Coalitional Game Theory

Lecture 22

Quiz

In words, what does it mean to say that a way of dividing payoffs in a coalitional game is "in the core"?

Coalitional Game Theory

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- How does coalitional game theory differ from noncooperative game theory?
- What is the transferrable utility assumption?
- What are the key questions asked by CGT? (1) which coalition will form; (2) how to divide payoffs.

Voting Game

Our first example considers a social choice setting.

Example (Voting game)

The parliament of Micronesia is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

More generally, in a voting game, there is a set of agents N and a set of coalitions $\mathcal{W} \subseteq 2^N$ that are *winning* coalitions, that is, coalitions that are sufficient for the passage of the bill if all its members choose to do so. To each coalition $S \in \mathcal{W}$, we assign $v(S) = 1$, and to the others we assign $v(S) = 0$.

Airport Game

Our second example concerns sharing the cost of a public good.

Example (Airport game)

A number of cities need airport capacity. If a new regional airport is built the cities will have to share its cost, which will depend on the largest aircraft that the runway can accommodate. Otherwise each city will have to build its own airport.
This situation can be modeled as a coalitional game (N, v), where N is the set of cities, and $v(S)$ is the sum of the costs of building runways for each city in S minus the cost of the largest runway required by any city in S.

Minimum Spanning Tree

Next, consider a situation in which agents need to get connected to the public good in order to enjoy its benefit. One such setting is the problem of multicast cost sharing.

Example (Minimum spanning tree game)

A group of customers must be connected to a critical service provided by some central facility, such as a power plant or an emergency switchboard. In order to be served, a customer must either be directly connected to the facility or be connected to some other connected customer. Let us model the customers and the facility as nodes on a graph, and the possible connections as edges with associated costs.
This situation can be modeled as a coalitional game (N, v). N is the set of customers, and $v(S)$ is the cost of connecting all customers in S directly to the facility minus the cost of the minimum spanning tree that spans both the customers in S and the facility.

Auction

Finally, consider an efficient auction mechanism. Our previous analysis treated the set of participating agents as given. We might instead want to determine if the seller would prefer to exclude some interested agents to obtain higher payments. To find out, we can model the auction as a coalitional game.

Example (Auction game)

Let N_B be the set of bidders, and let 0 be the seller. The agents in the coalitional game are $N = N_B \cup \{0\}$. Choosing a coalition means running the auction with the appropriate set of agents. The value of a coalition S is the sum of agents' utilities for the efficient allocation when the set of participating agents is restricted to S. A coalition that does not include the seller has value 0, because in this case a trade cannot occur.

Shapley Value

Axioms:

- Symmetry: two agents who are "interchangeable" (always contribute the same amount to every coalition) should receive the same payments
- Dummy: an agent who always contributes the same amount as he achieves on his own should get that amount
- Additivity: if we remodel the setting as two coalitional games with payoffs split somehow (e.g., half and half) between the games, agents get paid the sum of what they're paid in the new games.

The Shapley value is the unique payoff division that satisfies these axioms.

Understanding the Shapley Value

Definition (Shapley value)

Given a coalitional game (N, v), the Shapley value of player i is given by

$$\phi_i(N, v) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \Big[v(S \cup \{i\}) - v(S) \Big].$$

This captures the "average marginal contribution" of agent i, averaging over all the different sequences according to which the grand coalition could be built up from the empty coalition.

- imagine that the coalition is assembled by starting with the empty set and adding one agent at a time, with the agent to be added chosen uniformly at random.

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- Within any such sequence of additions, look at agent i's marginal contribution at the time he is added.

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- If he is added to the set S, his contribution is $[v(S \cup \{i\}) - v(S)]$.

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- Now multiply this quantity by the $|S|!$ different ways the set S could have been formed prior to agent i's addition and by the $(|N| - |S| - 1)!$ different ways the remaining agents could be added afterward.

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- Finally, sum over all possible sets S and obtain an average by dividing by $|N|!$, the number of possible orderings of all the agents.

Shapley Value Example

Consider the Voting game:

- A, B, C, and D have 45, 25, 15, and 15 votes
- 51 votes are required to pass the \$100 million bill
- A is in most winning coalitions, but doesn't win alone
- B, C, D are *interchangeable*: they always provide the same marginal benefit to each coalition
    - they add \$100 million to the coalitions $\{B, C\}$, $\{C, D\}$, $\{B, D\}$ that do not include them already and to $\{A\}$
    - they add \$0 to all other coalitions
- Grinding through the Shapley value calculation (see the book), we get the payoff division $(50, 16.66, 16.66, 16.66)$, which adds up to the entire \$100 million.

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  - Voting game: while A does not have a unilateral motivation to vote for a different split, A and B have incentive to defect and divide the $\$100$ million between them (e.g., $(75, 25)$).
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Definition (Core)

A payoff vector x is in the core of a coalitional game (N, v) iff

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S).$$

- The sum of payoffs to the agents in any subcoalition S is at least as large as the amount that these agents could earn by forming a coalition on their own.

Existence and Uniqueness

1. Is the core always nonempty?

2. Is the core always unique?

Existence and Uniqueness

1. Is the core always nonempty? No.
   - Consider again the voting game.
   - The set of minimal coalitions that meet the required 51 votes is $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, and $\{B, C, D\}$.
   - If the sum of the payoffs to parties B, C, and D is less than \$100 million, then this set of agents has incentive to deviate.
   - If B, C, and D get the entire payoff of \$100 million, then A will receive \$0 and will have incentive to form a coalition with whichever of B, C, and D obtained the smallest payoff.
   - Thus, the core is empty for this game.

2. Is the core always unique?

Existence and Uniqueness

1. Is the core always nonempty?
2. Is the core always unique? No.
   - Consider changing the example so that an 80% majority is required
   - The minimal winning coalitions are now $\{A, B, C\}$ and $\{A, B, D\}$.
   - Any complete distribution of the \$100 million among A and B now belongs to the core
     - all winning coalitions must have both the support of these two parties.