Lifted inference in relational graphical models and (potentially) probabilistic programs

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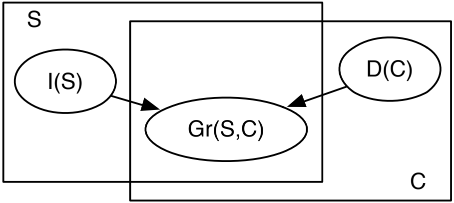
Outline

1. Relational Graphical Models
2. Exact Inference
   - Recursive Conditioning
   - Lifted Inference
   - Lifted Recursive Conditioning
3. Lifting Probabilistic Programs (?)

Plate Notation



- S, C logical variables representing students, courses
- the set of individuals of a type is called a population
- $I(S)$, $Gr(S,C)$, $D(C)$ are parametrized random variables
- Specify $P(I(S))$, $P(D(C))$, $P(Gr(S,C) \mid I(S), D(C))$

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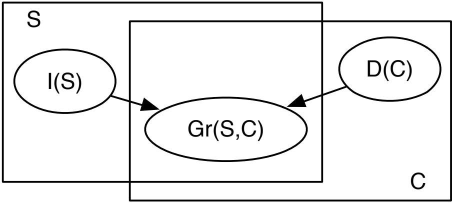

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Grounding:

- for every student s, there is a random variable $I(s)$
- for every course c, there is a random variable $D(c)$
- for every s, c pair there is a random variable $Gr(s,c)$

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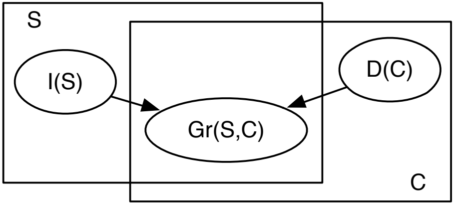


- With 1000 students and 100 courses, grounding contains
  - 1000 $I(s)$ variables
  - 100 $D(C)$ variables
  - 100000 $Gr(s, c)$ variables

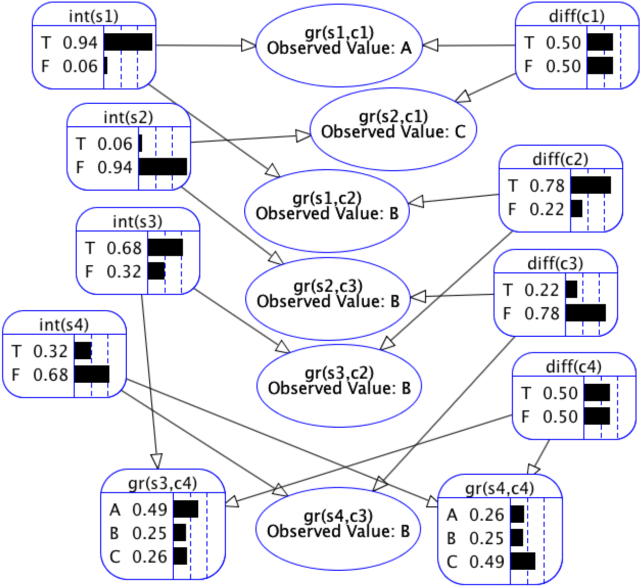
  total: 101100 variables
- Suppose Gr has 3 possible values. Numbers to be specified to define the probabilities:
  1 for $I(s)$, 1 for $D(C)$, 8 for $Gr(S, C) = 10$ parameters.

Example: Predicting Relations

Student	*Course*	*Grade*
s_1	c_1	A
s_2	c_1	C
s_1	c_2	B
s_2	c_3	B
s_3	c_2	B
s_4	c_3	B
s_3	c_4	?
s_4	c_4	?

- Students s_3 and s_4 have the same averages, on courses with the same averages.
- Which student would you expect to better?

Example: Predicting Relations



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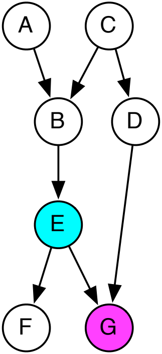
Why Exact Inference?

Why do we care about exact inference?

- Gold standard
- Size of problems amenable to exact inference is growing
- Learning for inference
- Basis for efficient approximate inference:
  - Rao-Blackwellization
  - Variational Methods

Inference via factorization in graphical models



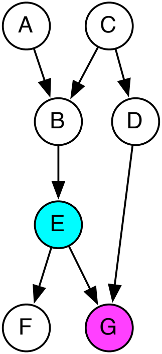
$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

$$P(E \wedge g)$$

$$= \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC)$$

$$P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED)$$

Inference via factorization in graphical models



$$P(E \mid g) = \frac{P(E \wedge g)}{\sum_E P(E \wedge g)}$$

$$\begin{aligned}
&P(E \wedge g) \\
&= \sum_F \sum_B \sum_C \sum_A \sum_D P(A)P(B \mid AC) \\
&\quad P(C)P(D \mid C)P(E \mid B)P(F \mid E)P(g \mid ED) \\
&= \left(\sum_F P(F \mid E)\right) \\
&\quad \sum_B P(E \mid B) \sum_C \left(P(C)\left(\sum_A P(A)P(B \mid AC)\right)\right. \\
&\quad \left.\left(\sum_D P(D \mid C)P(g \mid ED)\right)\right)
\end{aligned}$$

Recursive Conditioning

- Variable elimination is the dynamic programming variant of recursive conditioning.
- Recursive Conditioning is the search variant of variable elimination
- They do the same additions and multiplications.
- Complexity $O(nr^t)$, for n variables, range size r, and treewidth t.

Recursive Conditioning

procedure $rc(Con:$ context$, Fs:$ set of factors$)$:
  if $\exists v$ such that $\langle\langle Con, Fs\rangle, v\rangle \in cache$
    return v
  else if $vars(Con) \not\subseteq vars(Fs)$
    return $rc(\{X = v \in Con : X \in vars(Fs)\}, Fs)$
  else if $\exists F \in Fs$ such that $vars(F) \subseteq vars(Con)$
    return $eval(F, Con) \times rc(Con, Fs \setminus \{F\})$
  else if $Fs = Fs_1 \uplus Fs_2$ where $vars(Fs_1) \cap vars(Fs_2) \subseteq vars(Con)$
    return $rc(Con, Fs_1) \times rc(Con, Fs_2)$
  else select variable $X \in vars(Fs)$
    $sum \leftarrow 0$
    for each $v \in domain(X)$
      $sum \leftarrow sum + rc(Con \cup \{X = v\}, Fs)$
    $cache \leftarrow cache \cup \{\langle\langle Con, Fs\rangle, sum\rangle\}$
    return sum

Recursive Conditioning

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Lifted Inference

- Idea: treat those individuals about which you have the same information as a block; just count them.
- Use the ideas from lifted theorem proving - no need to ground.
- Potential to be exponentially faster in the number of non-differentialed individuals.
- Relies on knowing the number of individuals (the population size).

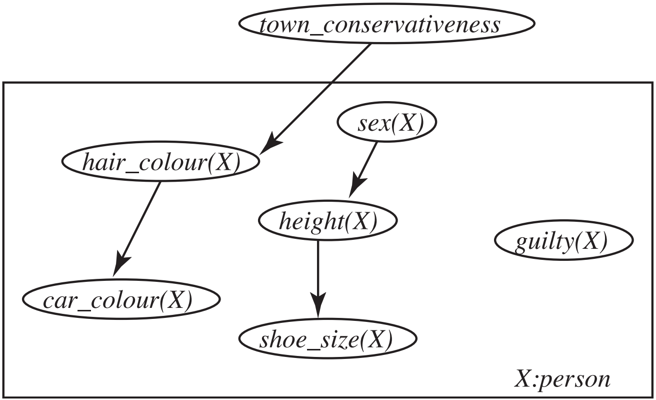
Queries depend on population size

Suppose we observe:

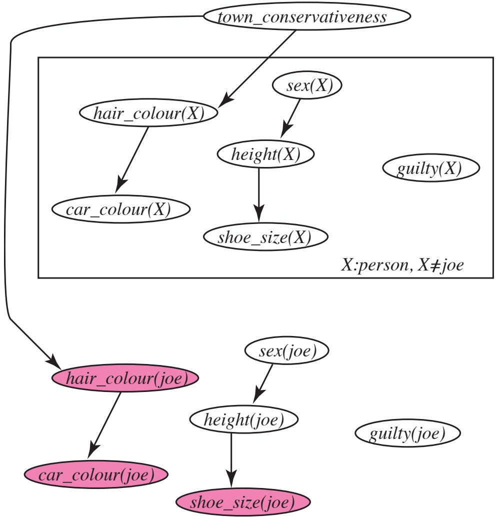
- Joe has purple hair, a purple car, and has big feet.
- A person with purple hair, a purple car, and who is very tall was seen committing a crime.

What is the probability that Joe is guilty?

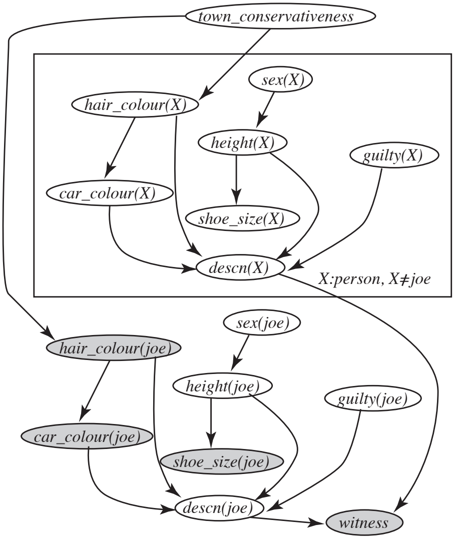
Background parametrized belief network



Observing information about Joe



Observing Joe and the crime



Parametric Factors

A parametric factor (parfactor) is a triple $\langle C, V, t\rangle$ where

- C is a set of inequality constraints on parameters,
- V is a set of parametrized random variables
- t is a table representing a factor from the random variables to the non-negative reals.

$$\left\langle \{X \neq sue\}, \{interested(X), boring\}, \begin{array}{|ll|l|} \hline interested & boring & Val \\ \hline yes & yes & 0.001 \\ yes & no & 0.01 \\ & \ldots & \\ \hline \end{array} \right\rangle$$

Factored Parametric Factors

A factored parametric factor is a triple $\langle C, V, t\rangle$ where

- C is a set of inequality constraints on parameters,
- V an assignment to parametrized random variables
- t number

Parfactor:

$$\left\langle \{X \neq sue\}, \{interested(X), boring\}, \begin{array}{|ll|l|} \hline interested & boring & Val \\ \hline yes & yes & 0.001 \\ yes & no & 0.01 \\ & \ldots & \\ \hline \end{array} \right\rangle$$

becomes

$\langle\{X \neq sue\}, interested(X) \wedge boring, 0.001\rangle$

$\langle\{X \neq sue\}, interested(X) \wedge \neg boring, 0.01\rangle$

...

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Lifted Recursive Conditioning

$lrc(Con, Fs)$

- Con is a set of assignments to random variables and counts to assignments of instances of relations. e.g.:

$$\begin{aligned}\{\neg A,\ &\#_x F(x) \wedge G(x) = 7,\\ &\#_x F(x) \wedge \neg G(x) = 5,\\ &\#_x \neg F(x) \wedge G(x) = 18,\\ &\#_x \neg F(x) \wedge \neg G(x) = 0\}\end{aligned}$$

- Fs is a set of factored parametrized factors, e.g.,

$$\begin{aligned}\{\ &\langle\{\}, \neg A \wedge \neg F(x) \wedge G(x), 0.1\rangle\ ,\\ &\langle\{\}, A \wedge \neg F(x) \wedge G(x), 0.2\rangle\ ,\\ &\langle\{\}, F(x) \wedge G(y), 0.3\rangle\ ,\\ &\langle\{\}, F(x) \wedge H(x), 0.4\rangle\}\end{aligned}$$

Evaluating ParFactors

Con:

$$\{\neg A,\ \#_x F(x) \wedge G(x) = 7,$$
$$\#_x F(x) \wedge \neg G(x) = 5,$$
$$\#_x \neg F(x) \wedge G(x) = 18,$$
$$\#_x \neg F(x) \wedge \neg G(x) = 0\}$$

Fs:

$$\{\ \langle \{\}, \neg A \wedge \neg F(x) \wedge G(x), 0.1\rangle\ ,$$
$$\langle \{\}, A \wedge \neg F(x) \wedge G(x), 0.2\rangle\ ,$$
$$\langle \{\}, F(x) \wedge G(y), 0.3\rangle\ ,$$
$$\langle \{\}, F(x) \wedge H(x), 0.4\rangle\}$$

$lrc(Con, Fs)$ returns:

Evaluating ParFactors

Con:

$$\{\neg A,\ \#_x F(x) \wedge G(x) = 7, \\ \#_x F(x) \wedge \neg G(x) = 5, \\ \#_x \neg F(x) \wedge G(x) = 18, \\ \#_x \neg F(x) \wedge \neg G(x) = 0\}$$

Fs:

$$\{\ \langle \{\}, \neg A \wedge \neg F(x) \wedge G(x), 0.1 \rangle\ , \\ \langle \{\}, A \wedge \neg F(x) \wedge G(x), 0.2 \rangle\ , \\ \langle \{\}, F(x) \wedge G(y), 0.3 \rangle\ , \\ \langle \{\}, F(x) \wedge H(x), 0.4 \rangle \}$$

$lrc(Con, Fs)$ returns:

$$0.1^{18} * 0.3^{12*25} * lrc(Con, \{\langle \{\}, F(x) \wedge H(x), 0.4 \rangle\})$$

Branching

Con:

$$
\begin{aligned}
\{\neg A,\ & \#_x F(x) \land G(x) = 7, \\
& \#_x F(x) \land \neg G(x) = 5, \\
& \#_x \neg F(x) \land G(x) = 18, \\
& \#_x \neg F(x) \land \neg G(x) = 0\}
\end{aligned}
$$

Fs:

$$\{\, \langle \{\}, F(x) \land H(x), 0.4 \rangle \,, \ldots \,\}$$

Branching on H for the 7 "x" individuals s.th. $F(x) \land G(x)$:
$lrc(Con, Fs) =$

Branching

Con:

$$\begin{aligned}\{\neg A, \ &\#_x F(x) \land G(x) = 7,\\ &\#_x F(x) \land \neg G(x) = 5,\\ &\#_x \neg F(x) \land G(x) = 18,\\ &\#_x \neg F(x) \land \neg G(x) = 0\}\end{aligned}$$

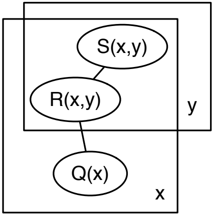
Fs:

$$\{ \langle \{\}, F(x) \land H(x), 0.4 \rangle, \ldots \}$$

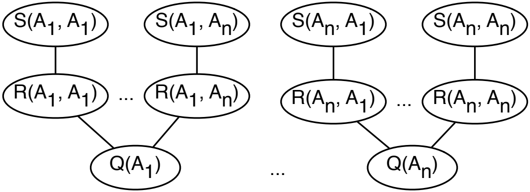
Branching on H for the 7 "x" individuals s.th. $F(x) \land G(x)$:
$lrc(Con, Fs) =$

$$\sum_{i=0}^{7} \binom{7}{i} lrc(\{\neg A, \quad \begin{aligned}&\#_x F(x) \land G(x) \land H(x) = i,\\ &\#_x F(x) \land G(x) \land \neg H(x) = 7 - i,\\ &\#_x F(x) \land \neg G(x) = 5, \ldots\}, Fs)\end{aligned}$$

Recognizing Disconnectedness



Relational Model



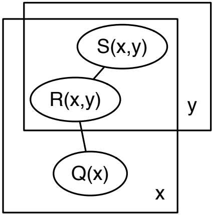
Grounding

Parfactors *Fs*:

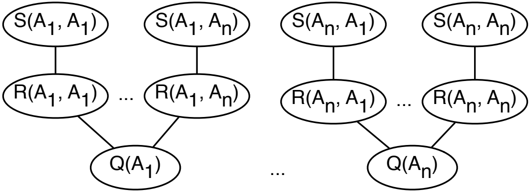
$$\{\langle\{\},\{S(x,y),R(x,y)\},t_1\rangle$$
$$\langle\{\},\{Q(x),R(x,y)\},t_2\rangle\}$$

$$lrc(Con, Fs) =$$

Recognizing Disconnectedness



Relational Model



Grounding

Parfactors Fs:

$$\{\langle\{\}, \{S(x,y), R(x,y)\}, t_1\rangle$$
$$\langle\{\}, \{Q(x), R(x,y)\}, t_2\rangle\}$$

$lrc(Con, Fs) = lrc(Con, Fs\{x/C\})^n$

...now we only have unary predicates

Observations and Queries

- Observations become the initial context.
  Observations can be ground or lifted.
- $P(q|obs) = rc(q \wedge obs, Fs)/(rc(q \wedge obs, Fs) + rc(\neg q \wedge obs, Fs))$
  calls can share the cache
- "How many?" queries are also allowed

Complexity

As the population size n of undifferentiated individuals increases:

- If grounding is polynomial — instances must be disconnected — lifted inference is constant in n (taking r^n for real r)
- Otherwise, for unary relations, grounding is exponential and lifted inference is polynomial.
- If non-unary relations become unary, above holds.
- Otherwise, ground an argument.
  Always exponentially better than grounding everything.

What we can and cannot lift

We can lift a model that consists just of

$$\langle \{x, z\}, \{F(x), \neg G(z)\}, \alpha_4 \rangle$$

or just of

$$\langle \{x, y, z\}, \{F(x, z), G(y, z)\}, \alpha_2 \rangle$$

or just of

$$\langle \{x, y, z\}, \{F(x, z), G(y, z), H(y)\}, \alpha_3 \rangle$$

We cannot lift (still exponential) a model that consists just of:

$$\langle \{x, y, z, w\}, \{F(x, z), G(y, z), H(y, w)\}, \alpha_3 \rangle$$

or

$$\langle \{x, y, z\}, \{F(x, z), G(y, z), H(y, x)\}, \alpha_3 \rangle$$

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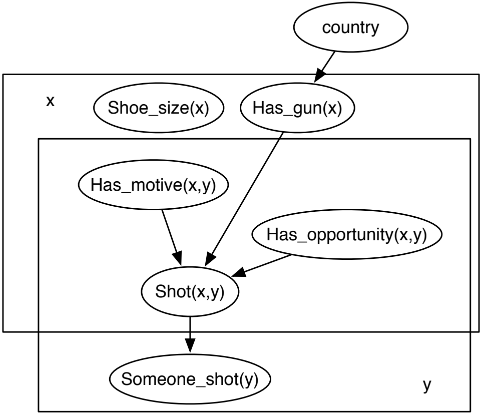
Example: Predicting Relations



Fred has unusual shoe size. Someone with unusual shoe size shot Joe. What is the probability Fred shot Joe?

Probabilistic Program

```
america := draw(0.2)
for x in range(0,1000000):
   size_23_shoe[x] := draw(0.00001)
   if america: has_gun[x] := draw(0.7)
      else: has_gun[x] := draw(0.02)
   for y in range(0,1000000):
      has_motive[x,y] := draw(0.001)
      has_opp[x,y] := draw(0.05)
      if has_motive[x,y] and has_gun[x] and has_opp[x,y]:
         actually_shot[x,y] := draw(0.1)
      if actually_shot[x,y]:
         someone_shot[y] := True
observe someone_shot[joe]
observe size_23_shoe[fred]
query actually_shot[fred,joe]
```

Lifting probabilistic programs?

- When we create many instances of one object, just create the “generic object”
- When we have to branch on a value; just count the qualitatively different answers
- If caching states in MCMC, assignments with the same counts can be treated as the same
- If computing some parts analytically, this provides one more technique in the toolbox

Conclusion

- Often probabilities depend on the number of individuals (even if not observed).
- Lifting exploits symmetry / exchangeability in relational models.
- Unary relations (properties) can be lifted. Binary relations cannot all be.
- Approximate lifted inference looks for cases that are approximately exchangeable or uses lifting in approximate algorithms
- Probabilistic logic programs use lifted inference.
  Can other probabilistic programming languages?