Reasoning Under Uncertainty: Variable Elimination

CPSC 322 – Uncertainty 6

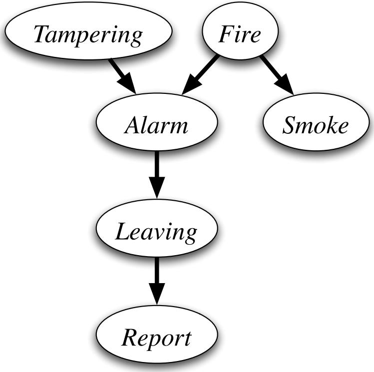
Textbook §6.4

Lecture Overview

1. Recap
2. Observing Variables
3. Belief Network Inference
4. Factors

Example: Fire Diagnosis

The fire diagnosis belief network:



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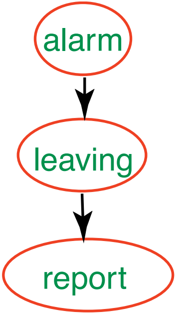
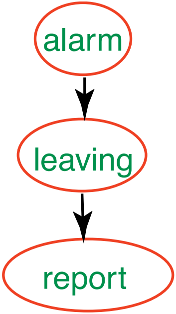
Chain



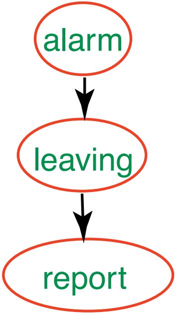
- $alarm$ and $report$ are independent:

Chain



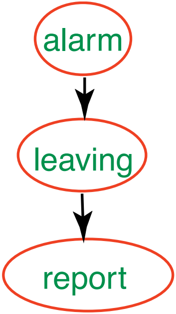
- $alarm$ and $report$ are independent: false.

Chain



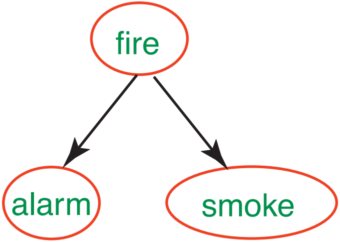
- $alarm$ and $report$ are independent: false.
- $alarm$ and $report$ are independent given $leaving$:

Chain



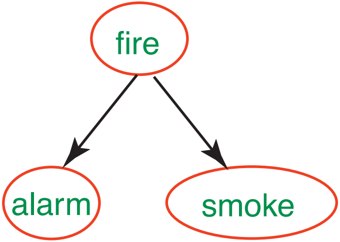
- *alarm* and *report* are independent: false.
- *alarm* and *report* are independent given *leaving*: true.
- Intuitively, the only way that the *alarm* affects *report* is by affecting *leaving*.

Common ancestors



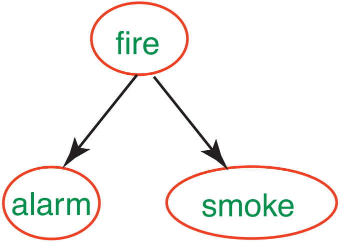
- *alarm* and *smoke* are independent:

Common ancestors



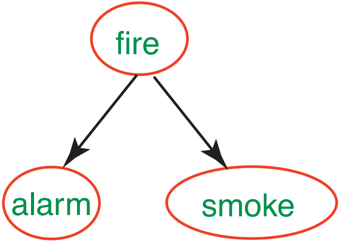
- $alarm$ and $smoke$ are independent: false.

Common ancestors



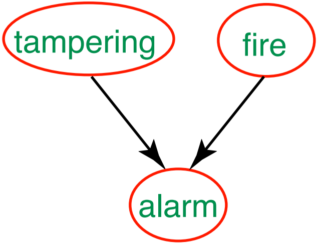
- *alarm* and *smoke* are independent: false.
- *alarm* and *smoke* are independent given *fire*:

Common ancestors



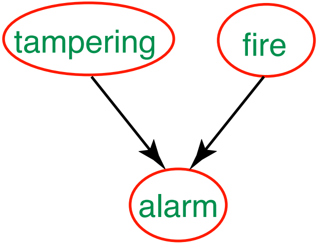
- *alarm* and *smoke* are independent: false.
- *alarm* and *smoke* are independent given *fire*: true.
- Intuitively, *fire* can explain *alarm* and *smoke*; learning one can affect the other by changing your belief in *fire*.

Common descendants



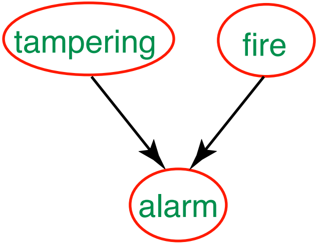
- $tampering$ and $fire$ are independent:

Common descendants



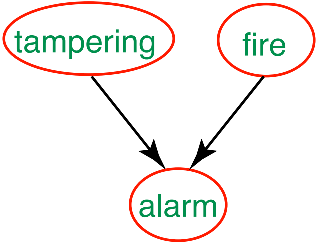
- $tampering$ and $fire$ are independent: true.

Common descendants



- $tampering$ and $fire$ are independent: true.
- $tampering$ and $fire$ are independent given $alarm$:

Common descendants



- $tampering$ and $fire$ are independent: true.
- $tampering$ and $fire$ are independent given $alarm$: false.
- Intuitively, $tampering$ can explain away $fire$

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Belief Network Inference

- Our goal: compute probabilities of variables in a belief network
- Two cases:
  1. the unconditional (prior) distribution over one or more variables
  2. the posterior distribution over one or more variables, conditioned on one or more observed variables

Evidence

- If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j$:

$$
\begin{aligned}
& P(Z \mid Y_1 = v_1, \ldots, Y_j = v_j) \\
= \quad & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{P(Y_1 = v_1, \ldots, Y_j = v_j)} \\
= \quad & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{\sum_Z P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}.
\end{aligned}
$$

- So the computation reduces to the probability of $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$.

Belief Network Inference

- Our goal: compute probabilities of variables in a belief network
- Two cases:
  1. the unconditional (prior) distribution over one or more variables
  2. the posterior distribution over one or more variables, conditioned on one or more observed variables
- To address both cases, we only need a computational solution to case 1
- Our method: exploiting the structure of the network to efficiently eliminate (sum out) the non-observed, non-query variables one at a time.

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Factors

- A factor is a representation of a function from a tuple of random variables into a number.
- We will write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.
- A factor denotes a distribution over the given tuple of variables in some (unspecified) context
  - e.g., $P(X_1, X_2)$ is a factor $f(X_1, X_2)$
  - e.g., $P(X_1, X_2, X_3 = v_3)$ is a factor $f(X_1, X_2)$
  - e.g., $P(X_1, X_3 = v_3 | X_2)$ is a factor $f(X_1, X_2)$

Manipulating Factors

- We can make new factors out of an existing factor
- Our first operation: we can assign some or all of the variables of a factor.
  - $f(X_1 = v_1, X_2, \ldots, X_j)$, where $v_1 \in dom(X_1)$, is a factor on $X_2, \ldots, X_j$.
  - $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.
- The former is also written as $f(X_1, X_2, \ldots, X_j)_{X_1 = v_1, \ldots, X_j = v_j}$

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

$$r(X{=}t, Y{=}f, Z{=}f) = 0.8$$

Summing out variables

Our second operation: we can sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$\begin{aligned}\left(\sum_{X_1} f\right)(X_2, \ldots, X_j) \\ = \quad f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)\end{aligned}$$

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Multiplying factors

- Our third operation: factors can be multiplied together.
- The product of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z}) \quad = \quad f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

- Note: it's defined on all $\overline{X}, \overline{Y}, \overline{Z}$ triples, obtained by multiplying together the appropriate pair of entries from f_1 and f_2.

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 \times f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32