Efficient Mechanism Design

Bandwidth Allocation in Computer Network

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Game Theory Course Presentation
April 1st, 2014

Efficient Mechanism Design

Efficient Mechanism Design focus on the mechanism that lead to efficient allocation!

Quick-fire Question

Review

Price of Anarchy?

Review

Price of Anarchy (PoA): PoA is a measure of the extent to which system efficiency degrades due to selfish behaviour of its agents.

Define s as a strategy profile, S as the set of all strategy profiles and $E \subseteq S$ is the set of strategies in equilibrium.

For Welfare function W / Cost function C.

$$PoA = \frac{\max_{s \in S} W(s)}{\min_{s \in E} W(s)} = \frac{\max_{s \in E} C(s)}{\min_{s \in S} C(s)}$$

Note: PoA$\geq$ 1, and the smaller, the better.

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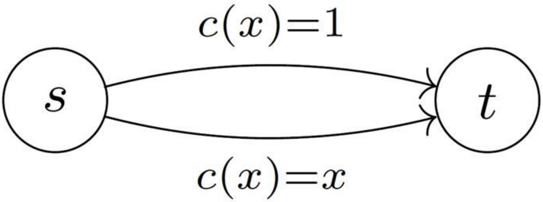

Figure : Pigou's example: selfish routing problem.

Name: Steve

Position: CEO of a big Internet Provider

Personality:

- Cares more about the **best use** of network resources (efficient allocation) than **money** in his pocket (revenue maximization)
- No price discrimination, and charge each user **the same price** for network resource per unit.

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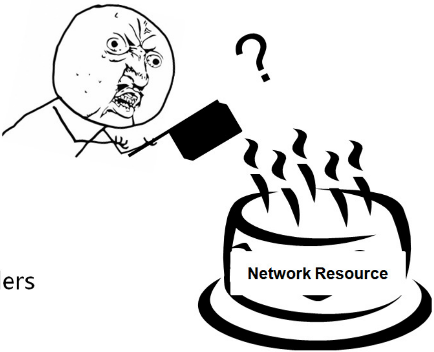
Task

multiple smaller Internet Providers



Problem Formulation

- A communciation link of capacity $C > 0$
- R users
- User r get capacity d_r.
- User r receives a utility $U_r(d_r)$
- $U_r(d_r)$ is concave, strictly increasing and continuously differentiable with domain $d_r > 0$

Given a complete knowledge and centralized control of the system, the optimization problem becomes

$$\begin{aligned} maximize \; & \sum_r U_r(d_r) \\ \text{subject to}: & \sum_r d_r \leq C; \\ & d_r \geq 0, r = 1, ..., R. \end{aligned} \tag{1}$$

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Suggested Mechanism

Proportional Allocation Mechanism: Each user r gives a payment w_r ($w_r \geq 0$) to Steve . Given the vector $\mathbf{w} = (w_1, \cdots, w_r)$, Steve chooses a capacity allocation $\mathbf{d} = (d_1, \cdots, d_r)$. Each user is charges with the same price $\mu > 0$, leading to $d_r = \frac{w_r}{\mu}$.

$$\sum_r \frac{w_r}{\mu} = C \Rightarrow \mu = \frac{\sum_r w_r}{C}$$

Quick-fire Question

Suggested Mechanism

Direct?

Logic flow of the analysis

- Price-taking Agent Model: Users do not anticipate the effect of their actions on the prices of the link per unit (μ), and they consider the price to be fixed and they select the best declarations w_r given μ.

$\Downarrow$ relaxation

- Price-Anticipating Agent Model: Users can anticipate the effects of their actions.

Proportional Allocation Mechanism: Price-taking Agent Model

Given a price $\mu > 0$, user r try to maximize its payoff function for $w_r \geq 0$:

$$P_r(w_r; \mu) = U_r\left(\frac{w_r}{\mu}\right) - w_r \; (\textit{Quasilinear in } w_r)$$

A pair $(\mathbf{w}, \mu)$ is a *competitive equilibrium* if users maximize their payoff

$$P_r(w_r; \mu) \geq P_r(\hat{w}_r; \mu) \;\; \forall \hat{w}_r \geq 0, \; r$$

[Kelly 2007] shows that when users are **price-takers**, there **exists** a competitive equilibrium, and the resulting allocation **solves** the optimization problem (1)

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Theorem

[KELLY 1997] Assume that for each user r, the utility function U_r is concave, strictly increasing, and continuously differentiable. Then there exists a competitive equilibrium, i.e., a vector $\mathbf{w} = (w_1, \cdots, w_r) \geq 0$ and a scalar $\mu > 0$ satisfying

$$P_r(w_r; \mu) \geq P_r(\hat{w}_r; \mu) \;\; \forall \hat{w}_r \geq 0, \; r$$

$$\mu = \frac{\sum_r w_r}{C}$$

In this case, the scalar μ is uniquely determined, and the vector $\mathbf{d} = \frac{\mathbf{w}}{\mu}$ is a solution to the optimization problem (1). If the functions U_r are strictly concave, then $\mathbf{w}$ is uniquely determined as well.

Proof

Step 1: Aim: Find the equivalent/optimality condition for the competitive equilibrium.
Given $\mu > 0$, **w** satisfy

$$P_r(w_r;\mu) \geq P_r(\hat{w}_r;\mu) \;\; \forall \hat{w}_r \geq 0, \; r$$

if and only if

$$\frac{dP_r(w_r;\mu)}{dw_r} = 0 \;\; \text{if } w_r > 0$$
$$\frac{dP_r(0;\mu)}{dw_r} \leq 0 \;\; \text{if } w_r = 0$$

(P_r is also concave) namely

$$U_r'\left(\frac{w_r}{\mu}\right) = \mu \;\; \text{if } w_r > 0$$
$$U_r'(0) \leq \mu \quad \text{if } w_r = 0$$

Proof

Step 2: Aim: There exists a **d** that satisfies constraints of similar form .
What We know: at least one optimal solution to the optimization problem exists (Why?)
Lagrangian:

$$L(\mathbf{d}, \mu) = \sum_r U_r(d_r) - \mu \left(\sum_r d_r - C \right)$$

Slater constraint qualification $\surd \Rightarrow$ existence of μ $\surd$
so the optimal **d** will satisfy

$$\begin{aligned} U_r^{'}(d_r) &= \mu \quad \textit{if } d_r > 0 \\ U_r^{'}(0) &\leq \mu \quad \textit{if } d_r = 0 \\ \sum_r d_r &= C. \end{aligned}$$

There exists a pair $(\mathbf{d}, \mu)$ that satisfy the constraints above, and μ is unique and $\mu > 0$.
Quick-fire Question

Proof

- Step 3: If the pair ($\mathbf{d}$,μ) satifies constraint in Step 2, let $\mathbf{w} = \mu\mathbf{d}$. and ($\mathbf{w}$,$\mu$) will satisfy the constraint in Step 1 (i.e. competitive equilibrium)
- Step 4: If $\mathbf{w}$ and $\mu > 0$ satisfy constraint in step 1 (i.e. competitive equilibrium), let $\mathbf{d} = \frac{\mathbf{w}}{\mu}$, and ($\mathbf{d}$,$\mu$) will satisfies constraints in Step 2.
- Step 5: Complete the proof.

Proportional Allocation Mechanism: Price-Anticipating Agent Model

Now the agents know that they can affect the price!

It is possible to show that a Nash equilibrium exists and that is unique.

Theorem

[Johari 2004] Let $R \geq 2$, let d^{CE} be an allocation profile achievable in competitive equilibrium and let d^{NE} be the unique allocation profile achievable in Nash equilibrium. Then any profile of valuation functions U_r for which $\forall\ r, U_r(0) \geq 0$ satisfies

$$\sum_r U_r(d_r^{NE}) \geq \frac{3}{4} \sum_i U_r(d_r^{CE}).$$

Quick-fire Question

Proportional Allocation Mechanism: Price-Anticipating Agent Model

In other words, the price of anarchy is $\frac{4}{3}$. Even in the worst case, the strategic behaviour by agents will only cause a small reduction in social welfare.

Proportional Allocation Mechanism: Price-Anticipating Model

Something Else:

- Not bad!
- It achieves minimal price of anarchy, as compared to a broad family of mechanisms in which
  - *agents' declarations are a single scalar*;
  - *the mechanism charges all users the same rate.*
- When mechanism is allowed to charge users at different prices, a VCG-like mechanism can be used to achieve full effciency.

Summary

- In a game where users of a congested single resource anticipate the effect of their actions on the price of the resource, the aggregate utility received by the users is at least 3/4 of the maximum possible aggregate utility.

References

Johari 2004 Johari, Ramesh, and John N. Tsitsiklis. "Efficiency loss in a network resource allocation game." Mathematics of Operations Research 29.3 (2004): 407-435.

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Question?