Generalized Fast Approximate Energy Minimization via Graph Cuts: α-Expansion β-Shrink Moves

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Introduction

- We address the energy minimization problem:

$$\min_{x \in \{1,2,\dots,N\}^p} \sum_{i \in \mathcal{V}} E_i(x_i) + \sum_{(i,j) \in \mathcal{A}} E_{ij}(x_i, x_j)$$

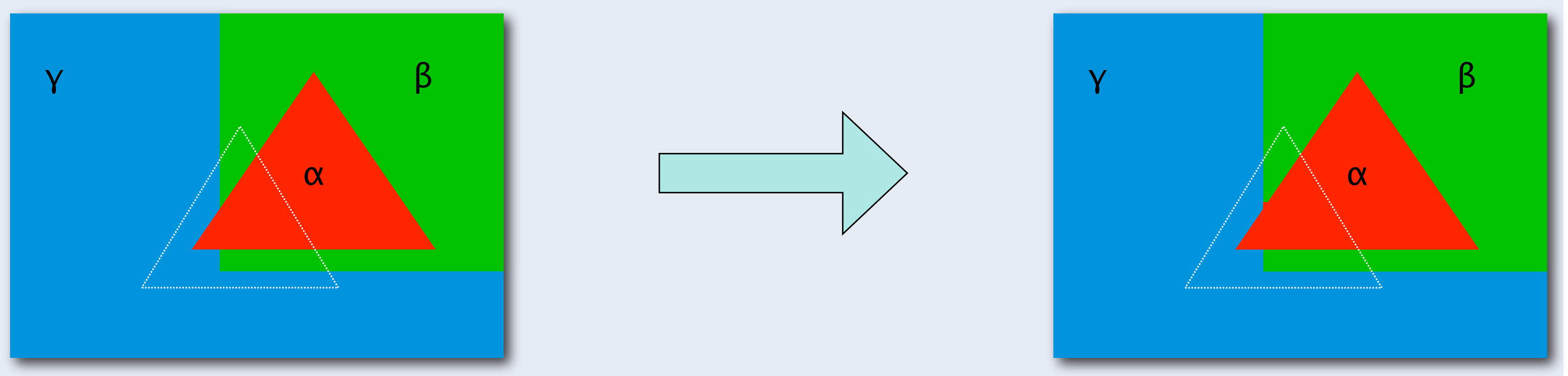
- Equivalent to MAP estimation in graphical models.
- Solvable in polynomial-time for binary variables if energies satisfy:

$$E_{ij}(1,1) + E_{ij}(2,2) \leq E_{ij}(2,1) + E_{ij}(1,2)$$

- For non-binary problems, αβ-swap and α-expansion moves find strong local optima by solving a sequence of such binary problems.
- But which one should we use?
- We propose a generalization of both, that:
  - Can be computed in polynomial-time.
  - Locally dominates them both.

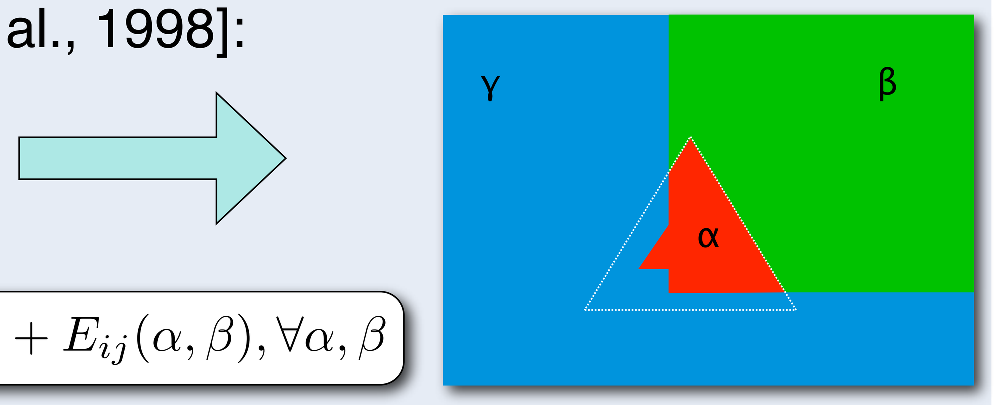
Approximate Energy Minimization

- Given x, each iteration of a descent method minimizes the energy among a set of *moves*.
- ICM Moves updates one variable [Besag, 1986]:



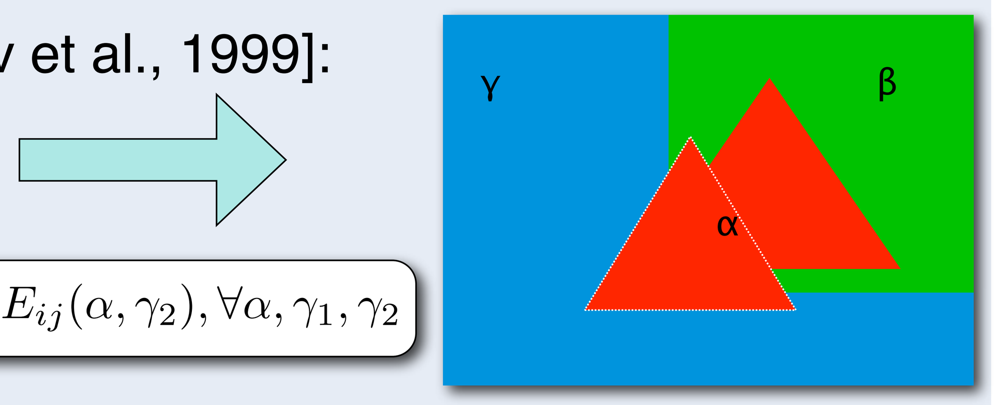
- αβ-Swap Moves [Boykov et al., 1998]:
  - Replace any α by β.
  - Replace any β by α.
  - Polynomial-time if:

$$E_{ij}(\alpha,\alpha) + E_{ij}(\beta,\beta) \leq E_{ij}(\beta,\alpha) + E_{ij}(\alpha,\beta), \forall \alpha, \beta$$



- α-Expansion Moves [Boykov et al., 1999]:
  - Replace anything by α.
  - Polynomial-time if:

$$E_{ij}(\alpha,\alpha) + E_{ij}(\gamma_1,\gamma_2) \leq E_{ij}(\gamma_1,\alpha) + E_{ij}(\alpha,\gamma_2), \forall \alpha, \gamma_1, \gamma_2$$



Local Dominance of Iterative Algorithms

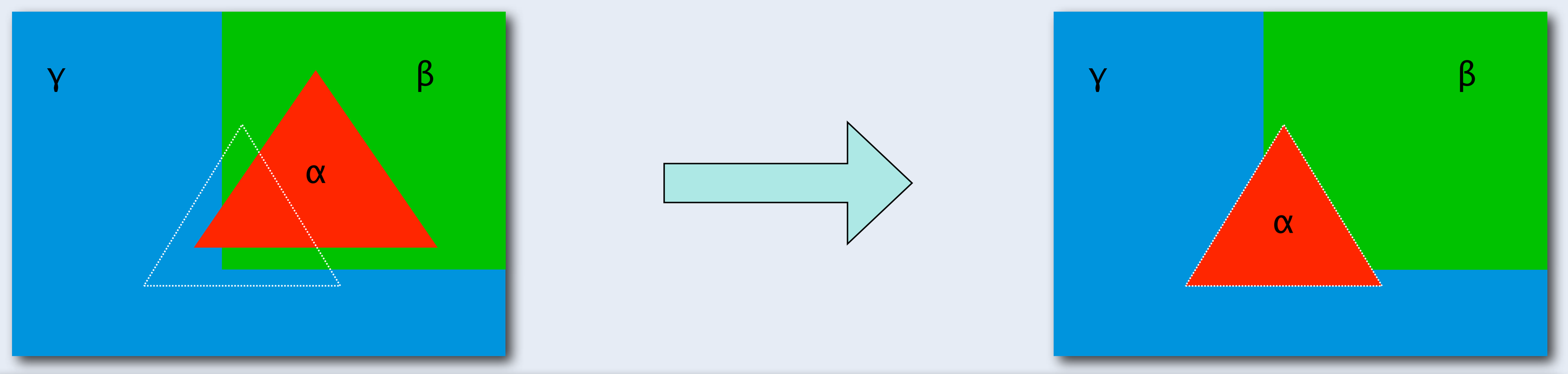
- We say that move set A dominates move set B if:
  - Optimizing over A *never does worse*.
  - Optimizing over A *can do better*.
- A may *escape from optima* with respect to B

Proposition 1.
αβ-Swaps and α-Expansions both *dominate* ICM.

Proposition 2.
αβ-Swaps *do not* dominate α-Expansions, and
α-Expansions *do not* dominate αβ-Swaps.

α-Expansion β-Shrink Moves

- α-Expansion β-Shrink Moves:
  - Replace anything by α.
  - Replace any α by β.



Proposition 3.
α-Expansion β-Shrink Moves *dominate* αβ-Swaps, and
α-Expansion β-Shrink Moves *dominate* α-Expansions.

Proposition 4.
α-Expansion β-Shrink Moves can be computed in polynomial-time if:

$$E_{ij}(\alpha,\alpha) + E_{ij}(\gamma_1,\gamma_2) \leq E_{ij}(\gamma_1,\alpha) + E_{ij}(\alpha,\gamma_2), \forall \alpha, \gamma_1, \gamma_2$$

- *Same condition* as α-expansions.
- *Same worst-case runtime* as α-expansions.

Problems with Many States

- In some applications we can't consider $O(N^2)$ α and β combinations.
- We can define a *mapping* from each α to a β, like β = min{α+1,N}. (prematurely expands the next value of α into the current α region)
- Reduces the number of combinations to O(N).
- *Still dominates* α-expansions.

Truncation for Non-Submodular Potentials

- In some problems Proposition 4 is not satisfied.
- We can define a *modified energy* where [Rother et al., 2005]:
  - moves can be computed in polynomial-time.
  - moves guaranteed to not increase the energy.
- For example, if $x_i \neq \alpha$ and $x_j \neq \alpha$ then replace $E_{ij}(x_i,x_j)$ with:

$$\bar{E}_{ij}(x_i,x_j) = \min\{E_{ij}(x_i,x_j), E_{ij}(\alpha,x_j) + E_{ij}(x_i,\alpha) - E_{ij}(\alpha,\alpha)\}$$

Computer Vision Experiments

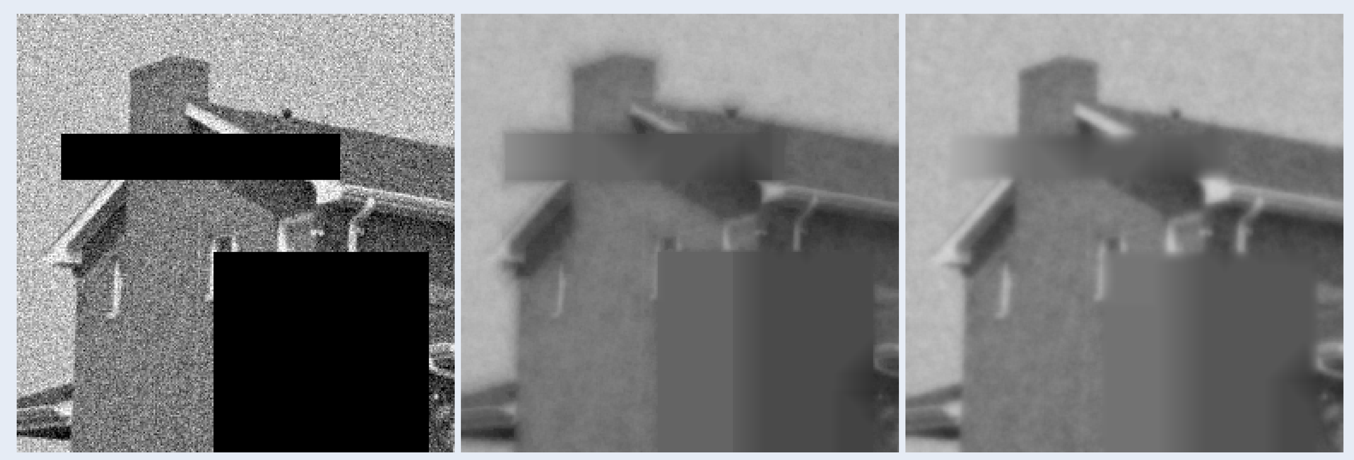
- Relative energy of local minima starting with variables set to state 1, on all non-binary data sets from Szeliski et al., [2008]:

Name	αβ-Swap	α-Expansion	Random β	β = α − 1	β = α + 1	All β
Family	1.0203	1	0.9998	1	0.9998	0.9998
Pano	1.3182	1	1.0006	1	1	1
Tsukuba	1.0315	1	1.0012	1	1.0000	1.0000
Venus	1.8561	1	1.0015	0.9992	0.9979	0.9968
Teddy	1.0037	1	0.9998	1	1.0007	0.9999
Penguin	1.1283	1	1.0037	0.9936	0.9793	0.9758
House	0.7065	1	0.7841	0.9973	0.7038	0.7032

- Relative energy of local minima starting with α-expansion optima:

Name	Random β	β = α − 1	β = α + 1
Family	0.9998	1	0.9998
Pano	1	1	1
Tsukuba	1	1	1
Venus	1.0000	0.9992	0.9979
Teddy	1	1	0.9999
Penguin	0.9998	0.9902	0.9775
House	0.8050	0.9971	0.7038

- Local minimum with respect to α-expansions and improved local minimum with β = min{α+1,N}:



Discussion

- Unlike previous generalizations, the new moves:
  - require no additional assumptions,
  - can be computed in polynomial-time.
- We expect the moves can be extended to higher-order potentials and other scenarios where α-expansions have been used.