Game Theoretic Approach for Elastic and Inelastic Demand Management in Microgrid

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Abstract

Smart grid, which consists of many small microgrids, leads to a more stable and secure grid. In this paper, we proposed a game theoretic approach using a novel combinational pricing signal for the demand side management in a microgrid. We classified the appliance in a smart grid into appliance with elastic energy demand and appliance with inelastic appliance. We use game-theoretic approach to analyze the interaction between the microgrid operator and end users as well as among end users themself. We formulate the problem as a single leader, multiple followers Stackelberg game. An unique Stackelberg-Nash equilibrium is derived under two-fold pricing at first and then extended to case of uniform pricing scheme.

1 Introduction

Smart grid, which uses two-way flows of electricity and information to make the energy generation, delivery and consumption more automated, reliable and efficient, is regarded as the next generation power grid. Instead of conveying power from a few centralized generators to many end-users in a traditional power grid, Smart grid typically use distributed energy generation [1] with renewable resources to make the grid more secure, efficient and "green".

A distributed grouping of electricity generation and loads is called microgrid [2]. A microgrid usually connects to a traditional centralized grid (macrogrid). The energy generation in the microgird could be renewable energy resources such as solar panels and small wind turbines for the environmental friendly purpose. However, these renewable energy resources are

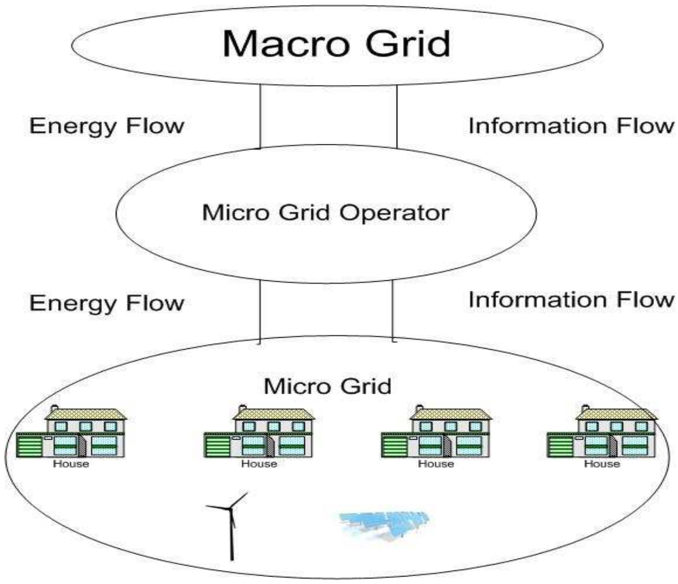

Figure 1: The infrastructure of microgrid system

quite unstable [3]. Consequently, an important goal for the microgrid operator is to keep the balance between the energy generation and demand.

Besides traditional technology to adjust the energy generation such as dispatching or energy storage in microgrid, Demand Side Management (DSM) could also contribute to keep the balance between generation and demand [4]. Among various DSM methods such as TOU and intentionally blackout, real-time pricing is one of the most important DSM strategies. It can be used to achieve different objectives. [4] sets the real-time price to be marginal cost of the energy generation so that the energy demand is shaped and the social welfare is maximized. [5] calculates the optimal real-time pricing based on utility maximization. [6] models the energy retailer as a self-interest agent which uses the real-time pricing to maximize profit for itself. However, different goals sometimes contradict with each other. In this paper, we use a sophisticated pricing to make a tradeoff between two goals: profit maximization and keeping match between generation and consumption. We model the microgrid as a subscriber of a macrogrid. Microgird sometimes could be disconnected with the macrogrid to form a "Islanded Microgrid". At the same time, the microgrid has several solar panels and small wind turbines to generate energy and is in charge of multiple Home Energy Management System (HEMS) in the microgrid. Figure 1 shows the hierarchy of the system.

Another important problem in DSM is how to model the different characteristics of appliance. Most of the literature models that the end-user has an elastic energy demand and the the utility function is convex. A convex utility function could make the problem tractable and lead to graceful and rigid mathematic solutions. Many appliance do have an elastic energy demand, such as heaters, air conditioners, etc. However, some of the appliance, such as desktop PC, TV, stereos, elevators and lights actually have inelastic energy demand and their utility functions should be modeled as a non-convex function. In this paper, we model the utility function of appliance with an inelastic energy demand (for brevity, we use inelastic appliance to refer to an appliance with inelastic energy demand in the following) as a sigmoid function and include both elastic and inelastic appliance in our problem. To the best of our knowledge, our work is the first one classifying appliance into elastic and inelastic for DSM.

In this paper, we propose a novel combinational pricing strategy for the DSM in a microgrid to make the tradeoff between two important goals in DSM. We use a single leader, multiple followers stackelberg game to model the interaction between the microgrid and the HEMS as well as the interaction among different HEMS. We acquire a unique Stackelberg-Nash equilibrium and derived a close form expression for the equilibrium under two-fold pricing. We extend our result into uniform pricing.

The rest of the paper is organized as follows. We explain our system model, especially pricing strategy and utility function in section II. We model the interaction between microgrid operator and HEMS as a single leader, multiple follower Stackelberg game in section III. In section IV, we derive a unique Stackelberg-Nash equilibrium under two-fold pricing. In section V, we extend our results into uniform pricing. Finally, we conclude our paper in section VII.

2 System Model

We consider a scenario where a microgrid is in charge of multiple HEMS. The microgrid owns several solar panels and small wind turbines. We denote the renewable energy generation as (S_{re}) in the microgrid. Due to the unstable of the energy generation, the microgrid needs to purchase energy from the macrogrid when its energy generation is lower than the energy consumption and sell extra energy in the verse condition. We assume that microgrid purchase energy on a real-time wholesale market from the macrogrid on a ten-minute basis. The microgrid charges the HEMS through a fast changing real-time pricing. Microgrid could broadcast its real-time price through a two-way communication system between the microgrid operator and the HEMS which is implemented in the smart grid.

2.1 Pricing Strategy

Generally speaking, there are three different pricing signals for the network pricing problem: flat pricing, usage pricing and congestion pricing [8]. Usage pricing, which is proportional to the amount of the energy consumption, is widely used in smart grid. However, a single pricing signal has limited functions. In this paper, we propose a novel combinational pricing strategy which is a combination of usage pricing and "mismatch pricing" in smart grid.

By "mismatch", we refer to the gap between planned energy supply and real energy consumption, which will cause non-negligible cost to the microgrid system and users, such as voltage disturbance, potential damage to the equipment and the possibility of blackout. This cost is a function with an increasing order on the gap between planned energy supply and consumption. We assume that the microgrid shift this cost to the HEMS in the system through a real-time mismatch pricing, so that HEMS will automatically adjust their energy consumption on the elastic appliance to keep the balance between energy generation and consumption. Assume there are N HEMS in total. The mismatch pricing for i_{th} HEMS can be written as:

$$\frac{k_1}{N}\left(k_2 - \sum_{i=1}^{i=N}\sum_{a\in Ei} x_{i,a}\right)^2 \tag{1}$$

where k_2 is the planned energy generation for the elastic appliance energy usage, E_i is the set of elastic appliances of the i_{th} HEMS. The HEMS have incentive to adjust their elastic energy usage to keep the balance between planned generation and consumption to reduce this mismatch pricing.

We model the microgrid as a self-interest agent maximizing its own profit. It collect revenue from the HEMS through usage pricing. We consider both the two-fold pricing strategy and uniform pricing strategy, which can be regarded as a special case of two-fold pricing. The final pricing signal in our design is the sum of usage pricing and mismatch pricing:

$$P_i = P_e \times \sum_{a\in Ei} x_{i,a} + P_{ie} \times \sum_{a\in IEi} x_{i,a} + \frac{k_1}{N}\left(k_2 - \sum_{i=1}^{i=N}\sum_{a\in Ei} x_{i,a}\right)^2 \tag{2}$$

where P_e / P_{ie} is the usage price per unit of energy for elastic / inelastic appliance, E_i / IE_i is the set of appliance which has elastic / inelastic energy demand for the i_{th} HEMS, $x_{i,a}$ is the energy consumption for the a_{th} appliance of the i_{th} HEMS, P_c is the mismatch pricing.

2.2 Utility Function

Most of the previous work model energy demand of the appliance as elastic and the utility function of the appliance as a convex, continuous function [7],[9],[4],[6]. The convex utility function could make the problem tractable and lead to graceful math solutions to the problem but is limited to model the appliances with an elastic energy demand. In fact, some of the appliance in the smart grid could have inelastic energy demand. For example, the desktop PC, TV, stereos, elevators and lights should be modeled as inelastic appliance. They just consume the energy they required. Additional energy cannot benefit them. The inelastic appliance could be modeled [10],[11],[12] as a sigmoid function or discontinuous function, which can be regarded as a special case of the sigmoid function. In this paper, we model the utility function of the elastic appliance as a convex functions and utility function of inelastic appliance as a sigmoid function.

- Elastic

  When an appliance has an elastic energy demand, such as air conditioner and heater, they perform better when there is more energy. We model their utility function as

  $$U_{i,a} = w_{i,a} log\left(x_{i,a}+1\right), x_{i,a}^{min} \leq x_{i,a} \leq x_{i,a}^{max} \tag{3}$$

  Where $U_{i,a}$ is the utility for a_{th} appliance of i_{th} HEMS, $m_{i,a}$ is the minimum energy requirement for this appliance.

- Inelastic

  For the inelastic appliance, we can model their utility as sigmoid function

  $$U_{i,a} = \frac{w_{i,a}}{1+e^{-(b_{i,a}x_{i,a}+d_{i,a})}}, x_{i,a}^{min} \leq x_{i,a} \leq x_{i,a}^{max} \tag{4}$$

  with an inflection point as

  $$x_{i,a}^{in} = \frac{-d_{i,a}}{b_{i,a}} \tag{5}$$

Correspondingly, we can divide the appliance of i_{th} HEMS into two sets, E_i, IE_i, where the appliance in E_i is elastic and the appliances in IE_i is inelastic. We have $E_i \cap IE_i = \emptyset$ and $E_i \cup IE_i = A_i$.

Fig2 shows an example for these two different utility functions.

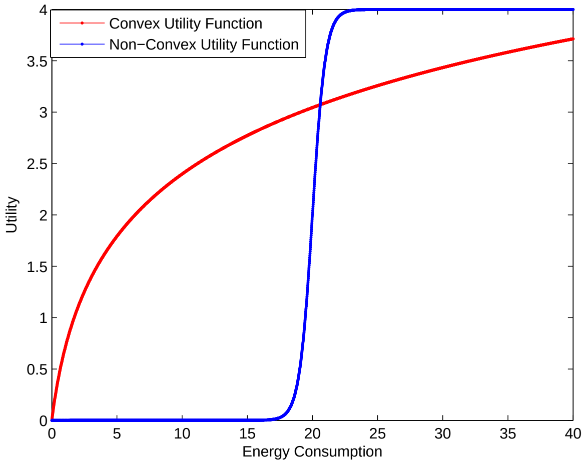

Figure 2: Utility functions $U_{i,a}$ versus energy consumption. The convex utility function is $U_{i,a} = log\left(x_{i,a} + 1\right)$. The non-convex utility function is $U_{i,a} = \frac{4}{1+e^{-\left(2x_{i,a}-40\right)}}$

3 Game Theoretic Approach

We model the interaction between the microgrid operator and the HEMS as a stackelberg game. The leader is the microgrid operator. It maximize the profit and shapes the load through pricing signals. The HEMS in the system are followers. They response to the pricing signal by shaping their load to maximize their utility.

Assume there are N HEMS in the system. The i_{th} HEMS objective can be written as

$$\sum_{a \in A_i} U_{i,a} - P \tag{6}$$

Note that the we use the mismatch pricing to encourage the users to adjust their *elastic load* to match the energy generation of the microgrid network. Because the mismatch pricing does not depend on the strategy of the inelastic appliance, we can divide the optimization problem into two independent problem as:

- Problem 1 (inelastic application)

$$\arg\max_{x_{i,a}} \sum_{a \in IE_i} U_{i,a} - P_{ie} \sum_{a \in IE_i} x_{i,a} \tag{7}$$

$$s.t. \quad x_{i,a}^{min} \leq x_{i,a} \leq x_{i,a}^{max}$$

- Problem 2 (elastic application)

$$\arg\max_{x_{i,a}} \sum_{a \in E_i} U_{i,a} - P_e \sum_{a \in E_i} x_{i,a} - \frac{k_1}{N} \left(k_2 - \sum_{i=1}^{i=N} \sum_{a \in E_i} x_{i,a} \right)^2 \tag{8}$$
$$s.t. \quad x_{i,a}^{min} \leq x_{i,a} \leq x_{i,a}^{max}$$

Assume that microgrid operator is a self-interest agent. Its goal is to maximize the profit. We assume that microgrid operator are not allowed to benefit from the mismatching pricing. In other words, the parameters k_1 have to be set according to the actual cost of the mismatch. The microgrid operator could select optimal P_e and P_{ie} to maximize its profit. We assume that the *variable cost* for the renewable energy generation in the microgrid could be neglected. The additional profit / cost for the microgrid comes from selling / purchasing energy from the real-time market of macrogrid. We assume that the price (P_m) is the same for selling energy and purchasing energy. Consequently, the objective of the microgrid operator is:

- Problem 3:

$$\arg\max_{P_e, P_{ie}} \begin{pmatrix} & P_e \times \sum_{i=1}^{i=N} \sum_{a \in E_i} x_{i,a} \\ + & P_{ie} \times \sum_{i=1}^{i=N} \sum_{a \in IE_i} x_{i,a} \\ - & P_m \times \left(\left(D_e \left(P_e \right) + D_{ie} \left(P_{ie} \right) \right) - S_{re} \right) \end{pmatrix} \tag{9}$$
$$s.t. \quad D_e, D_{ie} \geq 0$$

where D_e / D_{ie} is the total demand of the elastic / inelastic appliance.

4 Stackelberg-Nash Equilibrium with Two-fold Pricing

In this section, we assume that the microgrid operator sets two-fold price for elastic appliance and inelastic appliance. In this case, the problem is more tractable because the two prices could be optimized independently. We derive a unique Stackelberg-Nash equilibrium in this scenario. In the next section, we extend our conclusion to the scenario where microgrid operator sets uniform price for the elastic and inelastic appliance.

Stackelberg equilibrium in our problem is actually a subgame perfect nash equilibrium in an two-level extensive game. We first consider the first level.

4.1 Demand Response of Inelastic Appliance

To acquire the demand response of inelastic application, we need to solve problem 1 for the i_{th} HEMS. We define

$$s_{i,a}\left(x_{i,a}, P_{ie}\right) = U_{i,a} - P_{ie} \times x_{i,a} \tag{10}$$

It is easy to deduce that optimization problem 7 is equivalent to a set of independent optimization problems:

$$x^{*}_{i,a}\left(P_{ie}\right) = \arg\max_{x_{i,a}} s_{i,a}\left(x^{*}_{i,a}, P_{ie}\right), \forall a \in IE_i \tag{11}$$

Note that $s_{i,a}$ is still a sigmoid function with the same inflection point $x^{in}_{i,a}$. We can divide this optimization problem into two problems.

- Problem 1.1
$$x^{h}_{i,a}\left(P_{ie}\right) = \arg\max_{x_{i,a}} s_{i,a}\left(x_{i,a}, P_{ie}\right), \tag{12}$$
$$s.t. \quad x^{in}_{i,a} \leq x_{i,a} \leq x^{max}_{i,a}$$

- Problem 1.2:
$$x^{l}_{i,a}\left(P_{ie}\right) = \arg\max_{x_{i,a}} s_{i,a}\left(x_{i,a}, P_{ie}\right), \tag{13}$$
$$s.t. \quad x^{min}_{i,a} \leq x_{i,a} \leq x^{in}_{i,a}$$

For problem 1.1, since $s_{i,a}\left(x_{i,a}, P_u\right)$ is concave for $x_{i,a} \in \left[x^{in}_{i,a}, x^{max}_{i,a}\right]$, we can take derivative to get

$$s'_{i,a} = \frac{w_{i,a} b_{i,a} e^{-\left(b_{i,a} x_{i,a} + d_{i,a}\right)}}{\left[1 + e^{-\left(b_{i,a} x_{i,a} + d_{i,a}\right)}\right]^2} - P_u = 0 \tag{14}$$

For convenience, we denote $bw_{i,a} = w_{i,a} b_{i,a}$. We can consider two cases:

- Case 1 if $bw_{i,a} < 4P_u$, we have $s'_{i,a} < 0$. The function is decreasing on the domain. Thus, we have
$$x^{h}_{i,a} = x^{in}_{i,a} \tag{15}$$

- Case 2 if $bw_{i,a} \geq 4P_u$, since $0 \leq e^{-\left(b_{i,a} x_{i,a} + d_{i,a}\right)} \leq 1$, we can get

$$e^{-\left(bw_{i,a} x_{i,a} + d_{i,a}\right)} = \frac{-\left(2P_u - bw_{i,a}\right) - \sqrt[2]{\left(2P_u - bw_{i,a}\right)^2 - 4P_u^2}}{2P_u} \tag{16}$$

Thus, we get

$$x_{i,a}^{h}=\left[\frac{-d_{i,a}-ln\left(\frac{-(2P_u-bw_{i,a})-\sqrt[2]{(2P_u-bw_{i,a})^2-4P_u^2}}{2P_u}\right)}{b_{i,a}}\right] \tag{17}$$

Of course, we still need to check if this value is in the range of $x_{i,a}^{in}\leq x_{i,a}\leq x_{i,a}^{max}$.

Based on the analysis of two cases above, we have:

$$x_{i,a}^{h}\left(P_u\right)=\begin{cases} x_{i,a}^{in} & \text{if} \quad P_u>\frac{bw_{i,a}}{4} \\ x_{i,a}^{max} & \text{if} \quad P_u\leq\frac{bw_{i,a}e^{-\left(b_{i,a}x_{i,a}^{max}+d_{i,a}\right)}}{\left[1+e^{-\left(b_{i,a}x_{i,a}^{max}+d_{i,a}\right)}\right]^2} \\ \frac{-d_{i,a}-ln\left(\frac{-\left(2P_u-bw_{i,a}\right)-\sqrt[2]{\left(2P_u-bw_{i,a}\right)^2-4P_u^2}}{2P_u}\right)}{b_{i,a}} & \text{other} \end{cases} \tag{18}$$

To solve problem 1.2, note that $s_{i,a}$ is a convex function for $x_{i,a}^{min}\leq x_{i,a}\leq x_{i,a}^{in}$. The optimal solution must exist in $\left\{x_{i,a}^{min},x_{i,a}^{in}\right\}$. For inelastic appliances, we assume $x_{i,a}^{min}=0$. Intuitively, we have $U_{i,a}\left(0\right)=0$, because the utility should be 0 when there is no power. We can get the utility at the inflection point as

$$U_{i,a}\left(x_{i,a}^{in}\right)=\frac{w_{i,a}}{2}-P_u\frac{-d_{i,a}}{b_{i,a}} \tag{19}$$

Thus, we have:

$$x_{i,a}^{l}\left(P_u\right)=\begin{cases} x_{i,a}^{min} & \text{if} P_u>\frac{-bw_{i,a}}{2d_{i,a}} \\ x_{i,a}^{in} & \text{other} \end{cases} \tag{20}$$

Now the problem is how to combine $x_{i,a}^{l}$ and $x_{i,a}^{h}$ to get the optimal solution $x_{i,a}^{*}$ to the original problem. For example, assume $\frac{bw_{i,a}}{-2d_{i,a}}\geq\frac{bw_{i,a}}{4}$, it is easy to get

$$x_{i,a}^{*}\left(P_u\right)=\begin{cases} x_{i,a}^{min} & \text{if} \quad P_u>\frac{-bw_{i,a}}{2d_{i,a}} \\ x_{i,a}^{in} & \text{if} \quad \frac{-bw_{i,a}}{2d_{i,a}}\geq P_u>\frac{bw_{i,a}}{4} \\ x_{i,a}^{max} & \text{if} \quad P_u\leq\frac{bw_{i,a}e^{-\left(b_{i,a}x_{i,a}^{max}+d_{i,a}\right)}}{\left[1+e^{-\left(b_{i,a}x_{i,a}^{max}+d_{i,a}\right)}\right]^2} \\ \frac{-d_{i,a}-ln\left(\frac{-\left(2P_u-bw_{i,a}\right)-\sqrt[2]{\left(2P_u-bw_{i,a}\right)^2-4P_u^2}}{2P_u}\right)}{b_{i,a}} & \text{other} \end{cases} \tag{21}$$

However, if we consider the practical utility characteristics of inelastic appliance such as TV and PC, we typically have $\frac{bw_{i,a}}{-2d_{i,a}} < \frac{bw_{i,a}}{4}$. In this case, the combination results is:

$$x^*_{i,a}\left(P_u\right) = \begin{cases} x^{min}_{i,a} & \text{if} \quad P_u > P'_u \\ x^{max}_{i,a} & \text{if} \quad P_u \leq \frac{bw_{i,a}e^{-\left(b_{i,a}x^{max}_{i,a}+d_{i,a}\right)}}{\left[1+e^{-\left(b_{i,a}x^{max}_{i,a}+d_{i,a}\right)}\right]^2} \\ \frac{-d_{i,a}-ln\left(\frac{-\left(2P_u-bw_{i,a}\right)-\sqrt[2]{\left(2P_u-bw_{i,a}\right)^2-4P_u^2}}{2P_u}\right)}{b_{i,a}} & \text{other} \end{cases} \tag{22}$$

where P'_u can be acquired by solving:

$$\frac{bw_{i,a}e^{-\left(b_{i,a}x'_{i,a}+d_{i,a}\right)}}{\left[1+e^{-\left(b_{i,a}x'_{i,a}+d_{i,a}\right)}\right]^2} - P'_u = 0 \tag{23}$$

$$P'_u x'_{i,a} = \frac{w_{i,a}}{1+e^{-\left(b_{i,a}x'_{i,a}+d_{i,a}\right)}} \tag{24}$$

To make this problem tractable, we consider the practical characteristic of these inelastic appliance. As far as the inelastic appliance we considered, $|b|, |d|$ should be quite large (the utility function is more like a discontinuous function) and $x^{max}_{i,a}$ should be quite close to the inflection point (high voltage is bad for PC and TV), we write it as

$$x^{max}_{i,a} = x^{in}_{i,a} + \epsilon \tag{25}$$

where ϵ is a very small number. We have $U_{i,a}\left(x^{max}\right) \approx 1$ because $|b|, |d|$ are quite large. From the prospective of the leader, we have $x^{max}_{i,a} \approx x^{in}_{i,a}$ because we assume that ϵ is a very small number. Now we can approximate our solution as:

$$x^*_{i,a}\left(P_u\right) = \begin{cases} x^{min}_{i,a} & \text{if} P_u > \frac{-bw_{i,a}}{d_{i,a}} \\ x^{in}_{i,a} & \text{other} \end{cases} \tag{26}$$

Assume that the end-users tell their preference to HEMS by setting priority on different appliance. For example, the user could set PC and TV to be high priority and the democratize lights to be low priority. Assume there are M priorities on the HEMS. For the appliance set with the same priority, they can be regarded as one appliance in total. For all the appliance with inelastic energy demand, assuming that they are already sorted in the order $\frac{-bw_{1,1}}{d_{1,1}} = \cdots = \frac{-bw_{N,1}}{d_{N,1}} < \cdots < \frac{-bw_{1,M}}{d_{1,M}} = \cdots = \frac{-bw_{N,M}}{d_{N,M}}$, then the total energy demand from appliance with inelastic energy demand is given by equation 27.

$$D_{ie}\left(P_u\right)=\begin{cases}\sum_{i=1}^{i=N}\sum_{a=1}^{a=M}x_{i,a}^{min} & \text{if} P_u>\frac{-b_{N,M}}{d_{N,M}}\\ \sum_{i=1}^{i=N}\sum_{a=1}^{a=M-1}x_{i,a}^{min}+\sum_{i=1}^{i=N}\sum_{a=M}^{a=M}x_{i,a}^{in} & \text{if}\frac{-b_{N,M}}{d_{N,M}}\geq P_u>\frac{-b_{N,M-1}}{d_{N,M-1}}\\ \sum_{i=1}^{i=N}\sum_{a=1}^{a=M-2}x_{i,a}^{min}+\sum_{i=1}^{i=N}\sum_{a=M-1}^{a=M}x_{i,a}^{in} & \text{if}\frac{-b_{N,M-1}}{d_{N,M-1}}\geq P_u>\frac{-b_{N,M-2}}{d_{N,M-2}}\\ \vdots & \vdots\\ \sum_{i=1}^{i=N}\sum_{a=1}^{a=M}x_{i,a}^{in} & \text{if} P_u<\frac{-b_{N,1}}{d_{N,1}}\end{cases} \quad (27)$$

4.2 Demand Response of elastic Appliance

To acquire demand response of elastic appliance is quite different from the procedure for inelastic appliance. The most difficult part is that mismatch pricing signal for an arbitrary HEMS is dependent on the strategy of other HEMS. In this case, we cannot divide the problems into independent subproblems like what we did for the inelastic appliance. For the elastic appliance, different HEMS form a non-cooperative game among themselves, where Nash Equilibrium (NE) is a candidate solution.

To acquire the elastic appliance, we need to solve problem 2. Note that adding $\sum_{j=1,j\neq i}^{N}\sum_{a\in E_j}U_{j,a}-P_e\sum_{j=1,j\neq i}^{N}\sum_{a\in E_i}x_{i,a}$ to the objective function of the i_{th} HEMS will not change the optimal strategy of i_{th} HEMS. So all the HEMS have the identical objective function as

$$OF'=\sum_{i=1}^{N}\sum_{a\in E_i}U_{i,a}-\frac{k_1}{N}\left(k_2-\sum_{i=1}^{i=N}\sum_{a\in Ei}x_{i,a}\right)^2-P_u\times\sum_{i=1}^{N}\sum_{a\in E_i}x_{i,a} \quad (28)$$

Obviously, $U_{i,a}$ should not relate to $x_{j,b},(j,b)\neq(i,a)$ because the appliance utility should not relate to how much energy other appliances consumed. Consequently, we have

$$\frac{\partial^2 OF'}{\partial x_{i,a}\partial x_{j,b}}=-\frac{k_1}{N}<0,i,j\in\{1,\ldots,N\} \quad (29)$$

Because all the appliance is E_i is elastic, $U_{i,a}$ is always concave for different types of appliances, we have

$$\frac{\partial^2 OF'}{\partial x_{i,a}\partial x_{i,a}}=\frac{\partial^2 Ux_{i,a}}{\partial x_{i,a}\partial x_{i,a}}-\frac{k_1}{N}<0,i\in\{1,\ldots,N\} \quad (30)$$

So the OF' is strictly concave. This optimization problem has a unique solution. The strategy of the i_{th} HEMS in the NE can be acquired by solving

$$\frac{\partial OF'}{\partial x_{i,a}}=0,\forall a\in A_i \quad (31)$$

To get the NE of the game, we need equation 31 holds for the strategy of every HEMS. Thus, we get

$$\frac{\partial U_{i,a}}{\partial x_{i,a}} = \frac{\partial U_{j,b}}{\partial x_{j,b}}, a \in A_i, b \in A_j, \forall i, j \in N \tag{32}$$

Given our utility function for the elastic appliances, we have

$$\frac{w_{i,a}}{x_{i,a}+1} = \frac{w_{j,b}}{x_{j,b}+1} \tag{33}$$

Define $y_{i,a} = x_{i,a}+1$, $y = \sum_{i=1}^{i=N} \sum_{a\in Ei} y_{i,a}$, $m = \sum_{i=1}^{i=N} \sum_{a\in Ei} 1$,$w = \sum_{i=1}^{i=N} \sum_{a\in Ei} w_{i,a}$, we can change equation 31 as

$$\frac{\partial OF'}{\partial x_{i,a}} = g(y) = \frac{w}{y} - \frac{2k_1}{N}(y - k_2 - m) - P_e = 0 \tag{34}$$

$$s.t. y \geq m$$

It is quite easy to solve this equation to get:

$$y = \frac{\frac{2k_1}{N}(k_2+m) - P_e + \sqrt[2]{\left(\frac{2k_1}{N}(k_2+m) - P_e\right)^2 + \frac{8k_1 w}{N}}}{\frac{2k_1}{N}} \tag{35}$$

$$s.t. P_e < \frac{w}{m} + \frac{2k_1k_2}{N}$$

For the energy demand of a single appliance, we have

$$x_{i,a} = \frac{w_{i,a}y}{w} - 1 \tag{36}$$

To guarantee $x_{i,a} \geq 0$, we need

$$\frac{w_{i,a}y}{w} \geq 1, \forall i, a \tag{37}$$

We assume that all the appliance have similar $w_{i,a}$, so that this constrain always hold. Then we can get the analytical expression of the total demand of the elastic appliance as

$$D_e = \frac{\frac{2k_1}{N}(k_2+m) - P_e + \sqrt[2]{\left(\frac{2k_1}{N}(k_2+m) - P_e\right)^2 + \frac{8k_1 w}{N}}}{\frac{2k_1}{N}} - m \tag{38}$$

$$s.t. P_e < \frac{w}{m} + \frac{2k_1k_2}{N}$$

4.3 Stackelberg Game Equilibrium

In this section, assume that two-fold pricing strategy is adopted for elastic and inelastic appliance. The problem of the microgrid operator is to choose the optimal P_e, P_{ie} so that profit is maximized. The problem has been formulated in problem 3. To solve problem 3, we can divide the problem into two independent problem to acquire optimal P_e and P_{ie} independently.

4.3.1 Optimal Price for Inelastic Appliance

Because for two-fold pricing scheme, optimal price for inelastic appliance is not related to the elastic appliance, the problem could be written as

$$\arg\max_{P_{ie}} \left(P_{ie} \times D_{ie} - P_m \times D_{ie}\right) \tag{39}$$

According to equation 27, the demand response of inelastic appliances is a step function. It is obviously that the optimal solution exists in the set $\frac{-b_{N,1}}{d_{N,1}}, \frac{-b_{N,2}}{d_{N,2}}, \ldots, \frac{-b_{N,M}}{d_{N,M}}$, because we have $\frac{\partial D_{ie}}{\partial P_{ie}} = 0$ for the each interval of equation 27. Then we can search the set to get the optimal $P*_{ie}$, which requires a low computational complexity as $O\left(n\right)$, where n is the number of points in the set.

4.3.2 Optimal Price for Elastic Appliance

To acquire the optimal $P*_e$, we first write the problem as:

$$\arg\max_{P_e} \left(P_e \times D_e - P_m \times D_e\right) \tag{40}$$

Note that it is quite difficult to solve this problem if we directly substitute equation 38 to the above equation. Instead, we can substitute equation 34 to get

$$\arg\max_{y} \left\{\left[\frac{w}{y} - \frac{2k_1}{N}\left(y - k_2 - m\right) - P_m\right] \times \left(y - m\right)\right\} \tag{41}$$

We take derivative on y to get

$$-\frac{4k_1}{N}y^3 + \left[\left(k_2 + 2m\right)\frac{2k_1}{N} - P_b\right]y^2 + mw = 0 \tag{42}$$

We can get the discriminant [**?**] of this cubic function as:

$$\Delta = -4\left[\left(k_2 + 2m\right)\frac{2k_1}{N} - P_b\right]^3 mw - 27\left(\frac{4k_1}{N}\right)^2 m^2w^2 \tag{43}$$

Assume $P_b < (k_2 + 2m) \frac{2k_1}{N} + 3 \left(\frac{4k_1}{N}\right)$, we have

$$\Delta < 0$$

The only real solution y_e^* is equation 44

$$\begin{aligned} y_e^* = & + \frac{\left[(k_2+2m)\frac{2k_1}{N} - P_b\right]}{3\frac{4k_1}{N}} \\ & + \frac{\left\{\frac{1}{2}\left[2\left[(k_2+2m)\frac{2k_1}{N} - P_b\right]^3 + 27\left(\frac{4k_1}{N}\right)^2 mw + \sqrt{-27\left(\frac{4k_1}{N}\right)^2 \Delta}\right]\right\}^{\frac{1}{3}}}{3\frac{4k_1}{N}} \\ & + \frac{\left\{\frac{1}{2}\left[2\left[(k_2+2m)\frac{2k_1}{N} - P_b\right]^3 + 27\left(\frac{4k_1}{N}\right)^2 mw - \sqrt{-27\left(\frac{4k_1}{N}\right)^2 \Delta}\right]\right\}^{\frac{1}{3}}}{3\frac{4k_1}{N}} \end{aligned} \tag{44}$$

The optimal P_e is

$$P_e = \frac{w}{y_e^*} - \frac{2k_1}{N} \left(y_e^* - k_2 - m\right) \tag{45}$$

5 Stackelberg-Nash Equilibrium with Uniform Pricing

For the uniform pricing, we need to consider the whole problem, instead of dividing the problem into two subproblems. Assume the uniform price for both elastic and inelastic appliance is P_u. The problem could be formulated as:

$$\arg\max_{P_u} \begin{pmatrix} P_u \times \left(D_e\left(P_u\right) + D_{ie}\left(P_u\right)\right) \\ -P_m \times \left(D_e\left(P_u\right) + D_{ie}\left(P_u\right) - S_{re}\right) \end{pmatrix} \tag{46}$$

According to equation 27, we find that D_{ie} is not a continuous function, we need to check all the discontinuous points to find the best point. Then, for every segment of P_u, D_{ie} is a constant, we can treat D_{ie} be a constant and try to find a extreme point. We need to substitute every solution of D_{ie} to problem and follow the method to acquire P_e in the previous section. We can extend the equation 41 to :

$$\arg\max_{y} \left\{ \left[\frac{w}{y} - \frac{2k_1}{N}\left(y - k_2 - m\right) - P_m\right] \times \left(y - m + D_{ie}\right) \right\} \tag{47}$$

Then we follow the procedure in the previous section to get the extreme point P_u^j for every $j_t h$ interval. However, we still need to check that P_u^j falls in the interval so that the solution is valid. There could be as many as m valid extreme points or no valid extreme point at all. Besides these extreme points, we still need to check all the discontinuous point at $\frac{-b_{N,1}}{d_{N,1}}, \frac{-b_{N,2}}{d_{N,2}}, \ldots, \frac{-b_{N,M}}{d_{N,M}}$ in equation 27. Because we just search in a finite set, the computational complexity is still as low as $O(n)$, where n is the number of discontinuous points.

6 Conclusion

In this paper, we used the game-theoretic tools to analyze the interaction between the microgrid operator and end users as well as among end users themself. We considered both elastic and inelastic appliance. A single leader, multiple follower Stackelberg game is formulated. We acquired an unique Stackelberg-Nash equilibrium under both two-fold pricing and uniform pricing scheme. The future work is to use show our results using experimental results.

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