Voting System: elections

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Abstract

A voting system allows voters to choose between options. And, an election is an important voting system to select a cendidate. In 1951, Arrow's impossibility theorem in [1] showed that intuitively desirable criteria were mutually contradictory. Gibbard(1973) [2] and Satterthwaite(1975) [3] independently proved that for at least three alternatives, every Pareto optimal and non-manipulable choice rule is dictatorial. It natually follows that strategic voting is unavoidable in most voting systems. This survey is about how much manipulation power a voter has in every neutral voting system, and how to avoid it.

1 Introduction

In choosing new parliamentary representatives, most democracies use a voting system that selects among a group of candidates reported by the voters. The general result of Gibbard(1973) [2] and Satterthwaite(1975) [3] ensures that any voting system satisfying three conditions is vulnerable to manipulation. For example, Internet polls may show that a voter's first candidate has very low chance of winning comparing to the second candidate. As a result, the voter may cast on the second candidate. This means that, for any such voting system, there exist a profile and a voter who, by changing his preference, can induce a new profile resulting in an outcome which is better for him. This kind of manipulation may be undesirable for several reasons. First, the manipulating voter may benefit on the expense of others. Second, in order to obtain a good outcome, the right input should be given to the voting system. Finally, the impossibility of manipulation simplifies the decision process for the voters because they only have to know their own preferences.

In 2005, S. Maus, H. Peters and T. Storcken's [4] proposed a combinatoric argument to count the number of profiles that are manipulable. The analysis is applied to some specific voting systems. E. Friedgut, G Kalai and N. Nisan [5] establish a new low-bound on the total manipulation power. This result will be presented in section 2. Section 3 shows some approaches that are currently used. And, section 4 contains a conclusion.

2 Manipulation Power

2.1 Preliminaries and Notation

Let $[m]$ be a set of m alternatives, $m \geq 3$, over which n voters have preferences.

The preferences of the i-th voter are specified as $x_i \in L$, where L denotes the set of full orders over $[m]$. We view L as a subset of the space $\{0,1\}^{\binom{m}{2}}$ in which each bit denotes the preference between two alternatives. In L, these preferences must be transitive. For each voter i, $x_i \in \{0,1\}^{\binom{m}{2}}$, where for $a, b \in [m]$, $x_i^{a,b} = 1$ if and only if voter i prefers candidate a to candidate b.

Denote x_{-i} the vector x of preference when we want to single out voter i-th. Thus, $x = (x_i, x_{-i})$ and (x'_i, x_{-i}) : only the preference of i-th voter changed.

2.2 Social choice and social welfare function

- A social choice function is defined as $f : L^n \to [m]$.

A social choice function is called neutral if names of the alternatives do not matter. Formally, for any permutation δ of $[m]$, $f(\delta(x_1), .., \delta(x_n)) = \delta(f(x_1, .., x_n))$

Definition: Given any two functions f, g, we denote the distance between f and g as

$$\Delta(f, g) = Pr_{x \in L}[f(x) \neq g(x)]$$

If G is a family of such function we define $\Delta(f, G) = min_{g \in G} \Delta(f, g)$

- A generalized social welfare function is a function $F : L^n \to \{0, 1\}^{\binom{m}{2}}$. For every a, b, we denote $F^{a,b}$ the (a, b)-th bit output of F.

F is said to be neutral if it does not depend on the names of elements of $[m]$.
F is said to have a Condorcet winner on x, if there exists a that for all b, $F^{a,b}(x) = 1$.
F is said to satisfy indepence of irrelevant alternatives(IIA) if for all a, b, $F^{a,b}$ is in fact a function of just $x^{a,b} = (x_1^{a,b}, .., x_n^{a,b})$
Note that if F is both neutral and IIA, then F determined by a single boolean function $f : 0, 1^n \to 0, 1$, that is, $F^{a,b}(x^{a,b}) = f(x^{a,b})$ for all a, b.

2.3 Main theorem

Definition: A manipulation of a social choice function f of x is preference x'_i such that $f(x'_i, x_{-i})$ is prefered by voter i to $f(x)$.

If there exists a profitable manipulation for any voter i, then voter i may be better off by considering voting strategically. Clearly, by reporting x'_i as his preference rather than his true x_i is better for voter i, assuming that all other voters report their true preferences.

Definition: The manipulation power of voter i on a social choice function f, denote $M_i(f)$, is the probability that x'_i is a profitable manipulation of f by voter i where x_1, ..., x_n and x'_i are chosen uniformly at random in L.

In this definition, a uniform distribution is assumed for all preferences. This is certainly unrealistic, but a necessary choice for proving lower bound.

Theorem: There exists a constant $C > 0$ such that for every $\epsilon > 0$ the following holds. If f is a social choice function for n voters over 3 alternatives and $\Delta(f; g) > \epsilon$ for any dictatorship g, then f has total manipulability: $sum_{i=1}^{n} M_i(f) \geq C\epsilon^2$

The proof uses the work of G. Kalai [6] that obtained quantitative versions of Arrow's theorem using methods that involve the Fourier transform on the boolean hypercube. The proof has 2 further steps: first, a qualitative preserving redution to a variant of Gibbard-Satherwaite's theorem which allows multi-voter manipulation, and then, a directed isoperimetric Harris's inequality [7] that gives the bound on single voter manipulation.

3 Approaches

3.1 Criteria in [8]

Voting theorists design some desiable criteria for voting systems. Here are some common criteria:
1) Majority criterion: If there exists a majority that ranks (or rates) a single candidate higher than all other candidates, does that candidate always win?
2) Monotonicity criterion: Is it impossible to cause a winning candidate to lose by ranking him higher, or to cause a losing candidate to win by ranking him lower?
3) Consistency criterion: If the electorate is divided in two and a choice wins in both parts, does it always win overall?
4) Participation criterion: Is it always better to vote honestly than to not

vote? 5) Condorcet criterion: If a candidate beats every other candidate in pairwise comparison, does that candidate always win?
6) Condorcet loser criterion: If a candidate loses to every other candidate in pairwise comparison, does that candidate always lose?
7) Independence of irrelevant alternatives: Is the outcome the same after adding or removing non-winning candidates?
8) Independence of clone candidates: Is the outcome the same if candidates identical to existing candidates are added?
9) Reversal symmetry: If individual preferences of each voter are inverted, does the original winner never win?

It is impossible for a voting system to pass all these criteria. Hence, it is important to decide which criteria are important for the election when implementing a voting system. The following table shows the relationship between the above criteria and many voting system:

	1	2	3 & 4	5	6	7	8	9
Plurality	Yes	Yes	Yes	No	No	No	No	N/A
Borda count	No	Yes	Yes	No	Yes	No	No	Yes
Ranked Pairs	Yes	Yes	No	Yes	Yes	No	Yes	N/A
Runoff voting	Yes	No	No	No	Yes	No	No	N/A
Approval	N/A	Yes	Yes	No	No	Yes	N/A	Yes
Minimax	Yes	Yes	No	Yes	No	No	No	No
Range voting	No	Yes	Yes	No	No	Yes	Yes	Yes
IRV	Yes	No	No	No	Yes	No	Yes	No
Kemeny-Young	Yes	Yes	No	Yes	Yes	No	No	N/A
Schulze	Yes	Yes	No	Yes	Yes	No	Yes	Yes

It is possible to simulate large numbers of virtual elections on a computer and see how various voting systems compare in terms of voter satisfaction. Such simulations are sensitive to their assumptions, particularly with regards to voter strategy, but by varying the assumptions they can give repeatable measures that bracket the best and worst cases for a voting system. These simulations may indicate the fairness between political parties, effective representation of minority or special interest groups, political integration, effective voter participation and legitimacy.

3.2 Ranking Systems [9]

Ranking systems consider the setting in which the set of agents is the same as the set of candidates. Agents vote to express their opinions about each other, with the goal of determining a social ranking. Ranking systems have great practical use, such as, search engines and online auction. An interesting result is that a family of ranking algorithms for quasi-transitivity and ranked independence of irrelevant alternatives exists.

3.3 Computational Hardness Protocals

V. Conitzer and T. Sandholm's [10] gives a voting protocals where determining a beneficial manipulation is hard computationally. The new protocals can be NP-hard, #P-hard or PSPACE-hard to manipulate.

V. Conitzer and T. Sandholm's [11] shows that in protocals designed as above, there may be an efficient algorithm that often finds a successful manipulation (when it exists). For 3 candidates, a random manipulation is a good example that has a non-negligible probability of being profitable.

4 Conclusion

E. Friedgut, G Kalai and N. Nisan's theorem [5] implies that some voter has non-negligible manipulation power for all neutral voting systems. This is proved to 3 candidates case and they conject that some voter still has non-negligible manipulation power for more candidates.

V. Conitzer and T. Sandholm showed that in the worst case, their protocal is hard computationally. They also showed that it is not possible to design a voting system that finding a beneficial manipulation is usually hard.

In the real world, elections are implemented. Similar to mechanism design, we need to choose appropriate criteria for the voting system to satisfy.

Note that 2-candidates election does not have manipulability is most voting system.

References

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