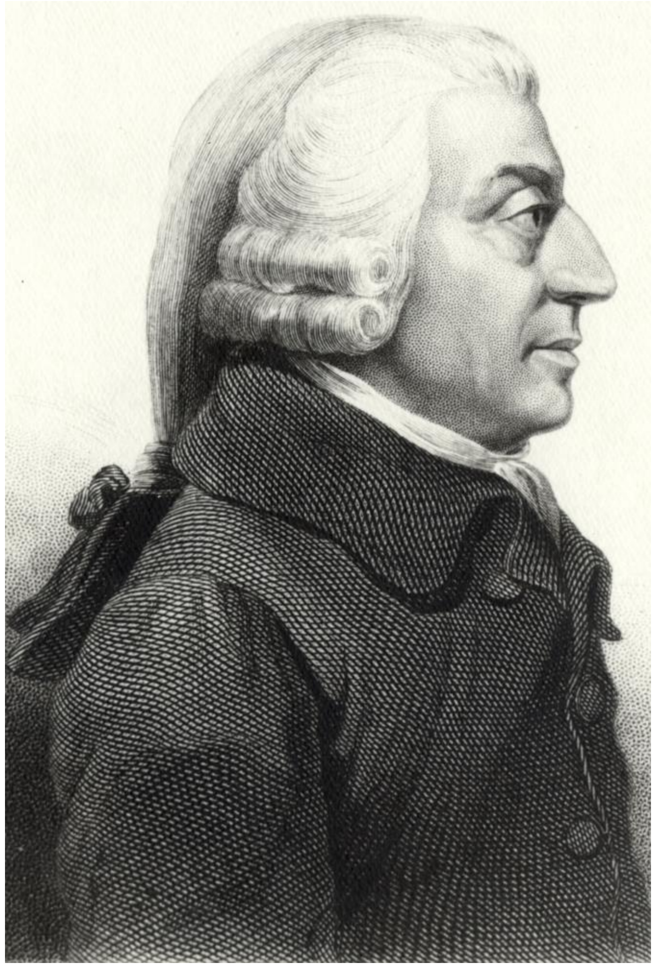
Price of Anarchy: Quantifying the Inefficiency of Equilibria

Zongxu Mu

“The Invisible Hand”



Source: http://en.wikipedia.org/wiki/File:AdamSmith.jpg

Equilibria and Efficiency

- Central to free market economics

The Wealth of Nations (Smith, 1776)

- “… led by an invisible hand to promote an end which was no part of his intention”
- Self-interest agents → social-efficient outcomes

Inefficiency of Equilibria

Inefficient equilibrium in markets:

- Of certain structures (e.g., monopoly)
- For certain kinds of goods (e.g., public goods)
- With externalities (e.g., pollution)
- …

Government interventions can be beneficial

- There is a price (efficiency lost) of "anarchy"

absence of order or government

-- Merriam Webster

Inefficiency of Equilibria

Nash equilibrium: DD

- Pareto-dominated
- The only non-Pareto-optimal outcome!

Pareto-optimality: a qualitative observation

A quantitative measure?

Prisoner's Dilemma

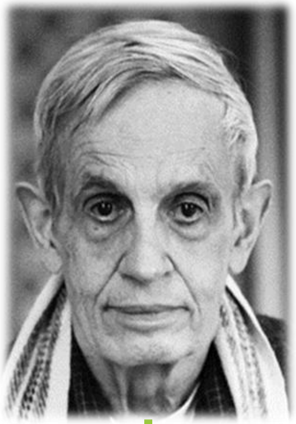
	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Outline

- Inefficiency of Equilibria
  - Pareto-optimality
  - Price of Anarchy
- Selfish Routing Games
  - Pigou’s example
  - Nonatomic games
- Other Applications
- Summary

Inefficiency of Equilibria – A Short History

Source: http://en.wikipedia.org/wiki/File:John_Forbes_Nash,_Jr._by_Peter_Badge.jpg



1951
Nash Equilibrium
Nash

Inefficiency of Equilibrium
Rapoport and Chammah
1965

Source: http://en.wikipedia.org/wiki/File:Anatol_Rapoport.jpg

Source: http://cgi.di.uoa.gr/~elias/images/elias-bio.jpg

1999
Origin of PoA: Coordination Ratio
Koutsoupias and Papadimitriou

Price of Anarchy
Papadimitriou
2001

Source: http://www.cs.berkeley.edu/~christos/index_files/image002.png

Inefficiency of Equilibria

Optimality in utilities?

- Utilities of different persons cannot be compared or summed up

Cost or payoff may also have concrete interpretations

- Money, network delay, …

Specific objective functions for "social cost"

- Utilitarian: $f(o) = \sum u_i$
- Egalitarian: $f(o) = \max(u_i)$

Inefficiency of Equilibria

Objective Function ➔ Quantify

Price of Anarchy

- Similar to approximation ratio

$$PoA = \frac{f(\text{worst equilibrium})}{f(\text{optimal outcome})}$$

$$= \frac{(-3)+(-3)}{(-1)+(-1)} = 3$$

Prisoner's Dilemma

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Price of Anarchy – Properties and Interests

Can be unbounded

- $d \to +\infty$
- $PoA = \frac{(-d)+(-d)}{(-1)+(-1)} = d \to +\infty$

Can be bounded

Is central control needed?

- Mechanism design

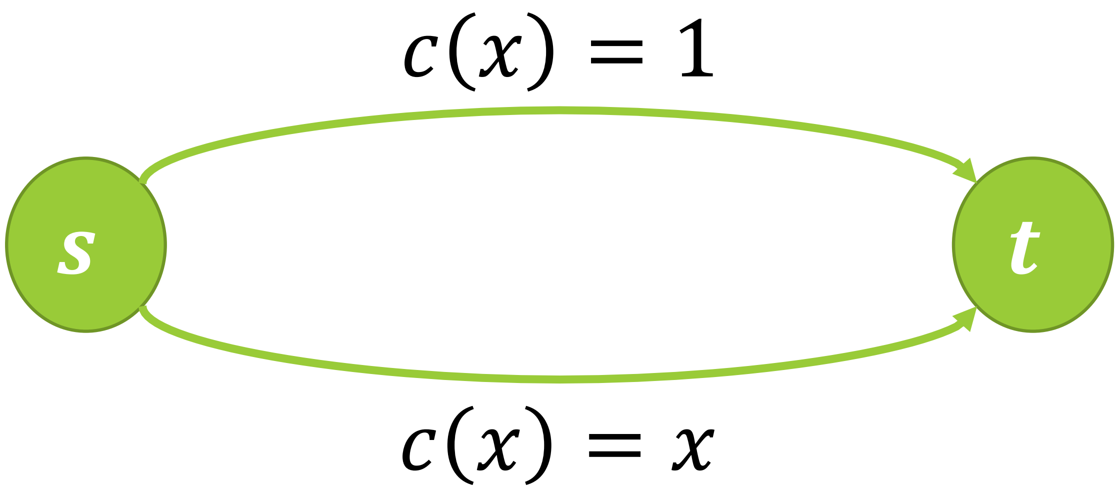
Prisoner's Dilemma

	C	D
C	$-1, -1$	$-d-1, 0$
D	$0, -d-1$	$-d, -d$

Selfish Routing Games

Pigou's (1920) example

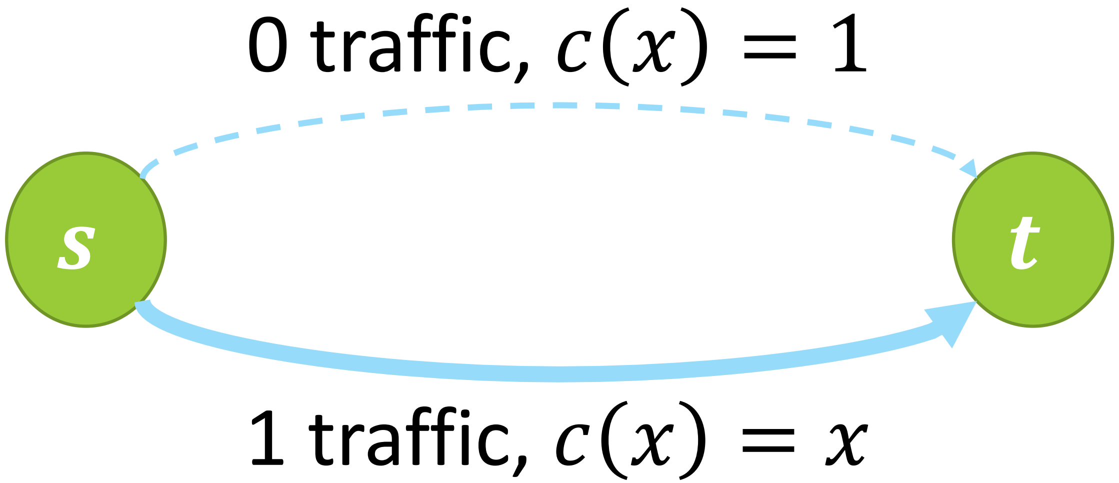
- s: source; t: sink
- $c(x)$: unit cost of an edge
- 1 unit of traffic in total
- **What is the Nash equilibrium?**



Selfish Routing Games

Pigou's (1920) example

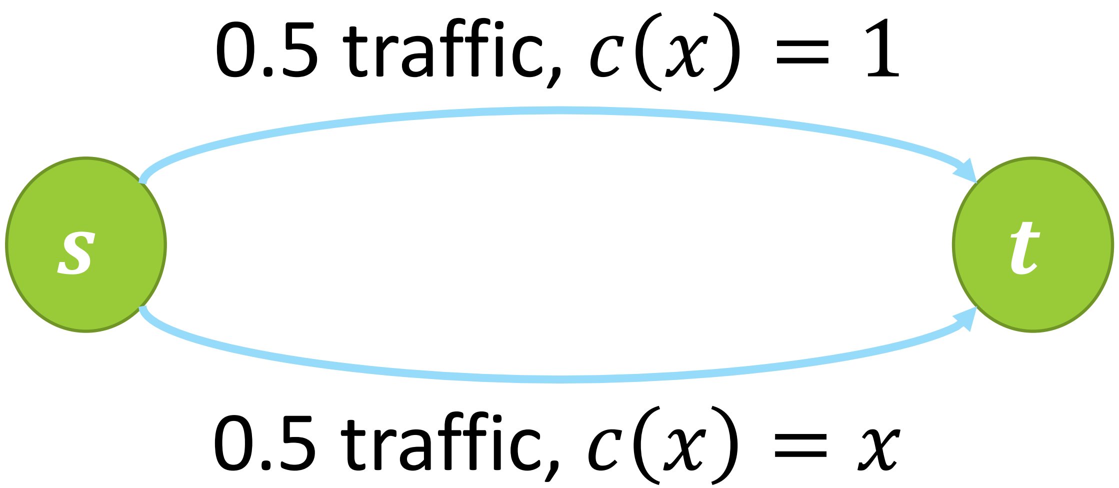
- Nash equilibrium:
  - All traffic on the lower edge
  - Total cost: $1 \times c(1) = 1$



Selfish Routing Games

Pigou's (1920) example

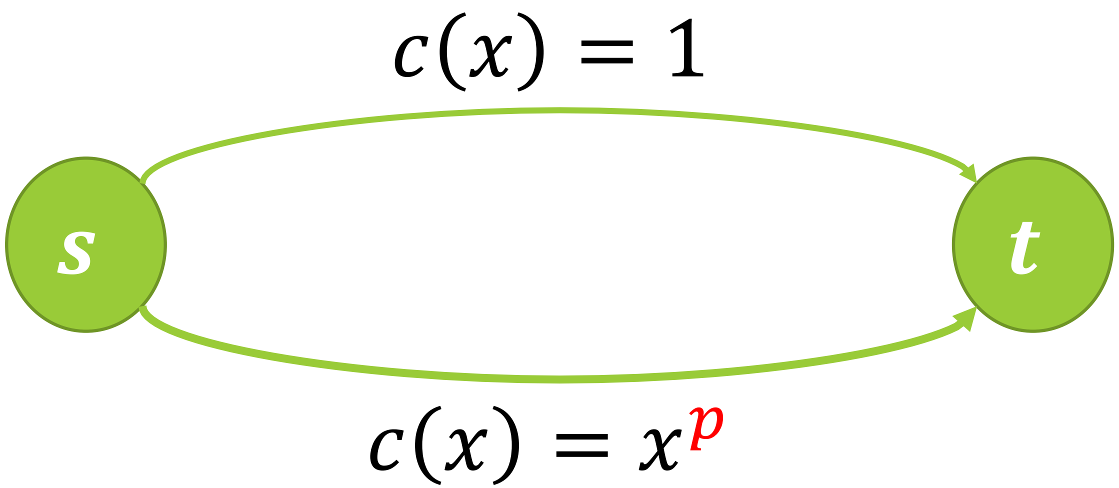
- Optimal solution:
  - Half traffic on each edge
  - Total cost: $0.5 \times 1 + 0.5 \times 0.5 = 0.75$
- Price of anarchy $= \frac{1}{0.75} = \frac{4}{3}$



Selfish Routing Games

Modified Pigou's example

- A small change in cost function



Selfish Routing Games

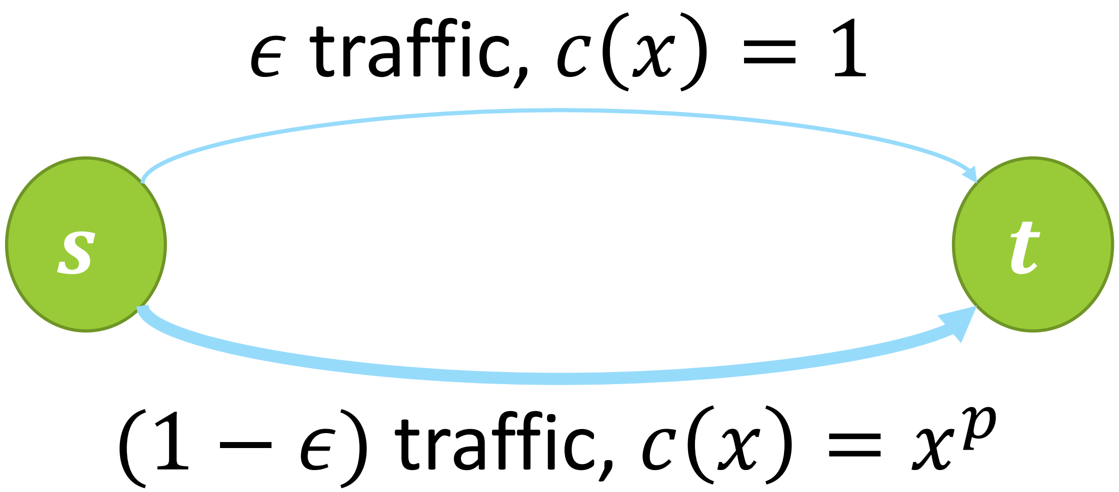
Modified Pigou's example

- Nash equilibrium:
  - All traffic on the lower edge
  - Total cost: $1 \times 1^1 = 1$

Selfish Routing Games

Modified Pigou's example

- **When is the cost optimized?**
  - $\epsilon \in [0,1]$: traffic on upper edge
  - Cost $= \epsilon + (1-\epsilon)^{p+1}$
  - Minimized when $\epsilon = 1 - (p+1)^{-\frac{1}{p}}$
- As $p \to \infty$, optimal cost $\to 0$



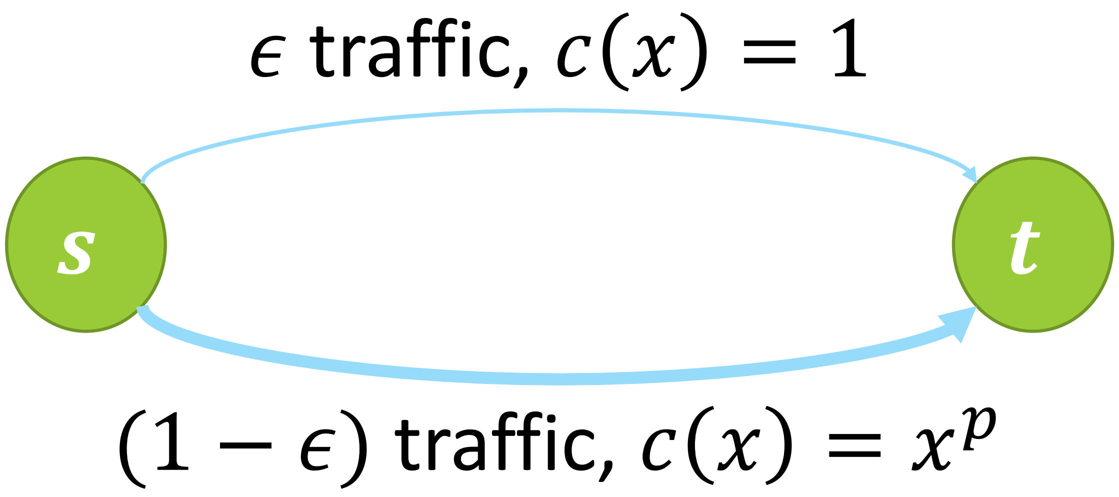
Selfish Routing Games

Modified Pigou's example

$$\text{PoA} = \frac{f(\textit{worst equilibrium})}{f(\textit{optimal outcome})}$$

As $p \to \infty$

- $f(\textit{optimal outcome}) \to 0$
- $\text{PoA} \to \infty$



Is That a Game?..

Players? → Some agents wanting their traffic get across

Actions? → Each agent can choose a path

Payoffs? → The utility is the negative of network delay

Familiar?? **Congestion games!!**

Selfish Routing Games

Atomic routing games

- Some players
- Each controls a non-negligible fraction of traffic

Nonatomic routing games

- Some players
- Each controls a negligible fraction of traffic

Oligopoly

Perfect competition

Selfish Routing Games

Marginal Social Cost

- Increase in total cost due to additional traffic
- Cost of x traffic: $x \cdot c(x)$
- Marginal cost function: $c^*(x) = \big(x \cdot c(x)\big)' = c(x) + x \cdot c'(x)$

Potential Function

- Use of integration in nonatomic games

Selfish Routing Games

General Equilibrium Properties

Nonatomic games

- At least one equilibrium flow
- Uniqueness of equilibrium

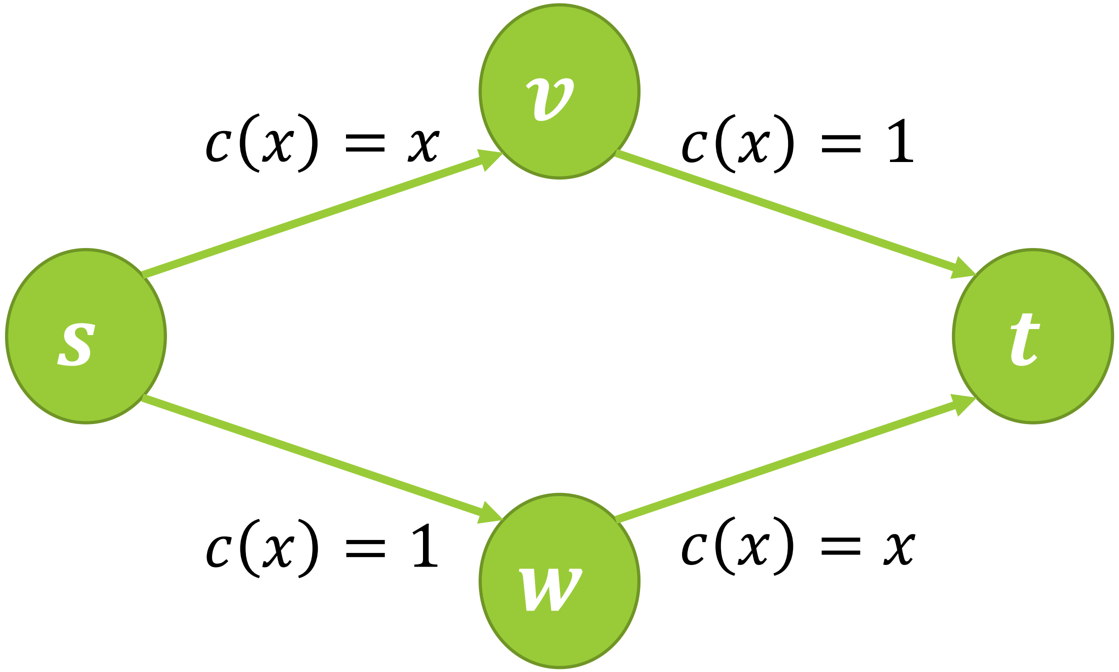
Atomic games

- Equilibrium flow exists
  - If all players control the same amount of traffic
  - With affine cost functions

Nonatomic Routing Games

Braess's Paradox in nonatomic routing games

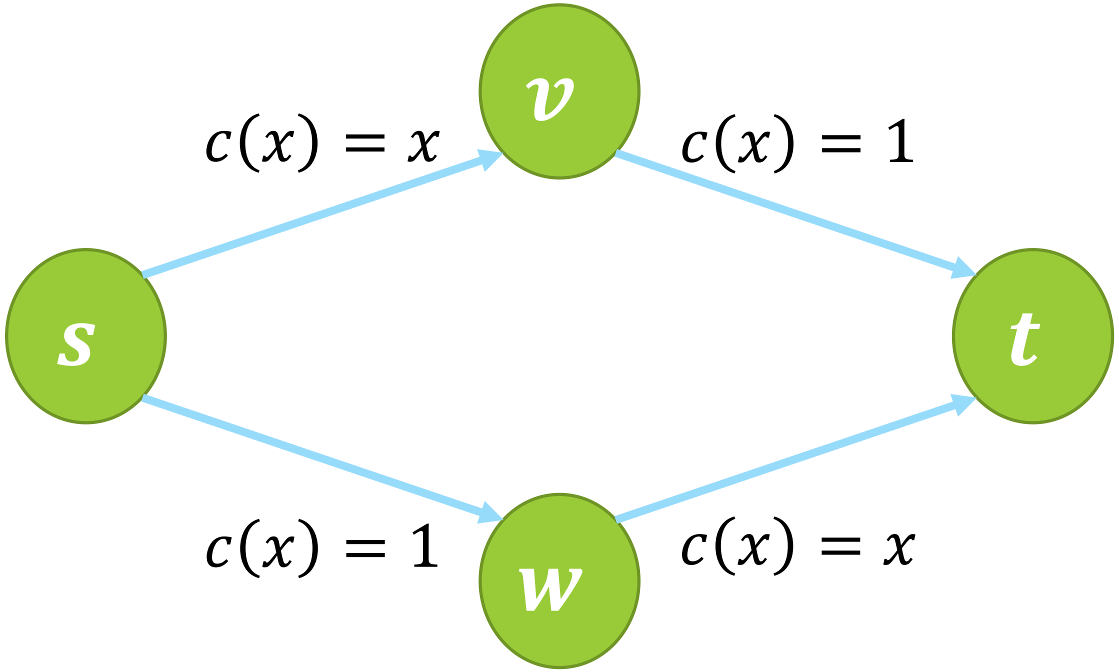
- 1 unit of total traffic



Nonatomic Routing Games

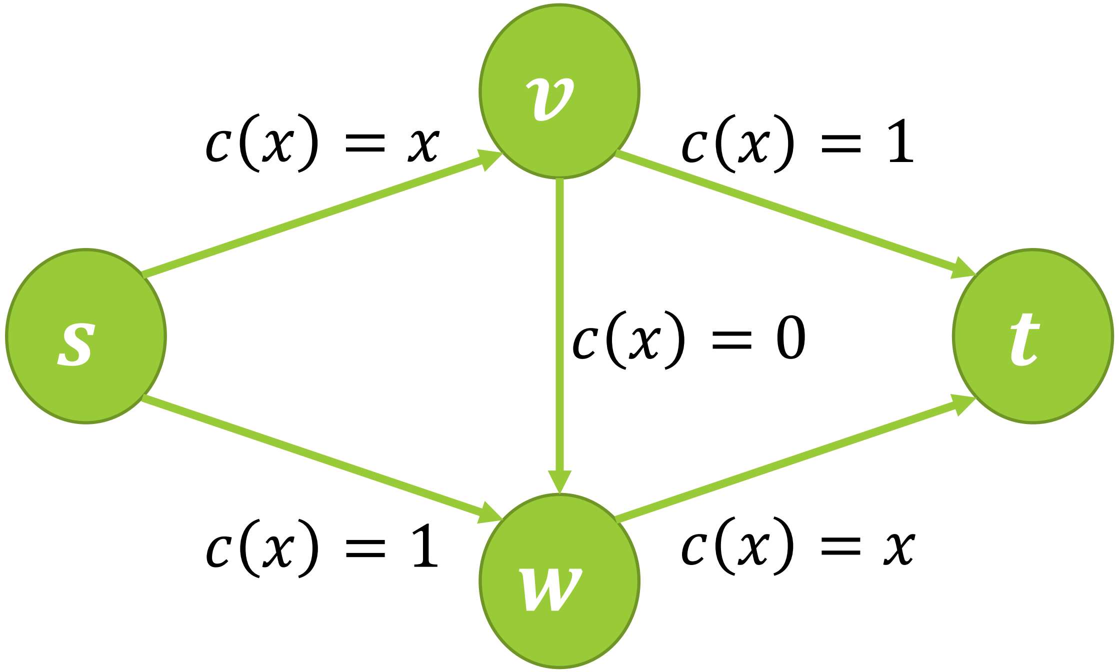
Braess's Paradox in nonatomic routing games

- Equilibrium:
  - $s \to v \to t$: 0.5 traffic
  - $s \to w \to t$: 0.5 traffic
- Cost $= 1.5$



Nonatomic Routing Games

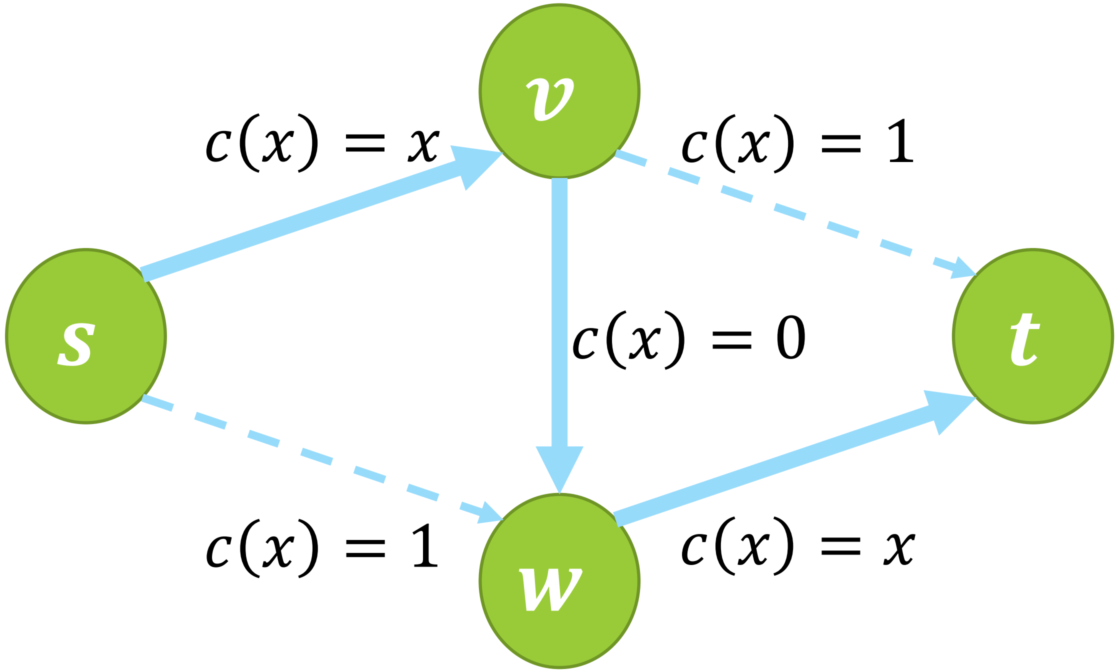
Braess's Paradox in nonatomic routing games



Nonatomic Routing Games

Braess's Paradox in nonatomic routing games

- Equilibrium:
  - $s \to v \to w \to t: 1$
- Cost $= 2$
- PoA $= \frac{2}{1.5} = \frac{4}{3}$



Nonatomic Routing Games

Price of anarchy

- Maximized in Pigou-like examples
- Dependent on "nonlinearity" of cost functions
  - Pigou bound: tight upper bound
- Independent of
  - Network size or structure
  - Number of different source-sink pairs

Polynomial degree $\leq p$
Non-negative coefficients

$$\left[1 - p \cdot (p+1)^{-\frac{p+1}{p}}\right]^{-1} \approx \frac{p}{\ln p}$$

$$p = 1 \Rightarrow \frac{4}{3}$$

Applications

Other games:

- Facility location
  - Pure Nash equilibrium exists
  - Price of anarchy is small
- Load balancing
  - Makespan scheduling
- Resource allocation
  - PoA as a design metric

Reduce PoA:

- Marginal cost pricing
  - Pigouvian taxes
- Capacity augmentation

Summary

Price of anarchy quantifies the inefficiency of equilibrium

- Ratio of “social cost” of worst equilibrium over optimum

Selfish routing is intensively studied

- Equilibrium flow always exists in nonatomic routing games
- Pigou’s example shows that PoA can be bounded or unbounded
- PoA depends on cost functions but not on other network properties

PoA presents in the study of other domains

References

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