Mixed Strategies; Maxmin

CPSC 532A Lecture 4

January 28, 2008

Lecture Overview

Example games

We saw a variety of example games:

- Zero-sum: matching pennies
- Pure cooperation: coordination
- General-sum: battle of the sexes; prisoner's dilemma

Pareto Optimality

- Sometimes, one outcome o is at least as good for every agent as another outcome o', and there is some agent who strictly prefers o to o'
  - in this case, it seems reasonable to say that o is better than o'
  - we say that o Pareto-dominates o'.

- An outcome o^* is Pareto-optimal if there is no other outcome that Pareto-dominates it.

Best Response, Nash equilibrium

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $a_{-i} = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle$.

- Best response: $a_i^* \in BR(a_{-i})$ iff
  $\forall a_i \in A_i,\ u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$

- Nash equilibrium: stable action profiles.
- $a = \langle a_1, \ldots, a_n \rangle$ is a ("pure strategy") Nash equilibrium iff
  $\forall i,\ a_i \in BR(a_{-i})$.

Lecture Overview

1. Recap
2. Mixed Strategies
3. Fun Game
4. Maxmin and Minmax

Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i.
  - pure strategy: only one action is played with positive probability
  - mixed strategy: more than one action is played with positive probability
    - these actions are called the support of the mixed strategy
- Let the set of all strategies for i be S_i
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$.

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
    - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

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- Best response:
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- Nash equilibrium:
  - $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i,\ s_i \in BR(s_{-i})$
- Every finite game has a Nash equilibrium! [Nash, 1950]
  - e.g., matching pennies: both players play heads/tails 50%/50%

Computing Mixed Nash Equilibria: Battle of the Sexes

	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support

Computing Mixed Nash Equilibria: Battle of the Sexes

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- Let player 2 play B with p, F with $1 - p$.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

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$$
\begin{aligned}
u_1(B) &= u_1(F) \\
2p + 0(1-p) &= 0p + 1(1-p) \\
p &= \frac{1}{3}
\end{aligned}
$$

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    - Why is player 1 willing to randomize?

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- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?
- Let player 1 play B with q, F with $1-q$.

$$u_2(B) = u_2(F)$$
$$q + 0(1-q) = 0q + 2(1-q)$$
$$q = \frac{2}{3}$$

- Thus the mixed strategies $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.

Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy? Different interpretations:

- Randomize to confuse your opponent
  - consider the matching pennies example
- Players randomize when they are uncertain about the other's action
  - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.

Lecture Overview

Fun Game!

	L	R
T	$80, 40$	$40, 80$
B	$40, 80$	$80, 40$

- Play once as each player, recording the strategy you follow.

Fun Game!

	L	R
T	$320, 40$	$40, 80$
B	$40, 80$	$80, 40$

- Play once as each player, recording the strategy you follow.

Fun Game!

	L	R
T	$44, 40$	$40, 80$
B	$40, 80$	$80, 40$

- Play once as each player, recording the strategy you follow.

Fun Game!

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T	$80, 40$; $320, 40$; $44, 40$	$40, 80$
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- What does row player do in equilibrium of this game?
  - row player randomizes 50-50 all the time
  - that's what it takes to make column player indifferent
- What happens when people play this game?
  - with payoff of 320, row player goes up essentially all the time
  - with payoff of 44, row player goes down essentially all the time

Lecture Overview

Maxmin Strategies

- Player i's maxmin strategy is a strategy that maximizes i's worst-case payoff, in the situation where all the other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to i.
- The maxmin value (or safety level) of the game for player i is that minimum amount of payoff guaranteed by a maxmin strategy.
- Why would i want to play a maxmin strategy?

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- Why would i want to play a maxmin strategy?
  - a conservative agent maximizing worst-case payoff
  - a paranoid agent who believes everyone is out to get him

Definition (Maxmin)

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$, and the maxmin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$.

Minmax Strategies

- Player i's minmax strategy against player $-i$ in a 2-player game is a strategy that minimizes $-i$'s best-case payoff, and the minmax value for i against $-i$ is payoff.
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Definition (Minmax, 2-player)

In a two-player game, the minmax strategy for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

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We can generalize to n players.

Definition (Minmax, n-player)

In an n-player game, the minmax strategy for player i against player $j \neq i$ is i's component of the mixed strategy profile s_{-j} in the expression $\arg\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$, where $-j$ denotes the set of players other than j. As before, the minmax value for player j is $\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$.

Minmax Theorem

Theorem (Minimax theorem (von Neumann, 1928))

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

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2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.

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2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.
3. Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).