Theorem 1 *An efficient social choice function $C : \mathbb{R}^{XN} \to X \times \mathbb{R}^N$ can be implemented in dominant strategies for agents with quasilinear utilities for all $v : O \to \mathbb{R}$ only if $p_i(v) = h(v_{-i}) - \sum_{j \neq i} v_j(x(v))$.*

Proof. From the revelation principle, we can assume that C is *truthfully* implementable in dominant strategies. Thus, from the definition of efficiency, the outcome must be chosen as

$$x = \arg\max_x \sum_i v_i(x) \tag{1}$$

We can write the payment function as

$$p_i(v) = h(v_i, v_{-i}) - \sum_{j \neq i} v_j(x(v)). \tag{2}$$

Observe that we can do this without loss of generality because h can be an arbitrary function that cancels out the second term. Now for contradiction, assume that there exist some v_i and v_i' such that $h(v_i, v_{-i}) \neq h(v_i', v_{-i})$.

Case 1: $x(v_i, v_{-i}) = x(v_i', v_{-i})$. Since C is truthfully implementable in dominant strategies, an agent i whose true valuation was v_i would be better off declaring v_i than v_i':

$$v_i(x(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(x(v_i', v_{-i})) - p_i(v_i', v_{-i}) \tag{3}$$

$$p_i(v_i, v_{-i}) \leq p_i(v_i', v_{-i}) \tag{4}$$

In the same way, an agent i whose true valuation was v_i' would be better off declaring v_i' than v_i:

$$v_i'(x(v_i', v_{-i})) - p_i(v_i', v_{-i}) \geq v_i'(x(v_i, v_{-i})) - p_i(v_i, v_{-i}) \tag{5}$$

$$p_i(v_i', v_{-i}) \leq p_i(v_i, v_{-i}) \tag{6}$$

Thus, we must have

$$p_i(v_i, v_{-i}) = p_i(v_i', v_{-i}) \tag{7}$$

$$h(v_i, v_{-i}) - \sum_{j \neq i} v_j(x(v_i, v_{-i}))) = h(v_i', v_{-i}) - \sum_{j \neq i} v_j(x(v_i', v_{-i})) \tag{8}$$

We are currently considering the case where $x(v_i, v_{-i}) = x(v_i', v_{-i})$. Thus we can write

$$h(v_i, v_{-i}) - \sum_{j \neq i} v_j(x(v_i, v_{-i}))) = h(v_i', v_{-i}) - \sum_{j \neq i} v_j(x(v_i, v_{-i})) \tag{9}$$

$$h(v_i, v_{-i}) = h(v_i', v_{-i}) \tag{10}$$

This is a contradiction.

Case 2: $x(v_i, v_{-i}) \neq x(v_i', v_{-i})$. Without loss of generality, let $h(v_i, v_{-i}) < h(v_i', v_{-i})$. Since this inequality is strict, there must exist some $\varepsilon \in \mathbb{R}^+$ such that $h(v_i, v_{-i}) < h(v_i', v_{-i}) - \varepsilon$.

Our mechanism must work for every v. Consider a case where i's valuation is

$$v_i''(x) = \begin{cases} -\sum_{j \neq i} v_j(x(v_i, v_{-i})) & x = x(v_i, v_{-i}) \\ -\sum_{j \neq i} v_j(x(v_i', v_{-i})) + \varepsilon & x = x(v_i', v_{-i}) \\ -\sum_{j \neq i} v_j(x) - \varepsilon & \text{for any other } x \end{cases} \tag{11}$$

Note that agent i still declares his valuations as real numbers; they just happen to satisfy the constraints given above. Also note that the ε used here is the same $\varepsilon \in \mathbb{R}^+$ mentioned above. From the fact that C is truthfully implementable in dominant strategies, an agent i whose true valuation was v_i'' would be better off declaring v_i'' than v_i:

$$v_i''(x(v_i'', v_{-i})) - p_i(v_i'', v_{-i}) \geq v_i''(x(v_i, v_{-i})) - p_i(v_i, v_{-i}) \tag{12}$$

Because our mechanism is efficient, it must pick the outcome that solves

$$f = \max_x \left(v_i''(x) + \sum_j v_j(x) \right). \tag{13}$$

Picking $x = x(v_i', v_{-i})$ gives $f = \varepsilon$; picking $x = x(v_i, v_{-i})$ gives $f = 0$, and any other x gives $f = -\varepsilon$. Therefore, we can conclude that

$$x(v_i'', v_{-i}) = x(v_i', v_{-i}). \tag{14}$$

Substituting Equation (14) into Equation (12), we get

$$v_i''(x(v_i', v_{-i})) - p_i(v_i'', v_{-i}) \geq v_i''(x(v_i, v_{-i})) - p_i(v_i, v_{-i}). \tag{15}$$

Expand Equation (15):

$$\begin{aligned}\left(-\sum_{j \neq i} v_j(x(v_i', v_{-i})) + \varepsilon \right) - \left(h(v_i'', v_{-i}) - \sum_{j \neq i} v_j(x(v_i'', v_{-i})) \right) \\ \geq \left(-\sum_{j \neq i} v_j(x(v_i, v_{-i})) \right) - \left(h(v_i, v_{-i}) - \sum_{j \neq i} v_j(x(v_i, v_{-i})) \right). \end{aligned} \tag{16}$$

We can use Equation (14) to replace $x(v_i'', v_{-i})$ by $x(v_i', v_{-i})$ on the LHS of Equation (16). The sums then cancel out, and the inequality simplifies to

$$h(v_i, v_{-i}) \geq h(v_i'', v_{-i}) - \varepsilon. \tag{17}$$

Since $x(v_i'', v_{-i}) = x(v_i', v_{-i})$, we can use the argument from Case 1 to show that

$$h(v_i'', v_{-i}) = h(v_i', v_{-i}). \tag{18}$$

Substituting Equation (18) into Equation (19), we get

$$h(v_i, v_{-i}) \geq h(v_i', v_{-i}) - \varepsilon. \tag{19}$$

This contradicts our initial assumption that $h(x(v_i, v_{-i})) < h(x(v_i', v_{-i})) - \varepsilon$. We have thus shown that there cannot exist v_i, v_i' such that $h(v_i, v_{-i}) \neq h(v_i', v_{-i})$.