# Potential Games: Theory and Application in Wireless Networks

Multiagent Systems
Course Project Report

April 24, 2008


## abstract

Potential games are subclass of strategic normal form games. Each potential game admits a potential function which is the key idea of potential games. Potential function characterizes special behavior of payoff functions. There are several notions of potential games. In this paper, I study some notions of potential games and explain the relation between potential games and potential functions. Among various properties of the potential games, I characterize two more important ones: existence of Nash equilibrium and convergence of the learning process to the Nash equilibrium. Two examples are presented in order to demonstrate the application of potential games in wireless network.


## 1 Introduction

Potential games were introduced in the seminal work of Monderer and Shapley in 1996 [1]. These games received increasing attention recently. Various notions of potential games introduced and studied in the literature. Generalized ordinal, ordinal, exact, and weighted potential games were introduced in [1]. Voorneveld *et al.* studied ordinal potential games and characterized several properties of these games. The notions of best-response potential games and pseudo-potential games were studied in [2] and [3], respectively.

A strategic game is a potential game if it admits a potential [1]. Potential functions quantify the difference in the payoff due to unilaterally deviation of each player either exactly (exact potential games), in sign (ordinal potential games), or deviation to the best response (best-response potential games). Potential function can be interpreted as a measure of the disagreement among players, or, equivalently as the drift towards the Nash equilibrium (NE). Potential function can replace the utility function of different players while preserves some of the game's structure like NE and best response.

[1]In physics, a function $P$ is a potential for vector $(\Pi_1, ..., \Pi_n)$ if $\frac{\partial \Pi_i(\mathbf{x})}{\partial x_i} = \frac{\partial P(\mathbf{x})}{\partial x_i}$, where $\mathbf{x} = \{x_1, ..., x_n\}$.

Potential games possess several special properties. The existence of the potential function guarantees these properties. Under some conditions, all potential games have pure strategy NE. More interestingly, under some conditions which are not too tight, every learning process based on best-response of the players converges to an NE. On other words, starting from an arbitrary point, the sequence of unilaterally best-responses of players reaches to an NE after finitely steps.

The focus in this paper is on ordinal and pseudo potential games which are more applicable in wireless networks. In this paper, I introduce some notions of potential games and describe how potential functions are defined and related to the potential games. Two special properties of potential games (i.e., existence of the NE and convergence of the learning process to NE) are studied with more details for ordinal and pseudo potential games. The rest of this paper is organized as follows: In Section 2, I review some notions of potential games. In section 3, some properties of the potential games are characterized. In Section 4, some examples of potential games in wireless network are presented. Conclusions are given in Section 5.

# 2 Potential Games

Let $\Gamma = \langle N, Y, u \rangle$ be a strategic normal form game with a finite number of players. The strategy space and payoff function of player $i$ are denoted by $Y_i$ and $u_i$ respectively. In the following subsections, I review some notions of potential games.

## 2.1 Ordinal Potential Games

$\Gamma$ is called an *ordinal potential game* if it admits an ordinal potential. A function $P : Y \rightarrow R$ is an ordinal potential for $\Gamma$ if for every $i \in N$ and for every $y_{-i} \in Y_{-i}$:

$$u_i(y_{-i}, x) - u_i(y_{-i}, z) > 0 \ \ iff \ \ P(y_{-i}, x) - P(y_{-i}, z) > 0 \ \ \forall x, z \in Y_i. \quad (1)$$

On other words, if player $i$ obtains a better (worse) payoff by unilaterally deviating from a strategy to another one, the potential function increases (decreases) with this deviation as well. For everywhere differentiable payoff functions and continuous action space, an equivalent condition for a game to be an ordinal potential is the existence of an ordinal potential function $P$ which satisfies:

$$\frac{\partial u_i(y)}{\partial y_i} > 0 \ \ iff \ \ \frac{\partial P(y)}{\partial y_i} > 0, \ \ \forall i \in N, \ y \in Y. \quad (2)$$

## 2.2 Weighted Potential Games

Let $w = (w_1, \cdots, w_n)$ be a vector of positive numbers which is called weights. A function $P : Y \rightarrow R$ is a weighted potential for $\Gamma$ if for every $i \in N$ and for every $y_{-i} \in Y_{-i}$

$$u_i(y_{-i}, x) - u_i(y_{-i}, z) = w_i \left( P(y_{-i}, x) - P(y_{-i}, z) \right), \quad \forall x,\ z \in Y_i. \tag{3}$$

$\Gamma$ is called a *weighted potential game* or simply $w$-potential if it admits a weighted potential. It can be easily verified that for everywhere differentiable payoff functions and continuous action space, an equivalent condition for a game to be weighted potential is the existence of a weighted potential function $P$ which satisfies:

$$\frac{\partial u_i(y)}{\partial y_i} = w_i \left( \frac{\partial P(y)}{\partial y_i} \right), \quad \forall i \in N,\ y \in Y. \tag{4}$$

### 2.3 Exact Potential Games

A function $P : Y \to R$ is an exact potential for $\Gamma$ if it is a weighted potential for $\Gamma$ with $w_i = 1,\ \forall i \in N$. $\Gamma$ is called an *exact potential game* if it admits an exact potential. In other words, a normal form game is called exact potential game if there exists a potential function which reflects the change in the utility accrued by every unilaterally deviating player. For example, Prisoner's Dilemma game G is an exact potential game with potential as follows:

$$G = \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

### 2.4 Pseudo-Potential Games

Game $\Gamma = \langle N, Y, u \rangle$ is a pseudo-potential game if there exists a pseudo-potential function $P : S \to R$ such that for all $i \in N$ and all $y_{-i} \in Y_{-i}$:

$$\arg \max_{y_i \in Y_i} u_i(y_i, y_{-i}) \ \supseteq \ \arg \max_{y_i \in Y_i} P(y_i, y_{-i}). \tag{5}$$

In other words, each player's best reply in the game $\Gamma^\star = \langle N, A, P \rangle$ is included in that of $\Gamma$. If the set of best replies of gems $\Gamma$ and $\Gamma^\star$ are identical, then game $\Gamma$ is called a best-response potential game [2]. If $P$ is strictly concave, then every pseudo-potential game is a best-response potential game. I study the pseudo-potential games, because a main class of games used in wireless networks is the game of weak strategic substitutes (WSTS) or complements (WSTC) with aggregation which is a pseudo-potential game. For more information about pseudo-potential games please refer to [4].

Game $\Gamma = \langle N, A, u \rangle$ is a game of strategic substitutes (STS) (complements(STC)) if the best response reaction of each player for the case that the other players turn more aggressive is to become less aggressive (more aggressive). A boarder view of STS/C games are games of weak strategic substitutes (WSTS) or complements (WSTC). WSTS/C are games in which there exists a selection from the best response correspondence of each player, which is nonincreasing (for WSTS), or nondecreasing (for WSTC). A more restricted class of these games is WSTS/C games with aggregation. In these games, the payoff of each player only depends upon his action and an aggregate of other players' actions. The simplest aggregation is additive aggregation defined as $\bar{y}_{-i} = \sum_{j \in N \setminus \{i\}} y_j$. In other words, $u_i$ only depends on $y_i$ and the summation of other players' actions. Consequently, the set of best replies of player $i$ for any choice of $y_{-i} \in Y_{-i}$ depends on $\bar{y}_{-i}$ which is denoted as $BR(\bar{y}_{-i})$.

Formally, game $\Gamma^\star = \langle N, A, P \rangle$ is a game of WSTS with aggregation if for every player $i$, there exists a function $b_i : \bar{Y}_{-i} \to Y_i$ such that (i) $b_i(\bar{y}_{-i}) \in BR(\bar{y}_{-i}) \in \bar{Y}_{-i}$, $\forall \bar{y}_{-i}$ and (ii) $b_i(\bar{y}_{-i}) \leq b_i(\bar{x}_{-i})$ whenever $\bar{y}_{-i} > \bar{x}_{-i}$. A game of WSTC with aggregation is defined exactly as above, except for replacing $\bar{y}_{-i} < \bar{x}_{-i}$ in (ii).

### 2.5 Relations Between Different Classes of Potential Games

The following equation relates different classes of potential games:

$$\begin{aligned} &\text{Exact potential} \subset \text{Weighted potential} \subset \text{Ordinal potential} \\ &\text{Ordinal potential} \subset \text{Best-response potential} \subset \text{Pseudo-potential.} \end{aligned}$$

There are some more notions of potential games such as: generalized ordinal potential games, generalized $\epsilon$-potential games, and quasi-potential games. However, they are not inside the scope of this research.

## 3 Properties of Potential Games

In this section, I will summarize some special properties of ordinal and pseudo potential games. Note that weighted potential and exact potential games are subsets of ordinal potential games. An ordinal potential game has all properties of pseudo-potential games. However, the properties of pseudo-potential games are more restricted. A normal form game is a finite game if the number of players is finite and each of them has a finite strategy space. The most interesting properties of potential games can be categorized in two parts: existence of Nash equilibrium and convergence of the learning process to the Nash equilibrium. In the following subsections, I discuss more about these properties separately.

### 3.1 Existence of Nash equilibrium

The following two remarks characterize the existence of Nash equilibrium in finite potential games.

*Remark 1.* If $P$ is the potential function of ordinal potential game $\Gamma = \langle N, Y, \{u_i\}_{i \in N} \rangle$, then the equilibrium set of $\Gamma$ coincides with the equilibrium set of coordination game $\Gamma^\star = \langle N, Y, \{P\}_{i \in N} \rangle$ [1].

*Remark 2.* Every finite ordinal potential game possesses a pure strategy Nash equilibrium [1].

Consequently, every $y^\star \in Y$ that maximizes $P(y)$ is a pure strategy equilibrium of $\Gamma$. However, the converse is not in general true. There might be pure or mixed strategy Nash equilibria that are just local maximum points of $P$. But, in addition, if the strategy space is convex and $P$ is continuously differentiable on the strategy space, then every Nash equilibrium of $\Gamma$ is a stationary point of $P$. If $P$ is concave, then every Nash equilibrium of $\Gamma$ is a maximum point of $P$. Such a Nash equilibrium is unique if $P$ is strictly concave. An important conclusion of Remark 1 is that: potential games can be studied using two different approaches: 1) the classical framework of game theory applied to game $\Gamma$; and 2) the framework of standard optimization theory applied to the potential function. For infinite ordinal potential games, the Nash equilibrium exists under some restrictions:

*Remark 3.* Every ordinal potential game possesses an $\epsilon$-equilibrium [2] [1].

*Remark 4.* Every pseudo potential game with compact strategy set and continuous potential function possesses a pure-strategy Nash equilibrium [4].

Note that the existence of a continuous payoff function does not guarantee the existence of continuous potential function [5]. In exact and weighted potential games, however, potential function is continuous if and only if the utility functions are continuous. For infinite potential games, the Nash equilibrium exists under more restricted conditions. However, these restrictions are not too tight. Interestingly, in practice, a large family of potential functions are continuous and the strategy space is compact. Note that all the relations described earlier among Nash equilibria and potential function still hold for infinite potential games. I refer interested readers to [5] for more details about Nash equilibria in infinite potential games.

## 3.2 Convergence to Nash equilibrium

A *path* in action set $Y$ is a sequence $\gamma = (y^0, y^1, \cdots)$ such that $y^k = (y^{k-1}_{-i}, x)$ for some $x \in Y_i$. In other words, in each step $k$, only one player is allowed to deviate. $y_0$ is the initial point of the path. Player $i$ is called the deviator in step $k$. $\gamma$ is an *improvement path* with respect to $\Gamma$ if $u_i(y^k) > u_i(y^{k-1})$ for all $k \geq 1$. $\Gamma$ has the *finite improvement property* (FIP) if *every* improvement path is finite.

*Remark 5.* Every finite ordinal potential game has the FIP [1].

It is obvious that every finite improvement path of the ordinal potential games must terminate in an equilibrium point. That is, the sequence of one-sided *better* replies converges to the equilibrium independent of the initial point. Note that the order at which players deviate to a better or best response can be deterministic or random and need not to be synchronized. It is the most interesting property of the potential games especially in order to distributively find the equilibrium of the self-organizing systems. Note that every finite game with the FIP is not an ordinal potential game. However, in [1], it is proved that every finite game has the FIP *iff* it has generalized ordinal potential. I

[2] A strategy $y^\star$ is $\epsilon$-equilibrium of game $\Gamma$ if no player can gain more than $\epsilon$ by unilaterally deviating from this strategy.

refer interested readers to [1] for more information about generalized potential games.

All the improvement paths in an infinite potential game may not be finite. Note that improvement paths are constructed based on better response not best response. However, there is no best-response cycle in infinite pseudo potential games [4]. Since ordinal potential games are subset of pseudo potential games, finite and infinite ordinal potential games have no best-response cycle. For ordinal potential games, Voorneveld *et al.* proved that there is no even better response (improvement) cycle [6].

A path $\gamma = (y^0, y^1, ...)$ is an $\epsilon$-improvement path with respect to $\Gamma$ if for all $k \geq 1$, $u_i(y^k) > u_i(y^{k-1}) + \epsilon$, where $i$ is the unique deviator at step $k$. The game $\Gamma$ has the approximate finite improvement property (AFIP) if for every $\epsilon > 0$, every $\epsilon$-improvement path is finite.

*Remark 6.* Every ordinal potential game with bounded payoff functions has the AFIP [1].

The stability property of NE for pseudo potential games is summarized in the following remarks:

*Remark 7.* For finite pseudo-potential games, the sequential best replies converges to an NE [3].

*Remark 8.* For infinite pseudo-potential games with convex strategy space and single-valued best reply [3], the sequence of *simultaneous* best replies converges to an NE of the game [3].

On other words, If players start with an arbitrary strategy profile and simultaneously deviates to their unique best replies in each period, the process terminates in an NE after *finitely* steps. To remove the conditions restricting general existence of NE and convergence of learning process to NE, one may quantize the action space and construct a discrete action space. In several applications, utilizing the quantization technique converts an infinite potential games to a finite potential games.

Ordinal and pseudo potential games have several other interesting properties which can be found in [1], [6], and [4].

# 4 Potential Games in Wireless Networks

The problem of noncooperative resource allocation in wireless cellular, ad hoc, and cognitive radio networks can be studied as a potential game. Existence of potential function enables us to obtain a fully distributed algorithm for resource allocation problem. In the next subsections, two examples are presented showing the application of pseudo and ordinal potential games in wireless networks. I tried to choose simple examples and explain in simple language to prevent

[3] Games with strictly multi-concave potential (Concave in each players' unilateral deviation) have single-valued best reply

discussion about networking concepts. However, more complicated example can be found in the literature.

## 4.1 Power Control in Cellular Networks [7, 8]

Consider a power-controlled cellular system [4]. In the system, every user is associated with a base station. I consider only downlink transmissions, because the uplink case can be treated similarly. Assume that there are $m$ network nodes. Let $p_i^t$ denote the transmission power at the downlink of node $i$. Let $g_{ij}$ denote the gain from the home base station of user $j$ to user $i$. Then, the interference power received at user $i$ from the downlink of user $j$ is $p_{ij}^r = g_{ij} p_j$. Let $\eta_i$ be the background noise received at node $i$. The quality of service of node $i$ is measured in terms of its signal to noise and interference ratio (SINR):

$$\alpha_i = \frac{g_{ii} p_i^t}{\sum_{j \neq i} g_{ij} p_j^t + \eta_i} = \frac{p_{ii}^r}{\sum_{j \neq i} p_{ij}^r + \eta_i} \tag{6}$$

Note that this model is general enough to represent CDMA [5] systems with matched-filter receivers or TDMA [6]/FDMA [7] systems. The problem of determining the transmission powers can be considered as a noncooperative game. The set of players is the set of network nodes, $M = 1, ..., m$. The action space of each player can be discrete or continuous. The utility of each player depends on the level of transmission power and SINR. On other words, the payoff function of node $i$ can be written as: $u_i(\alpha_i, p_i^t)$. Let $I_i$ denote the interference of other transmissions at node $i$:

$$I_i = \sum_{j \neq i} g_{ij} p_j^t = \sum_{j \neq i} p_{ij}^r. \tag{7}$$

Since SINR of node $i$ depends on $p_i^t$ and $I_i$, the payoff of player $i$ only depends on $p_i^t$ and $I_i$. A quasilinear form of the utility function is $u_i(\alpha_i, p_i^t) = R_i(\alpha_i) - c p_i^t$, where $c$ is the price constant. A common form of function $R_i$ is sigmoid function or $log(1 + \alpha_i)$. If function $R_i$ is monotonic, it can be verified that this game is a game of weak strategic complements/substitues with aggregation $I_i$. If $p = 0$ and utility function is increasing in terms of SINR, then the game is strategic complements and the best-replay is to always set the the power to the maximum allowable level. The role of $p$ is to transform the game from strategic complements to strategic substitutes to avoid excessive congestion. In general, if function $R$ is concave, then the game is of strategic substitutes whenever:

$$c I_i \geq -\frac{\partial^2 R}{\partial p_i^t} \alpha_i, \quad i = 1, ...m. \tag{8}$$

[4] The extension of this method for ad hoc network is straight forward.
[5] Code Division Multiple Access systems
[6] Time Division Multiple Access systems
[7] Frequency Division Multiple Access systems

If the problem is expressed in terms of received powers, then the aggregator would be additive (equation (7)). From the convergence of the best-response process, the Nash equilibrium of this game can be reached with finite number of steps.

### 4.2 Power Control in Cognitive Radio Networks [9, 10]

Consider a single cell network of wireless cognitive radios adjusting their transmit powers in an attempt to achieve a target SINR at a common base station. In general, this can be modeled as a myopic repeated game as follows. The set of power adapting cognitive radios, $N$, forms the player set. Each players action set, $A_i$, is defined as convex set $A_i = [0, a_{\max}]$. Each radios utility function is expressed as follows:

$$u_i(\mathbf{A}) = -\left|\hat{\gamma} - \frac{g_i a_i}{\frac{1}{K}\left\{\sum_{j\neq i} g_j a_j + \eta\right\}}\right| \tag{9}$$

where $a_i$ is the transmission power of radio $i$, $g_i$ is the link gain from node $i$ to base station, $\eta$ is the power of noise at base station, $K$ is the spreading gain, and $\hat{\gamma}$ is the target SINR. In [9], it is proved that game $\gamma = \langle N, \mathbf{P}, \{u_i\}_{i\in N}\rangle$ is an ordinal potential game with potential function as follows:

$$P(A) = 2\hat{\gamma}/K\left(\sum_i \sum_{k>i} g_i g_k p_i p_j\right) + \sum_i \left(-g_i^2 p_i^2 + 2\hat{\gamma}\eta g_i p_i/K\right). \tag{10}$$

This game is an ordinal potential game with convex strategy space and continuous potential function. Hence, the sequence of unilaterally best-responses of each player converges to the NE. This approach enables a distributed implementation of the power allocation problem for the cognitive radio networks.

## 5 Conclusion

In this paper, I reviewed the concept of potential games. Each potential game admits a potential function which is the key idea of potential games. Potential function reflects the behavior of payoff functions. Various notions of potential games were studied in this paper. It was showed how potential functions can represent the payoff function behavior of potential games. Existence of NE and convergence of the learning process to the NE were studied as two main properties of potential games. With two examples, I described how potential games can be used to address the power allocation problem in wireless networks. There exist several other examples demonstrating the way that potential games can be used for resource allocation in wireless networks.

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