Coalitional Game Theory

Lecture 22

Lecture Overview

Multiunit Demand

How does VCG behave when (some) bidders may want more than a single unit of the good?

- no longer a $k+1$st-price auction
- instead, all winning bidders who won the same number of units will pay the same amount as each other.
  - the change in social welfare from dropping any of these bidders is the same.
- Bidders who win different numbers of units will not necessarily pay the same per unit prices.
- However, bidders who win larger numbers of units will pay at least as much in total (not necessarily per unit) as bidders who won smaller numbers of units
  - their impact on social welfare will always be at least as great

Winner Determination for Multiunit Demand

- Let m be the number of units available, and let $\hat{v}_i(k)$ denote bidder i's declared valuation for being awarded k units.
- It's no longer computationally easy to identify the winners—now it's a (NP-complete) weighted knapsack problem:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in N} \sum_{1 \leq k \leq m} \hat{v}_i(k) x_{k,i} && (1) \\
\text{subject to} \quad & \sum_{i \in N} \sum_{1 \leq k \leq m} k \cdot x_{k,i} \leq m && (2) \\
& \sum_{1 \leq k \leq m} x_{k,i} \leq 1 \qquad \forall i \in N && (3) \\
& x_{k,i} = \{0, 1\} \qquad \forall 1 \leq k \leq m, i \in N && (4)
\end{aligned}
$$

Winner Determination for Multiunit Demand

$$\text{maximize} \quad \sum_{i \in N} \sum_{1 \leq k \leq m} \hat{v}_i(k) x_{k,i} \tag{1}$$

$$\text{subject to} \quad \sum_{i \in N} \sum_{1 \leq k \leq m} k \cdot x_{k,i} \leq m \tag{2}$$

$$\sum_{1 \leq k \leq m} x_{k,i} \leq 1 \qquad \forall i \in N \tag{3}$$

$$x_{k,i} = \{0, 1\} \qquad \forall 1 \leq k \leq m, i \in N \tag{4}$$

- $x_{k,i}$ indicates whether bidder i is allocated exactly k units
- maximize: sum of agents' valuations for the chosen allocation
- (2): number of units allocated does not exceed number available
- (3): no more than one $x_{\cdot,i}$ is nonzero for any i
- (4): all x's must be integers

Combinatorial auctions

- running a simultaneous ascending auction is inefficient
  - exposure problem
  - inefficiency due to fear of exposure
- if we want an efficient outcome, why not just run VCG?
  - unfortunately, it again requires solving an NP-complete problem
  - let there be n goods, m bids, sets C_j of XOR bids
  - weighted set packing problem:

$$\begin{aligned} \max \sum_{i=1}^{m} x_i p_i & \\ \text{subject to} \sum_{i|g \in S_i} x_i \leq 1 & \qquad \forall g \\ x_i \in \{0,1\} & \qquad \forall i \\ \sum_{k \in C_j} x_k \leq 1 & \qquad \forall j \end{aligned}$$

Combinatorial auctions

$$
\begin{aligned}
\max \sum_{i=1}^{m} x_i p_i & & \\
\text{subject to} \sum_{i|g\in S_i} x_i \leq 1 & & \forall g \\
x_i \in \{0,1\} & & \forall i \\
\sum_{k\in C_j} x_k \leq 1 & & \forall j
\end{aligned}
$$

- we don't need the XOR constraints
  - instead, we can introduce "dummy goods" that don't correspond to goods in the auction, but that enforce XOR constraints.
  - amounts to exactly the same thing: the first constraint has the same form as the third

Lecture Overview

Expressing a bid in combinatorial auctions: OR bidding

- Atomic bid: (S, p) means $v(S) = p$
  - implicitly, an "AND" of the singletons in S
- OR bid: combine atomic bids
- let v_1, v_2 be arbitrary valuations

$$(v_1 \vee v_2)(S) = \max_{\substack{R,T \subseteq S \\ R \cap T = \emptyset}} [v_1(R) + v_2(T)]$$

Theorem

OR bids can express all valuations that do not have any substitutability, and only these valuations.

XOR Bids

- XOR bidding: allow substitutabilities
  - $(v_1 XOR v_2)(S) = \max(v_1(S), v_2(S))$

Theorem

XOR bids can represent any valuation

- this isn't really surprising, since we can enumerate valuations
- however, this implies that they don't represent everything efficiently

Theorem

Additive valuations require linear space with OR, exponential space with XOR

- likewise with many other valuations: any in which the price is different for every bundle

Composite Bidding Languages

- OR-of-XOR
- sets of XOR bids, where the bidder is willing to get either one or zero from each set
  - $(\ldots XOR \ldots XOR \ldots) OR (\ldots) OR (\ldots)$

Theorem

Any downward sloping valuation can be represented using the OR-of-XOR language using at most m^2 atomic bids.

- XOR-of-OR
  - a set of OR atomic bids, where the bidder is willing to select from only one of these sets
- generalized OR/XOR
  - arbitrary nesting of OR and XOR

The OR* Language

- OR*
  - OR, but uses dummy goods to simulate XOR constraints

Theorem

OR-of-XOR size $k \Rightarrow$ *OR* size* k, $\leq k$ *dummy goods*

Theorem

Generalized OR/XOR size $k \Rightarrow$ *OR* size* k, $\leq k^2$ *dummy goods*

Corollary

XOR-of-OR size $k \Rightarrow$ *OR* size* k, $\leq k^2$ *dummy goods*

Advanced topics in combinatorial auctions

- iterative combinatorial auction mechanisms
  - reduce the amount bidders have to disclose / communication complexity
  - allow bidders to learn about each others' valuations: e.g., affiliated values
- non-VCG mechanisms for restricted valuation classes
  - these can rely on polynomial-time winner determination algorithms

Lecture Overview

Introduction

- Our focus is on what groups of agents, rather than individual agents, can achieve.
- Given a set of agents, a coalitional game defines how well each group (or *coalition*) of agents can do for itself.
- We are not concerned with:
  - how the agents make individual choices within a coalition;
  - how they coordinate;
- ...instead, we take the payoffs to a coalition as given.

Definition

- Transferable utility assumption:
  - the payoffs to a coalition may be freely redistributed among its members.
  - satisfied whenever there is a universal *currency* that is used for exchange in the system
  - means that each coalition can be assigned a single value as its payoff.

Definition (Coalitional game with transferable utility)

A coalitional game with transferable utility is a pair (N, v), where

- N is a finite set of players, indexed by i; and
- $v : 2^N \mapsto \mathbb{R}$ associates with each coalition $S \subseteq N$ a real-valued payoff $v(S)$ that the coalition's members can distribute among themselves. We assume that $v(\emptyset) = 0$.

Using Coalitional Game Theory

Questions we use coalitional game theory to answer:

1. Which coalition will form?
2. How should that coalition divide its payoff among its members?

The answer to (1) is often "the grand coalition"—the name given to the coalition of all the agents in N—though this can depend on having made the right choice about (2).

Lecture Overview

Voting Game

Our first example considers a social choice setting.

Example (Voting game)

The parliament of Micronesia is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

More generally, in a voting game, there is a set of agents N and a set of coalitions $\mathcal{W} \subseteq 2^N$ that are *winning* coalitions, that is, coalitions that are sufficient for the passage of the bill if all its members choose to do so. To each coalition $S \in \mathcal{W}$, we assign $v(S) = 1$, and to the others we assign $v(S) = 0$.

Airport Game

Our second example concerns sharing the cost of a public good, along the lines of the road-building referendum.

Example (Airport game)

A number of cities need airport capacity. If a new regional airport is built the cities will have to share its cost, which will depend on the largest aircraft that the runway can accommodate. Otherwise each city will have to build its own airport.
This situation can be modeled as a coalitional game (N, v), where N is the set of cities, and $v(S)$ is the sum of the costs of building runways for each city in S minus the cost of the largest runway required by any city in S.

Minimum Spanning Tree

Next, consider a situation in which agents need to get connected to the public good in order to enjoy its benefit. One such setting is the problem of multicast cost sharing.

Example (Minimum spanning tree game)

A group of customers must be connected to a critical service provided by some central facility, such as a power plant or an emergency switchboard. In order to be served, a customer must either be directly connected to the facility or be connected to some other connected customer. Let us model the customers and the facility as nodes on a graph, and the possible connections as edges with associated costs.

This situation can be modeled as a coalitional game (N, v). N is the set of customers, and $v(S)$ is the cost of connecting all customers in S directly to the facility minus the cost of the minimum spanning tree that spans both the customers in S and the facility.

Auction

Finally, consider an efficient auction mechanism. Our previous analysis treated the set of participating agents as given. We might instead want to determine if the seller would prefer to exclude some interested agents to obtain higher payments. To find out, we can model the auction as a coalitional game.

Example (Auction game)

Let N_B be the set of bidders, and let 0 be the seller. The agents in the coalitional game are $N = N_B \cup \{0\}$. Choosing a coalition means running the auction with the appropriate set of agents. The value of a coalition S is the sum of agents' utilities for the efficient allocation when the set of participating agents is restricted to S. A coalition that does not include the seller has value 0, because in this case a trade cannot occur.

Lecture Overview

Superadditive games

Definition (Superadditive game)

A game $G = (N, v)$ is superadditive if for all $S, T \subset N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$.

- Superadditivity is justified when coalitions can always work without interfering with one another
  - the value of two coalitions will be no less than the sum of their individual values.
  - implies that the grand coalition has the highest payoff
- All our examples are superadditive.

Convex games

An important subclass of superadditive games are the convex games.

Definition (Convex game)

A game $G = (N, v)$ is convex if for all $S, T \subset N$,
$v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.

- Convexity is a stronger condition than superadditivity.
  - However, convex games are not too rare in practice.
  - E.g., the airport game is convex.
- Convex games have a number of useful properties, as we will see later.