Auctions

Week 11

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- How do English auctions relate to other auction types?
- Why might a seller choose to use one auction type over another?

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- Revenue equivalence depends on risk neutrality. Would a risk-averse buyer prefer first or second price?
- What about a risk-averse seller?

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- What happens in the case where bidders' valuations come from different distributions?

Fun game

- Look at the jar of coins
- Bid for it using real money in a sealed-bid second-price auction.

Going beyond IPV

- common value model
  - motivation: oil well
  - winner's curse
  - things can be improved by revealing more information
- general model
  - IPV + common value
  - example motivation: private value plus resale

Affiliated Values

- Definition: a high value of one bidder's signal makes high values of other bidders' signals more likely
  - common value model is a special case
- generally, ascending auctions lead to higher expected prices than second price, which in turn leads to higher expected prices than first price
  - intuition: winner's gain depends on the privacy of his information.
  - The more the price paid depends on others' information (rather than expectations of others' information), the more closely this price is related to the winner's information, since valuations are affiliated
  - thus the winner loses the privacy of his information, and can extract a smaller "information rent"

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  - thus the winner loses the privacy of his information, and can extract a smaller "information rent"
- Linkage principle: if the seller has access to any private source of information which will be affiliated with the bidders' valuations, she should precommit to reveal it honestly.

Multiunit Auctions

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- what is VCG in this setting?
  - every unit is sold for the amount of the $k+1$st highest bid
- how else can we sell the goods?
  - pay-your-bid: "discriminatory" pricing, because bidders will pay different amounts for the same thing
  - lowest winning bid: very similar to VCG, but ensures that bidders don't pay zero if there are fewer bids than units for sale
  - sequential single-good auctions

Revenue Equivalence

Theorem (Revenue equivalence theorem, multiunit version)

Assume that each of n *risk-neutral agents has an independent private valuation for a single unit of* k *identical goods at auction, drawn from a common cumulative distribution* $F(v)$ *that is strictly increasing and atomless on* $[\underline{v}, \bar{v}]$*. Then any efficient auction mechanism in which any agent with valuation* $\underline{v}$ *has an expected utility of zero yields the same expected revenue, and hence results in any bidder with valuation* v_i *making the same expected payment.*

Sequential Auctions

Although we can apply the revelation principle, for greater intuition we can also use backward induction to derive the equilibrium strategies in finitely-repeated second-price auctions.

- everyone should bid honestly in the final auction
- we can also compute a bidder's expected utility (conditioned on type) in that auction
- in the second-last auction, bid the difference between valuation and the expected utility for losing
  - i.e., bid valuation minus the expected utility for playing the second auction
  - why: consider affine transformation of valuations subtracting this constant expected utility
- combining these last two auctions together, there's some expected utility to playing both of them
- now this is the "expected utility of losing"
- apply backward induction

Unlimited Supply

- consider MP3 downloads as an example of a multiunit good.
- They differ from the other examples we gave:
  - the seller can produce additional units at zero marginal cost
  - hence has an effectively unlimited supply of the good
  - The seller will not face any supply restrictions other than those she imposes herself.
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- How should such goods be auctioned?
  - the seller will have to artificially reduce supply
  - the *first* unit of the good might have been very expensive to produce!

Optimal Single Price

If we *knew* bidders' valuations but had to offer the goods at the same price to all bidders, it would be easy to compute the optimal single price.

Definition (Optimal single price)

The optimal single price is calculated as follows.

1. Order the bidders in descending order of valuation; let v_i denote the ith-highest valuation.
2. Calculate $opt \in \arg\max_{i \in \{1,\ldots,n\}} i \cdot v_i$.
3. The optimal single price is v_{opt}.

Random Sampling Auction

Definition (Random sampling optimal price auction)

The *random sampling optimal price auction* is defined as follows.

1. Randomly partition the set of bidders N into two sets, N_1 and N_2 (i.e., $N = N_1 \cup N_2$; $N_1 \cap N_2 = \emptyset$; each bidder has probability 0.5 of being assigned to each set).
2. Using the procedure above find p_1 and p_2, where p_i is the optimal single price to charge the set of bidders N_i.
3. Then set the allocation and payment rules as follows:
    - For each bidder $i \in N_1$, award a unit of the good if and only if $b_i \geq p_2$, and charge the bidder p_2;
    - For each bidder $j \in N_2$, award a unit of the good if and only if $b_j \geq p_1$, and charge the bidder p_1.

Results

Theorem

Random sampling optimal price auctions are dominant-strategy truthful, weakly budget balanced and ex post individually rational.

Theorem

The random sampling optimal price auction always yields expected revenue that is at least a $(\frac{1}{4.68})$ constant fraction of the revenue that would be achieved by charging bidders the optimal single price, subject to the constraint that at least two units of the good must be sold.

Multiunit Demand

How does VCG behave when (some) bidders may want more than a single unit of the good?

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How does VCG behave when (some) bidders may want more than a single unit of the good?

- no longer a $k+1$st-price auction
- instead, all winning bidders who won the same number of units will pay the same amount as each other.
  - the change in social welfare from dropping any of these bidders is the same.
- Bidders who win different numbers of units will not necessarily pay the same per unit prices.
- However, bidders who win larger numbers of units will pay at least as much in total (not necessarily per unit) as bidders who won smaller numbers of units
  - their impact on social welfare will always be at least as great

Winner Determination for Multiunit Demand

- Let m be the number of units available, and let $\hat{v}_i(k)$ denote bidder i's declared valuation for being awarded k units.
- It's no longer computationally easy to identify the winners—now it's a (NP-complete) weighted knapsack problem:

$$\text{maximize} \quad \sum_{i \in N} \sum_{1 \le k \le m} \hat{v}_i(k) x_{k,i} \tag{1}$$

$$\text{subject to} \quad \sum_{i \in N} \sum_{1 \le k \le m} k \cdot x_{k,i} \le m \tag{2}$$

$$\sum_{1 \le k \le m} x_{k,i} \le 1 \qquad \forall i \in N \tag{3}$$

$$x_{k,i} = \{0, 1\} \qquad \forall 1 \le k \le m, i \in N \tag{4}$$

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- $x_{k,i}$ indicates whether bidder i is allocated exactly k units
- maximize: sum of agents' valuations for the chosen allocation
- (2): number of units allocated does not exceed number available
- (3): no more than one $x_{\cdot,i}$ is nonzero for any i
- (4): all x's must be integers

Multiunit Valuations

How can bidders express their valuations in a multiunit auction?

- m homogeneous goods, let S denote some set
- general: let $p_1, \ldots, p_m$ be arbitrary, non-negative real numbers. Then $v(S) = \sum_{j=1}^{|S|} p_j$.
- downward sloping: general, but $p_1 \geq p_2 \geq \ldots \geq p_m$
- additive: $v(S) = c|S|$
- single-item: $v(S) = c$ if $s \neq \emptyset$; 0 otherwise
- fixed-budget: $v(S) = \min(c|S|, b)$
- majority: $v(S) = c$ if $|S| \geq m/2$, 0 otherwise

Valuations for heterogeneous goods

- now consider a case where multiple, heterogeneous goods are being sold.
- consider the sorts of valuations that agents could have in this case:
  - complementarity: for sets S and T, $v(S \cup T) > v(S) + v(T)$
    - e.g., a left shoe and a right shoe
  - substitutability: $v(S \cup T) < v(S) + v(T)$
    - e.g., two tickets to different movies playing at the same time
- substitutability is relatively easy to deal with
  - e.g., just sell the goods sequentially, or allow bid withdrawal
- complementarity is trickier...

Combinatorial auctions

- running a simultaneous ascending auction is inefficient
  - exposure problem
  - inefficiency due to fear of exposure
- if we want an efficient outcome, why not just run VCG?
  - unfortunately, it again requires solving an NP-complete problem
  - let there be n goods, m bids, sets C_j of XOR bids
  - weighted set packing problem:

$$\max \sum_{i=1}^{m} x_i p_i$$

$$\text{subject to} \sum_{i|g \in S_i} x_i \leq 1 \qquad \forall g$$

$$x_i \in \{0, 1\} \qquad \forall i$$

$$\sum_{k \in C_j} x_k \leq 1 \qquad \forall j$$

Winner determination problem

How do we deal with the computational complexity of the winner determination problem?

- Require bids to come from a restricted set, guaranteeing that the WDP can be solved in polynomial time
  - problem: these restricted sets are *very* restricted...
- Use heuristic methods to solve the problem
  - this works pretty well in practice, making it possible to solve WDPs with many hundreds of goods and thousands of bids.