) are partially disjoint discs from $\mathcal{D}$, and $sp(a, b)$ is nonempty. Take any point $p \in sp(a, b)$. Without loss of generality, assume $r_a \leq r_b$. Now, $sp(a, b)$ is a subset of the zeros of the function

$$f(x) = \big(d(x, \alpha_a) + r_a\big) - \big(d(x, \alpha_b) + r_b\big) ,$$

whereas $\langle\!\langle a, b\rangle\!\rangle$ is a subset of the zeros of the function

$$g(x) = \big(d(x, \alpha_a) - r_a\big) - \big(d(x, \alpha_b) - r_b\big) .$$

Note that if $r_a = r_b$, $g(p) = 0$. Otherwise, consider the line segment $\overline{p\alpha_a}$. Since $f(p) = 0$, and $r_b > r_a$, it is easy to show that $g(p) > 0$. Now,

$$\begin{aligned} g(\alpha_a) &= -r_a - \big(d(\alpha_a, \alpha_b) - r_b\big) \\ &= (r_b - r_a) - d(\alpha_a, \alpha_b) \end{aligned} ,$$

and since (D_a, D_b) are partially disjoint, we must have

$$d(\alpha_a, \alpha_b) > r_b - r_a ;$$

so $g(\alpha_a) < 0$. Thus there must exist some $p' \in \overline{p\alpha_a}$ such that $g(p') = 0$.

We now show that $p' \in \langle\!\langle a, b\rangle\!\rangle$. For p' not to be on $\langle\!\langle a, b\rangle\!\rangle$, there must exist some $D_k \in \mathcal{D} \setminus \{D_a, D_b\}$ such that the additively weighted distance from p' to α_k is less than that from p' to α_a. Note, however, that p was already at least as close to D_a as to any other disc (except possibly D_b), and moved directly towards D_a to get to p'; thus no such D_k can exist. Finally, since the additively weighted distance of p' to D_a and D_b is equal, and minimal over all discs in $\mathcal{D}$, p' must lie within $R_{\{a,b\}}$, a cell of $V^2(\mathcal{D})$. ☐

Lemma 9.5 *The guaranteed Delaunay edges for partially disjoint uncertain discs $\mathcal{D}$ can be constructed in $O(n \log n)$ time.*

Proof. We start by constructing $V^2(\mathcal{D})$, the order-2 Voronoi diagram of the discs. By Lemma 9.4, only those discs (D_a, D_b) which have a nonempty cell $R_{\{a,b\}}$ of $V^2(\mathcal{D})$ can be guaranteed Delaunay edges. For each such cell, we construct the hyperbolic arc A that will contain $sp(a, b)$ (if it exists), per Lemma 9.2. We now show how $sp(a, b)$ can be efficiently constructed from A.

If p is an endpoint (not at infinity) of $sp(a, b)$, there must exist a disc C centered at p, supported by $\{D_a, D_b\}$, which has outside-tangent one or more discs $Q \subset \mathcal{D}\setminus\{D_a, D_b\}$. Clearly, $p \in R_{\{a,b\}}$. Consider any $D_k \in Q$. We can shrink C with respect to D_k until it has outside-tangent at least one of $\{D_a, D_b\}$; without loss of generality, assume D_a. Note that each point on the path of centerpoints of the shrinking C is within $R_{\{a,b\}}$, and that the centerpoint of the final shrunken C is within $R_{\{a,c\}}$ as well. Thus $R_{\{a,b\}}$ and $R_{\{a,c\}}$ are neighbors in $V^2(\mathcal{D})$, and p lies on $\langle a, c\rangle$.

Thus, if after clipping A to the hyperbolic arcs containing $\langle a, c\rangle$ (resp., $\langle b, c\rangle$), for each neighboring cell $R_{\{a,c\}}$ (resp., $R_{\{b,c\}}$) of $R_{\{a,b\}}$, A is nonempty, it represents $sp(a, b)$, which (by Lemma 9.3) implies that (a, b) is a guaranteed Delaunay edge.

$V^2(\mathcal{D})$ contains $O(n)$ edges, and can be generated in $O(n \log n)$ time [8]. Each clipping operation requires constant time, and there are at most two of these for each edge in $V^2(\mathcal{D})$; thus the running time is dominated by the time spent constructing $V^2(\mathcal{D})$. $\square$

We now show how the algorithm of Lemma 9.5 can be modified to handle discs that may not be partially disjoint. The following lemmas will be required.

Lemma 9.6 *If D_a and D_b are uncertain discs from set $\mathcal{D}$, and $D_b \subseteq D_a$, then (a, b) is a guaranteed Delaunay edge iff no $D_{k\notin\{a,b\}} \in \mathcal{D}$ penetrates D_a.*

Proof. If no such D_k penetrates D_a, then consider any pair of points $(p_a \in D_a, p_b \in D_b)$. By following a procedure similar to that of Lemma 9.1, we can construct a disc within D_a that is supported by $\{p_a, p_b\}$; hence (a, b) exists. If, on the other hand, some D_k penetrates D_a, then there exists a disc $C \subset D_a$ of radius $\epsilon > 0$ centered at a point $p_k \in int(D_a) \cap D_k$ such that for any point $p_b \in D_b$ (with $p_b \neq p_k$), there exists a point $p_a \in C$ where p_k is interior to segment $\overline{p_a p_b}$. Thus (a, b) cannot be a guaranteed Delaunay edge. $\square$

We will make use of the fact that the centerpoint of a disc (that is not contained by another disc) lies within the standard Voronoi cell of the disc.

Lemma 9.7 *If $\mathcal{D}$ is a set of n partially disjoint discs, then the center of each $D_i \in \mathcal{D}$ lies in the interior of R_i.*

Proof. Assume by way of contradiction that for some D_i, α_i is not interior to R_i. There must then exist a disc $D_{j\neq i} \in \mathcal{D}$ such that $d(\alpha_i, \alpha_j) - r_j \leq d(\alpha_i, \alpha_i) - r_i$, which implies $d(\alpha_i, \alpha_j) \leq r_j - r_i$. Consider the point $p \in D_i$ farthest from α_j. Now,

$$\begin{aligned} d(p, \alpha_j) &= d(\alpha_i, \alpha_j) + r_i \\ &\leq (r_j - r_i) + r_i \\ &\leq r_j \ , \end{aligned}$$

a contradiction since the discs are partially disjoint. $\square$

Lemma 9.8 *If $\mathcal{D}$ is a set of partially disjoint discs, and $D_i \in \mathcal{D}$ is penetrated by at least one other disc in $\mathcal{D}$, then D_i is penetrated by one of its neighbors within $V(\mathcal{D})$.*

Proof. Suppose D_i is penetrated by some other D_j, yet is not penetrated by any neighbor in $V(\mathcal{D})$. Let p be the point of intersection of segment $\overline{\alpha_i \alpha_j}$ with the Voronoi bisector of D_i and D_j. Then

$$\begin{aligned} & d(\alpha_i, \alpha_j) < r_i + r_j \\ \Rightarrow\ & d(p, \alpha_i) + d(p, \alpha_j) < r_i + r_j \\ \Rightarrow\ & d(p, \alpha_i) - r_i < -(d(p, \alpha_j) - r_j) \\ \Rightarrow\ & d(p, \alpha_i) - r_i < -(d(p, \alpha_i) - r_i) \\ \Rightarrow\ & d(p, \alpha_i) < r_i \ ; \end{aligned}$$

thus p penetrates D_i, and by Lemma 9.7, so does every point on segment $S = \overline{\alpha_i p}$. Now, since D_j is not a neighbor to D_i, there must exist a point $q \in S$ that lies on the Voronoi bisector of D_i and some D_k that does not penetrate D_i. Hence,

$$\begin{aligned} & d(\alpha_i, q) - r_i = d(\alpha_k, q) - r_k \\ \Rightarrow\ & d(\alpha_i, q) - r_i \geq \big(d(\alpha_i, \alpha_k) - d(\alpha_i, q)\big) - r_k \\ \Rightarrow\ & 2d(\alpha_i, q) - r_i + r_k \geq d(\alpha_i, \alpha_k) \\ \Rightarrow\ & 2d(\alpha_i, q) - r_i \geq r_i + r_k \\ \Rightarrow\ & d(\alpha_i, q) \geq r_i \ , \end{aligned}$$

thus q does not penetrate D_i. But $q \in S$, so this is a contradiction. $\square$

Theorem 9.9 *The guaranteed Delaunay edges for uncertain discs $\mathcal{D}$ can be constructed in $O(n \log n)$ time, and this running time is optimal.*

Proof. Let $\mathcal{D}o \subseteq \mathcal{D}$ be those discs that are not contained by others, and $\mathcal{D}$c be $\mathcal{D} \setminus \mathcal{D}o$. We partition $\mathcal{D}$ into $\mathcal{D}$o and $\mathcal{D}$c, in $O(n \log n)$ time, by using the algorithm of [4] to construct $V(\mathcal{D})$, which as a side effect can detect all pairs $\{(D_o \in \mathcal{D}o, D_c \in \mathcal{D}c) \mid D_c \subset D_o\}$. We then use the algorithm of Lemma 9.5 on the subset $\mathcal{D}$o to generate an initial set of candidate guaranteed Delaunay edges. We assign each disc in $\mathcal{D}$o a flag indicating whether it is penetrated by any other discs in $\mathcal{D}$o. Lemma 9.7 ensures that each disc in $\mathcal{D}$o has a nonempty cell within $V(\mathcal{D})$, and Lemma 9.8 ensures that the flags can be initialized in linear time, by examining only the immediate neighbors of each disc. After these flags are initialized, we then examine each of the pairs (D_o, D_c) generated earlier. If D_o's flag is already set, we remove any guaranteed Delaunay edges incident to D_o (since D_o is penetrated by at least two discs, so by Lemma 9.6 it cannot be incident to such an edge); otherwise, we set D_o's flag, and add edge (D_o, D_c), since by Lemma 9.6, this is a guaranteed Delaunay edge (unless another (D_o, D_c') is found later, at which point edge (D_o, D_c) will be removed).

The running time for the algorithm, which is dominated by the time spent constructing $V(\mathcal{D})$ and $V^2(\mathcal{D})$, is $O(n \log n)$. This is optimal, since when the disc radii are all zero, the problem reduces to generating the Delaunay triangulation of the discs' centerpoints. □

10 Future Research

The guaranteed Voronoi diagrams introduced in this paper have focused mainly on disc-shaped, and to a lesser extent, polygon-shaped regions of uncertainty. Extending these results to more general regions of uncertainty is one direction of possible future research. Another area of future research is the investigation of guaranteed Voronoi diagrams in higher dimensions.

An applet demonstrating these diagrams is available at '`http://www.cs.ubc.ca/~jpsember/gv.html`'.

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