# Repeated Games

Week 5

# Course Logistics

- Midterm: Feb 26
- Project Outline Working Session: Feb 28
- I'm away next week; Bayesian games
- Quizzes starting next week.

# Repeated Games

- Finally, we get to repeat our games!
- Let's play 10 repetitions, aiming to maximize total payoff:
  - Battle of the sexes
  - Prisoner's dilemma
  - Rock-Paper-Scissors

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- Let's play 10 repetitions, aiming to maximize total payoff:
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- Now let's do it again, $1/4$ probability of stopping after every round (2 coin flips both come up heads)

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- Does a finitely-repeated game have an induced normal form?
- What can we say about the equilibria of these games?

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- What is a pure strategy in an infinitely-repeated game?

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- What does the folk theorem say intuitively?
- How many would find it useful to go through the proof together?

# Lecture Overview

1. Folk Theorem
2. Games played by automata

# Definitions

- Consider any $n$-player game $G = (N, A, u)$ and any payoff vector $r = (r_1, r_2, \ldots, r_n)$.
- Let $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$.
  - $i$'s minmax value: the amount of utility $i$ can get when $-i$ play a minmax strategy against him

## Definition

A payoff profile $r$ is enforceable if $r_i \geq v_i$.

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A payoff profile $r$ is feasible if there exist rational, non-negative values $\alpha_a$ such that for all $i$, we can express $r_i$ as $\sum_{a \in A} \alpha u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$.

- a payoff profile is feasible if it is a convex, rational combination of the outcomes in $G$.

# Folk Theorem

**Theorem (Folk Theorem)**

*Consider any $n$-player game $G$ and any payoff vector $(r_1, r_2, \ldots, r_n)$.*

1. *If $r$ is the payoff in any Nash equilibrium of the infinitely repeated $G$ with average rewards, then for each player $i$, $r_i$ is enforceable.*
2. *If $r$ is both feasible and enforceable, then $r$ is the payoff in some Nash equilibrium of the infinitely repeated $G$ with average rewards.*

# Folk Theorem (Part 1)

Payoff in Nash $\rightarrow$ enforceable

**Part 1:** Suppose $r$ is not enforceable, i.e. $r_i < v_i$ for some $i$. Then consider a deviation of this player $i$ to $b_i(s_{-i}(h))$ for any history $h$ of the repeated game, where $b_i$ is any best-response action in the stage game and $s_{-i}(h)$ is the equilibrium strategy of other players given the current history $h$. By definition of a minmax strategy, player $i$ will receive a payoff of at least $v_i$ in every stage game if he adopts this strategy, and so $i$'s average reward is also at least $v_i$. Thus $i$ cannot receive the payoff $r_i < v_i$ in any Nash equilibrium.

# Folk Theorem (Part 2)

Feasible and enforceable $\rightarrow$ Nash

**Part 2:** Since $r$ is a feasible payoff profile, we can write it as $r_i = \sum_{a \in A} \left(\frac{\beta_a}{\gamma}\right) u_i(a)$, where $\beta_a$ and $\gamma$ are non-negative integers.[1] Since the combination was convex, we have $\gamma = \sum_{a \in A} \beta_a$. We're going to construct a strategy profile that will cycle through all outcomes $a \in A$ of $G$ with cycles of length $\gamma$, each cycle repeating action $a$ exactly $\beta_a$ times. Let $(a^t)$ be such a sequence of outcomes. Let's define a strategy $s_i$ of player $i$ to be a trigger version of playing $(a^t)$: if nobody deviates, then $s_i$ plays $a_i^t$ in period $t$. However, if there was a period $t'$ in which some player $j \neq i$ deviated, then $s_i$ will play $(p_{-j})_i$, where $(p_{-j})$ is a solution to the minimization problem in the definition of $v_j$.

[1] Recall that $\alpha_a$ were required to be rational. So we can take $\gamma$ to be their common denominator.

# Folk Theorem (Part 2)

Feasible and enforceable $\rightarrow$ Nash

First observe that if everybody plays according to $s_i$, then, by construction, player $i$ receives average payoff of $r_i$ (look at averages over periods of length $\gamma$). Second, this strategy profile is a Nash equilibrium. Suppose everybody plays according to $s_i$, and player $j$ deviates at some point. Then, forever after, player $j$ will receive his $\min\max$ payoff $v_j \leq r_j$, rendering the deviation unprofitable.

# Lecture Overview

# Games played by automata

- One strategy for repeated games is to encode strategies as finite automata?
- What strategies work well? Let's play against each other, and against other students in the Coursera class.
- `http://gametheory.cs.ubc.ca`