Reasoning Under Uncertainty: Belief Networks

CPSC 322 – Uncertainty 4

Textbook §6.3

Lecture Overview

1. Recap
2. Belief Networks
3. Belief Network Examples

Marginal independence

Definition (marginal independence)

Random variable X is marginally independent of random variable Y if, for all $x_i \in dom(X)$, $y_j \in dom(Y)$ and $y_k \in dom(Y)$,

$$\begin{aligned} & P(X = x_i | Y = y_j) \\ = \quad & P(X = x_i | Y = y_k) \\ = \quad & P(X = x_i). \end{aligned}$$

That is, knowledge of Y's value doesn't affect your belief in the value of X.

Conditional Independence

- Sometimes, two random variables might not be marginally independent. However, they can *become* independent after we observe some third variable.

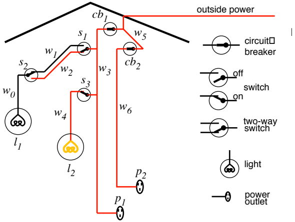
Definition

Random variable X is conditionally independent of random variable Y given random variable Z if, for all $x_i \in dom(X)$, $y_j \in dom(Y)$, $y_k \in dom(Y)$ and $z_m \in dom(Z)$,

$$\begin{aligned} P(X = x_i | Y = y_j \wedge Z = z_m) \\ &= P(X = x_i | Y = y_k \wedge Z = z_m) \\ &= P(X = x_i | Z = z_m). \end{aligned}$$

- That is, knowledge of Y's value doesn't affect your belief in the value of X, given a value of Z.

More examples of conditional independence



- Whether light $l1$ is lit is independent of the position of light switch $s2$ given whether there is power in wire w_0.
  - two random variables that are not marginally independent can still be conditionally independent
- Every other variable may be independent of whether light $l1$ is lit given whether there is power in wire w_0 and the status of light $l1$ (if it's ok, or if not, how it's broken).

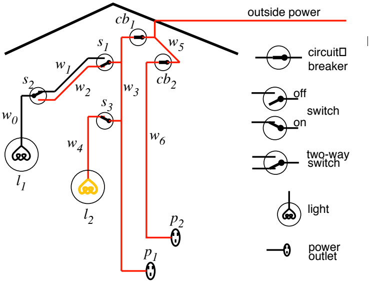
More examples of conditional independence

- The probability that the Canucks will win the Stanley Cup is independent of whether light $l1$ is lit given whether there is outside power.
  - sometimes, when two random variables are marginally independent, they're also conditionally independent given a third variable.
- But not always...
  - Let C_1 be the proposition that coin 1 is heads; let C_2 be the proposition that coin 2 is heads; let B be the proposition that coin 1 and coin 2 are both either heads or tails.
  - $P(C_1|C_2) = P(C_1)$: C_1 and C_2 are marginally independent.
  - But $P(C_1|C_2, B) \neq P(C_1|B)$: if I know both C_2 and B, I know C_1 exactly, but if I only know B I know nothing.
  - Hence C_1 and C_2 are *not* conditionally independent given B.

Lecture Overview

Idea of belief networks

Whether $l1$ is lit ($L1\_lit$) depends only on the status of the light ($L1\_st$) and whether there is power in wire $w0$. Thus, $L1\_lit$ is independent of the other variables given $L1\_st$ and $W0$. In a belief network, $W0$ and $L1\_st$ are parents of $L1\_lit$.



Similarly, $W0$ depends only on whether there is power in $w1$, whether there is power in $w2$, the position of switch $s2$ ($S2\_pos$), and the status of switch $s2$ ($S2\_st$).

Idea of belief networks

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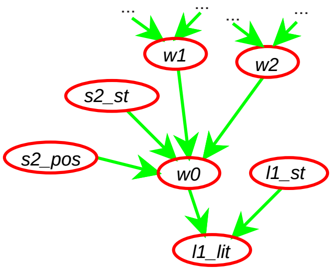

Similarly, $W0$ depends only on whether there is power in $w1$, whether there is power in $w2$, the position of switch $s2$ ($S2\_pos$), and the status of switch $s2$ ($S2\_st$).

Components of a belief network

Definition (belief network)

A belief network consists of:

- a directed acyclic graph with nodes labeled with random variables
- a domain for each random variable
- a set of conditional probability tables for each variable given its parents (which includes prior probabilities for nodes with no parents).

Constructing a belief network

Given a set of random variables, a belief network can be constructed as follows:

- Totally order the variables of interest: $X_1, \ldots, X_n$
- Theorem of probability theory (chain rule): $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | X_1, \ldots, X_{i-1})$
- The parents pX_i of X_i are those predecessors of X_i that render X_i independent of the other predecessors. That is, $pX_i \subseteq X_1, \ldots, X_{i-1}$ and $P(X_i | pX_i) = P(X_i | X_1, \ldots, X_{i-1})$
- So $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | pX_i)$

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Example: Fire Diagnosis

Suppose you want to diagnose whether there is a fire in a building

- you receive a noisy report about whether everyone is leaving the building.
- if everyone *is* leaving, this may have been caused by a fire alarm.
- if there is a fire alarm, it may have been caused by a fire or by tampering
- if there is a fire, there may be smoke

Example: Fire Diagnosis

First you choose the variables. In this case, all are boolean:

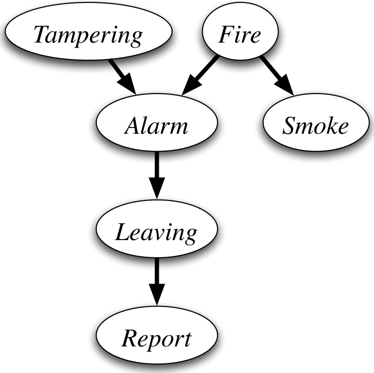
- Tampering is true when the alarm has been tampered with
- Fire is true when there is a fire
- Alarm is true when there is an alarm
- Smoke is true when there is smoke
- Leaving is true if there are lots of people leaving the building
- Report is true if the sensor reports that people are leaving the building

Example: Fire Diagnosis

- Next, you order the variables: $Fire$; $Tampering$; $Alarm$; $Smoke$; $Leaving$; $Report$.
- Now evaluate which variables are conditionally independent given their parents:
  - $Fire$ is independent of $Tampering$ (learning that one is true would not change your beliefs about the probability of the other)
  - $Alarm$ depends on both $Fire$ and $Tampering$: it could be caused by either or both.
  - $Smoke$ is caused by $Fire$, and so is independent of $Tampering$ and $Alarm$ given whether there is a $Fire$
  - $Leaving$ is caused by $Alarm$, and thus is independent of the other variables given $Alarm$.
  - $Report$ is caused by $Leaving$, and thus is independent of the other variables given $Leaving$.

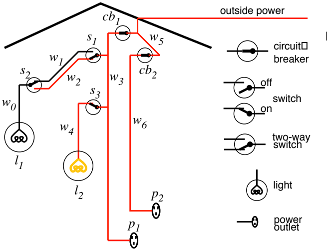
Example: Fire Diagnosis

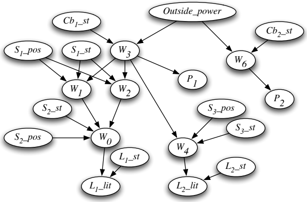
This corresponds to the following belief network:



Of course, we're not done until we also come up with conditional probability tables for each node in the graph.

Example: Circuit Diagnosis





The belief network also specifies:

- The domain of the variables: $W_0, \ldots, W_6 \in \{live, dead\}$ S_1_pos, S_2_pos, and S_3_pos have domain $\{up, down\}$ S_1_st has $\{ok, upside\_down, short, intermittent, broken\}$.
- Conditional probabilities, including: $P(W_1 = live | s_1\_pos = up \wedge S_1\_st = ok \wedge W_3 = live)$ $P(W_1 = live | s_1\_pos = up \wedge S_1\_st = ok \wedge W_3 = dead)$ $P(S_1\_pos = up)$ $P(S_1\_st = upside\_down)$

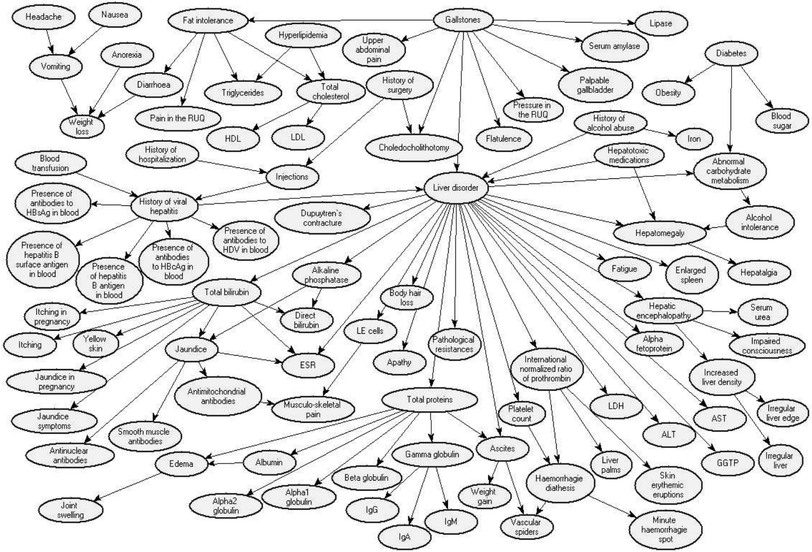
Example: Circuit Diagnosis

The power network can be used in a number of ways:

- Conditioning on the status of the switches and circuit breakers, whether there is outside power and the position of the switches, you can simulate the lighting.
- Given values for the switches, the outside power, and whether the lights are lit, you can determine the posterior probability that each switch or circuit breaker is ok or not.
- Given some switch positions and some outputs and some intermediate values, you can determine the probability of any other variable in the network.

Example: Liver Diagnosis

Source: Onisko et al., 1999



Belief network summary

- A belief network is a directed acyclic graph (DAG) where nodes are random variables.
  - A belief network is automatically acyclic by construction.
- The parents of a node n are those variables on which n directly depends.
- A belief network is a graphical representation of dependence and independence:
  - A variable is conditionally independent of its non-descendants given its parents.