CPSC 532L, Winter 2011
Homework #3

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1. **[20 points]** Consider the following voting scheme:

   Each agent submits a total preference ordering, along with additional information of that agent's utility for each outcome. (Obviously, the preference ordering reflects the ordering of the outcomes by utility, with the highest utility outcome being most preferred.) Let the utility for each outcome be an element of $[0, 100]$. The social welfare function orders each outcome by the sum of the utilities of that outcome for each agent. In the case of ties, the outcome with the earlier lexicographic ordering is preferred.

   (a) Is this Pareto efficient? Justify.

   (b) Is this independent of irrelevant utilities? (In this setting, define IIU as the proposition that the social ordering of o and o' does not change as long as agents do not change their utilities for o and o'.) Justify.

   (c) Is this non dictatorial? Justify.

   (d) Does this voting scheme contradict Arrow's theorem? If yes, explain why; if not, demonstrate that Arrow's theorem is not violated.

2. **[30 points]** Consider a potentially infinite outcome space $\mathcal{O} \subset [0, 1]$, and a finite set N of n agents. Denote the utility of an agent with type θ_i for outcome o as $u_i(o, \theta_i)$. Constrain the utility functions so that every agent has some unique, most-preferred outcome $b(\theta_i) \in \mathcal{O}$, and so that $|o' - b(\theta_i)| < |o'' - b(\theta_i)|$ implies that $u_i(o', \theta_i) > u_i(o'', \theta_i)$.

   Consider a direct mechanism which asks every agent to declare his most-preferred outcome and then selects the median outcome. (If there are an even number of agents, the mechanism chooses the larger of the two middle outcomes.)

   (a) Prove that truthtelling is a dominant strategy.

   (b) Prove that the mechanism selects a Pareto optimal outcome.

   (c) Prove that if the mechanism designer submits $n - 1$ "dummy preferences" of her own with any values she likes, and then runs the same mechanism on the $2n-1$ preferences, the dominant strategy is preserved.

   (d) As described so far, the mechanism selects the $\lceil \frac{n}{2} \rceil^{\text{th}}$-order statistic of the declared preferences. (The k^{th}-order statistic of a set of numbers is the k^{th}-largest number in the set.) Explain how to select dummy preferences in such a way that the mechanism selects the k^{th}-order statistic of the agents' declared preferences for any $k \in \{1, \ldots, n\}$. Of course, the dummy preferences must be set in a way that does not depend on the specific declarations made by the agents.

3. **[10 points]** Consider running the VCG mechanism over the set of agents $N = \{1, 2, 3\}$ with quasilinear utilities over the choice set $X = \{x_1, x_2, x_3\}$. Calculate the VCG outcome (the final choice and each agent's payment) and the utility for each agent when:

	x_1	x_2	x_3
v_1	4	1	3
v_2	2	7	4
v_3	4	3	0

(a) All the agents truthfully report their valuation functions, as given in the above table.

(b) Agent 2 changes his reported value for x_2 from 7 to 5.

4. **[30 points]** The VCG mechanism does not violate the Myerson-Satterthwaite Theorem because it is not budget balanced for general quasilinear preferences. But this seems like an easy enough problem to solve—we can just evenly redistribute any money that was collected by the mechanism (or, tax all agents equally if the net payment to the agents was positive). Below is a proposed, budget-balanced version of the VCG Mechanism. It converts what was p_i into a temporary variable t_i. Then, the new payments p_i contains an equal redistribution of the sum of the original payments.

   The *Budget-Balanced VCG Mechanism* is a direct mechanism $M(\hat{v}) = (x(\hat{v}), p_1(\hat{v}), \ldots, p_n(\hat{v}))$, where

   - $x(\hat{v}) = \arg\max_{x \in X} \sum_{i \in N} \hat{v}_i(x)$, and
   - $t_i(\hat{v}) = \max_{o \in O_{-i}} \sum_{j \neq i} \hat{v}_j(o) - \sum_{j \neq i} \hat{v}_j(x)$
   - $p_i(\hat{v}) = t_i - \frac{1}{n} \sum_i t_i(\hat{v})$

   (a) Show that this mechanism is not incentive compatible. One option is to prove this directly. Alternately, you can give two valuation functions for an agent i (one that gives his true valuation (v_i), and alternative one (v_i')) and a set of declared valuation functions ($\hat{v}_j, \hat{v}_k, \ldots$) for as many agents other than i that you need, and show that if all other agents declare these valuations, the utility for agent i is higher if he declares v_i' instead of v_i.

   (b) Although our first mechanism failed, we can use a similar idea to make VCG budget-balanced *ex-ante*. Assume that bidders valuations v_i are randomly drawn from some joint commonly known distribution. Consider the following mechanism:

   Ex-Ante Budget-Balanced VCG Mechanism is a direct mechanism $M(\hat{v}) = (x(\hat{v}), p_1(\hat{v}), \ldots, p_n(\hat{v}))$, where

   - $x(\hat{v}) = \arg\max_{x \in X} \sum_{i \in N} \hat{v}_i(x)$, and
   - $t_i(\hat{v}) = \max_{o \in O_{-i}} \sum_{j \neq i} \hat{v}_j(o) - \sum_{j \neq i} \hat{v}_j(x(\hat{v}))$
   - $p_i(\hat{v}) = t_i(\hat{v}) - \frac{1}{n} \sum_j \mathbb{E}_v[t_j(v)]$

   Prove that truth-telling is a dominant strategy in this new mechanism.

   (c) Show that this mechanism is *ex-ante* budged-balanced on expectation (i.e. expected total payment by all agents is zero).

Academic Honesty Form

For this assignment, it is acceptable to collaborate with other students provided that you write up your solutions independently. The only reference materials that you can use are the course notes and textbook, and the reference textbooks listed on the course web page. In particular, getting help from students or course materials from previous years is not acceptable.

List any people you collaborated with:

List any non-course materials you refered to:

Signature:

Fill in this page and include it with your assignment submission.